Analysis of some structured preconditioners for saddle point problems

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Collaborations

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Outline

- Statement of the problem

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- Some applications

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- Exact Structured Preconditioners
    1. Spectral properties
    2. Optimality properties
    3. Numerical results

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- Going inexact
    1. Computation: what we lose, what we gain
    2. Spectral analysis

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- Final remarks and work in progress

The Saddle point linear system

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},$$

B full column rank,

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$$\begin{bmatrix} A & B \\ B^T & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},$$

B full column rank,

C semidefinite matrix

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C semidefinite matrix

$$\mathcal{M}u = b$$

$\mathcal{M}$ (symmetric) indefinite matrix

General preconditioning strategy

- Find $\mathcal{P}$ such that

$$\mathcal{M}\mathcal{P}^{-1}\hat{u} = b \qquad \hat{u} = \mathcal{P}u$$

is easier (faster) to solve than $\mathcal{M}u = b$

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- A look at efficiency:
  - Dealing with $\mathcal{P}$ should be cheap
  - Storage for $\mathcal{P}$ should be low
  - Properties (algebraic/functional) exploited

Iterative solver

Projection methods

$$\mathcal{M}u = b \qquad n \times n$$

$$u_k \in \text{Range}(V_k) \subset \mathbb{R}^n, \qquad u_k = V_k y_k \quad y_k \in \mathbb{R}^k \quad k \ll n$$

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e.g. Petrov-Galerkin condition: $r_k = b - \mathcal{M}u_k$ s.t.

$$W_k^T r_k = 0$$

if $W_k = \mathcal{M}V_k \qquad \|r_k\| = \min_{y \in \mathbb{R}^k} \|r_0 - \mathcal{M}V_k y\|$

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V_k generated iteratively, $V_k \subset V_{k+1}$

$$\text{Range}(V_k) = \text{span}\{r_0, \mathcal{M}r_0, \ldots, \mathcal{M}^{k-1}r_0\}$$

Motivation

Numerical approximation of several application problems described by PDEs leads to structured indefinite systems

- (Navier–)Stokes problems
- Elasticity problems
- Mixed formulation of

  Dirichlet's problem

  Biharmonic problems
- Linearly constrained (nonlinear) programs
- . . .

The Magnetostatic problem

(3D) Maxwell equations: $\operatorname{div}\mathbf{B} = 0 \quad \operatorname{curl}\mathbf{H} = \mathbf{J}$

Constitutive law: $\mathbf{B} = \mu\mathbf{H}$

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Constitutive law: $\mathbf{B} = \mu\mathbf{H}$

Constrained quadratic programming formulation:

$$\min \frac{1}{2}\int_\Omega \mu^{-1}\,|\mathbf{B} - \mu\mathbf{H}|^2\,dx$$

with
$$\begin{array}{ll} \mathbf{B}\cdot\mathbf{n} = f_B & \text{on } \Gamma_B \\ \mathbf{H}\wedge\mathbf{n} = \mathbf{f}_H & \text{on } \Gamma_H \end{array}$$
and
$$\begin{array}{l} \operatorname{div}\mathbf{B} = 0 \\ \operatorname{curl}\mathbf{H} = \mathbf{J} \end{array}$$

boundary conditions are enforced, constraints: Lagrange multipliers +.

General formulation

$$\begin{cases} a(\mathbf{u},\mathbf{u}^*) + b(\mathbf{u}^*,\mathbf{p}) = 0 & \forall \mathbf{u}^* \in V \\ b(\mathbf{u},\mathbf{p}^*) \qquad\qquad = (\mathbf{g},\mathbf{p}^*) & \forall \mathbf{p}^* \in Q \end{cases}$$

- $a(.,.)$ *coercive* on $\mathcal{K}^b$:
  for $\mathcal{K}^b = \{\mathbf{u} \in V : \ b(\mathbf{u},\mathbf{q}) = 0 \ \forall \mathbf{q} \in Q\}$ $\qquad a(\mathbf{v},\mathbf{v}) \geq \alpha ||\mathbf{v}||_V, \quad \forall \mathbf{v} \in \mathcal{K}^b$
- *inf-sup* condition: $\beta > 0$ s.t. $\qquad \inf_{\mathbf{q}\in Q} \sup_{\mathbf{v}\in V} \frac{b(\mathbf{v},\mathbf{q})}{||\mathbf{v}||_V} \geq \beta ||\mathbf{q}||_{Q/\ker B}$

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In case of stabilization term:

$$\begin{cases} a(\mathbf{u},\mathbf{u}^*) + b(\mathbf{u}^*,\mathbf{p}) = 0 & \forall \mathbf{u}^* \in V \\ b(\mathbf{u},\mathbf{p}^*) - c(\mathbf{p}^*,\mathbf{p}) = (\mathbf{g},\mathbf{p}^*) & \forall \mathbf{p}^* \in Q \end{cases}$$

Discretization: Choice of suitable Finite Elements

Structured Preconditioners. I

A symmetric. Block diagonal (definite) preconditioner

$$\mathcal{P} = \begin{bmatrix} D & 0 \\ 0 & B^T D^{-1} B + C \end{bmatrix}, \qquad D \approx A$$

Rusten Winther (1992), Silvester Wathen (1993-1994), Klawonn (1998)
Fischer Ramage Silvester Wathen (1998...), . . .

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A symmetric. Block indefinite preconditioner

$$\mathcal{Q} = \begin{bmatrix} D & B \\ B^T & -C \end{bmatrix}$$

Ewing Lazarov Lu Vassilevski (1990), Braess Sarazin (1997) Golub Wathen (1998)
Vassilevski Lazarov (1996), Lukšan Vlček (1998-1999), Perugia S. Arioli (1999), Keller Gould Wathen (2000)
Perugia S. (2000), Gould Hribar Nocedal (2001), Rozloznik S. (2002)

Structured Preconditioners. II

A nonsymmetric, $C = 0$. Block triangular preconditioner

$$\mathcal{R} = \begin{bmatrix} A & B \\ 0 & -B^T A^{-1} B \end{bmatrix}$$

Murphy Golub Wathen (2000), Elman Silvester Wathen (2002), ...

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A nonsymmetric, $C = 0$. Block triangular preconditioner

$$\mathcal{R} = \begin{bmatrix} A & B \\ 0 & -B^T A^{-1} B \end{bmatrix}$$

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A nonsymmetric, $C \neq 0$. Block triangular preconditioner

$$\mathcal{R}_{\mathcal{C}} = \begin{bmatrix} A & B \\ 0 & C_0 \end{bmatrix}, \qquad C_0 \approx C$$

Bramble Pasciak (1988), Elman Silvester (1996), Klawonn (1998), Elman (1999), Krzyzanowski (2001), ...

The exactly preconditioned system

Block Indefinite preconditioner: .

$$\mathcal{M}\mathcal{Q}^{-1}\hat{u} = b$$

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} I & B \\ B^T & 0 \end{bmatrix}^{-1} \hat{u} = b \qquad C = 0, \ D = I$$

(practical implementation)

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(practical implementation)

$$\mathcal{M}\mathcal{Q}^{-1} \;=\; \begin{bmatrix} G & (A-I)B(B^TB)^{-1} \\ 0 & I \end{bmatrix}$$

$$G = I + (A-I)(I - B(B^TB)^{-1}B^T) \equiv I + (A-I)(I-\pi)$$

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$$\mathcal{M}\mathcal{R}^{-1} = \begin{bmatrix} I & 0 \\ B^T A^{-1} & I \end{bmatrix}$$

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Spectral properties

Consider

$$\mathcal{G} = \begin{bmatrix} G_{1,1} & G_{1,2} \\ 0 & G_{2,2} \end{bmatrix}$$

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For Preconditioner $\mathcal{R}$

$$\Lambda(\mathcal{M}\mathcal{R}^{-1}) = \{1\}$$

Convergence of GMRES

Preconditioner $\mathcal{R}$:

$$\mathcal{M}\mathcal{R}^{-1}\hat{u} = b$$

Exact solution in at most 2 iterations

Convergence of GMRES

Preconditioner $\mathcal{Q}$:

$$\mathcal{M}\mathcal{Q}^{-1}\hat{u} = b$$

Assume u_0 s.t. $r_0 = \begin{bmatrix} r_0^{(1)} \\ 0 \end{bmatrix}$

$$\frac{||r_k||}{||r_0||} \leq \kappa(Z) \min_{\phi \in \bar{\mathbb{P}}_k} \max_{\lambda \in \Lambda} |\phi(\lambda)| \qquad G = Z \Lambda Z^{-1}$$

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$$\kappa(Z) \leq (1 + ||\gamma||)^2 \qquad \text{with} \quad ||\gamma|| \leq \frac{||A||}{\min_i |\lambda_i - 1|},$$

Numerical example

2D Magnetostatic problem (cross-section of 3D pb.) (grid)

Size	QMR	QMR$(\mathcal{Q})$
2088	1212	11
3810	1442	11
9102	2919	12
14880	3407	11

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Size	QMR	QMR($\mathcal{Q}$)
1119	2368	15
2208	2825	13
4371	5191	17
8622	$>$10000	16
22675	$>$10000	25

Full 3D problem

Down to Earth

Difficulties in large real application problems

Preconditioner $\mathcal{R}$:

- A cannot be inverted explicitly
- $B^T A^{-1} B$ cannot be dealt with exactly

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Preconditioner $\mathcal{Q}$:

- Factorization of $B^T B + C$ is memory consuming
- Solutions with $B^T B + C$ are too expensive

An example

Inexact Preconditioners. I

$$\mathcal{R} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix} \qquad S = B^T A^{-1} B$$

approximated by

$$\widehat{\mathcal{R}} = \begin{bmatrix} \hat{A} & B \\ 0 & \hat{S} \end{bmatrix}$$

$\hat{A} \approx A \qquad \hat{S} \approx S$ cheap to deal with

Inexact Preconditioners. II

$$\mathcal{Q} = \begin{bmatrix} I & 0 \\ B^T & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -B^T B \end{bmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$$

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$B^T B$ approximated by $\hat{H}$

$$\widehat{\mathcal{Q}}^{-1} = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\hat{H} \end{bmatrix} \begin{bmatrix} I & 0 \\ -B^T & I \end{bmatrix}$$

Trade-offs

What we may loose

Optimality w.r.to meshsize

Independence of solver parameters

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No loss with spectrally equivalent matrices

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What we gain

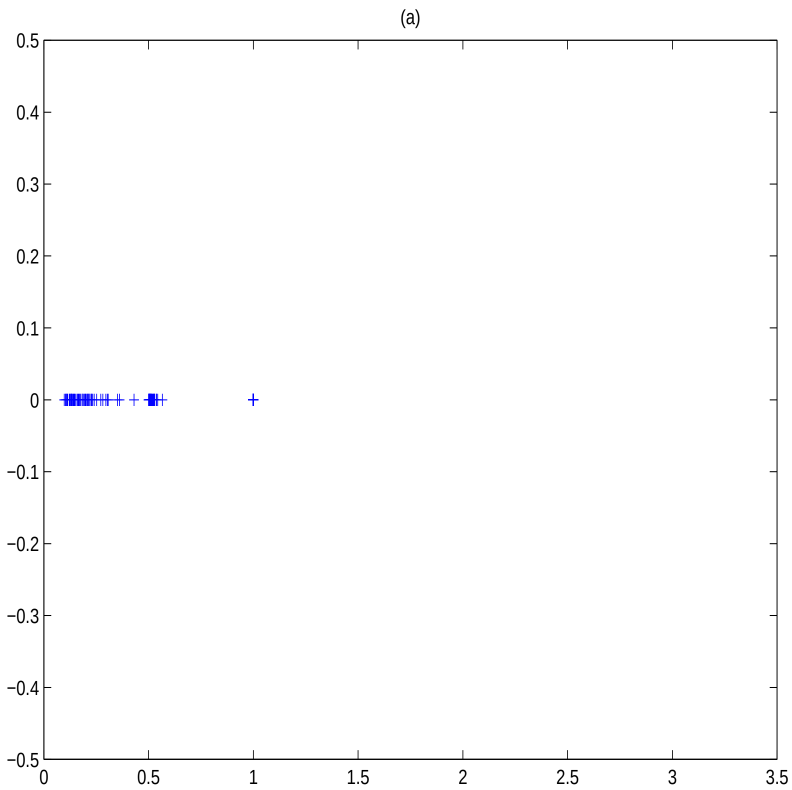
Lower memory requirements

Lower computational cost

An Example

Spectrum of perturbed problem

3D Magnetostatic problem



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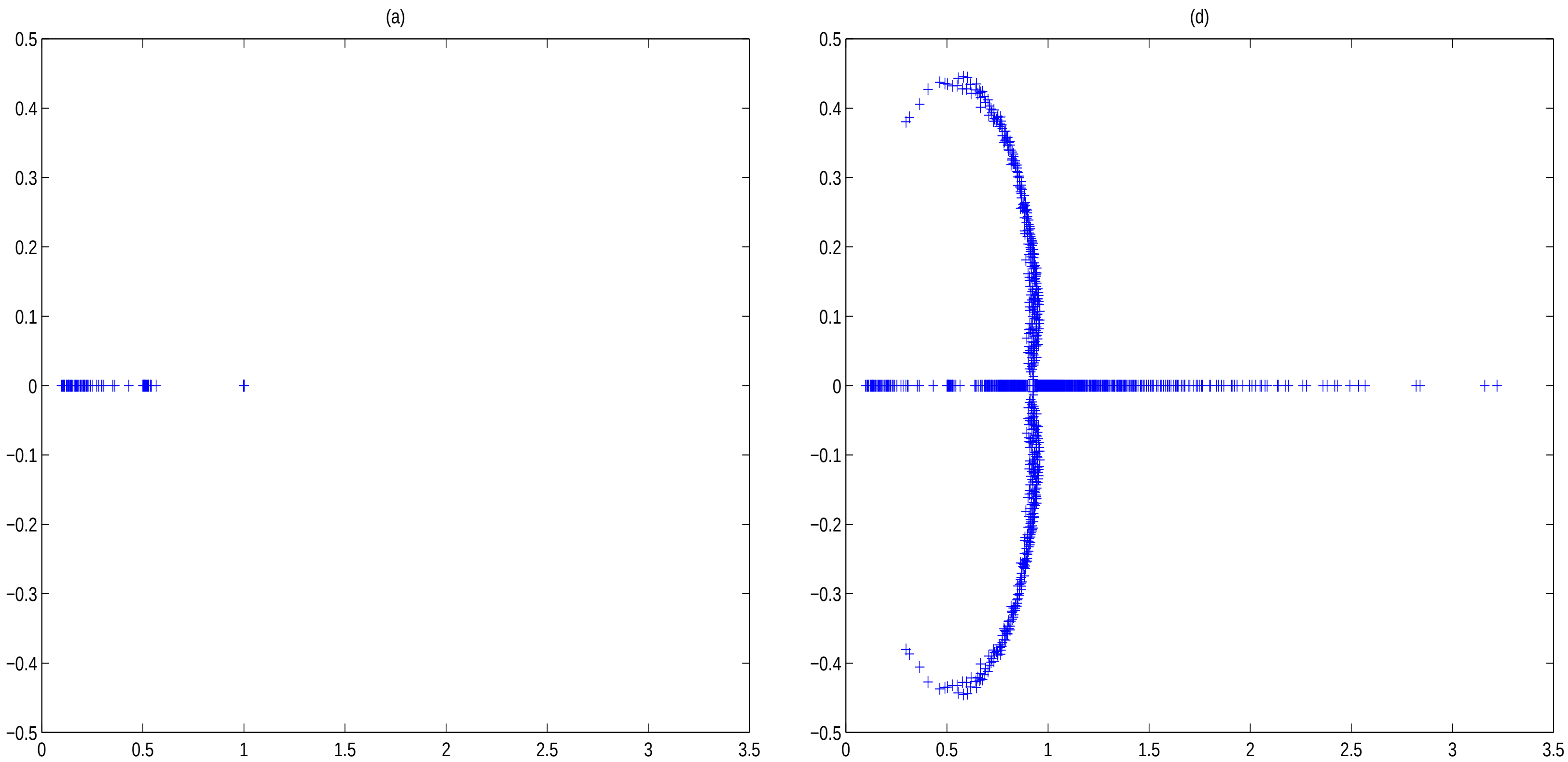
3D Magnetostatic problem



$$\|(B^T B + C) - \hat{H}\|_\infty \approx 2 \cdot 10^{-1} \|B^T B + C\|_\infty$$

Inexact $\Rightarrow$ *Structured Perturbation*

For both preconditioners

$$\widehat{\mathcal{P}} = \mathcal{P} + \mathcal{E} \qquad \mathcal{P} \quad \text{block triangular}$$

see e.g. Elman (1997), Perugia S. (2000)

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e.g. for $\mathcal{Q}$:

$$\widehat{\mathcal{Q}} \;=\; \begin{bmatrix} G & (A-I)B^T(B^TB)^{-1} \\ 0 & I \end{bmatrix} + \begin{bmatrix} E_{1,1} & E_{1,2} \\ E_{2,1} & E_{2,2} \end{bmatrix}$$

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$$= \begin{bmatrix} G + E_{1,1} & (A-I)B^T(B^TB)^{-1} + E_{1,2} \\ 0 & I + E_{2,2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ E_{2,1} & 0 \end{bmatrix}$$

Computational aspects

$$\mathcal{M}\mathcal{P}^{-1}\hat{u} = b \qquad \Rightarrow y_k = \mathcal{P}^{-1} v_k \quad \forall k$$

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$\Rightarrow$ Common strategy: Maintain same $\mathcal{P}_k$ accuracy $\forall k$

Counter–intuitive situation: Fixed accuracy not necessary
$\Rightarrow$ Relax $\mathcal{P}_k$ accuracy as convergence of u_k takes place

Experimental evidence: Bouras Frayssè Giraud (tr. 2000)

Theoretical explanation: Speijpen van den Eshof (tr. 2002), S. Szyld (tr. 2002)

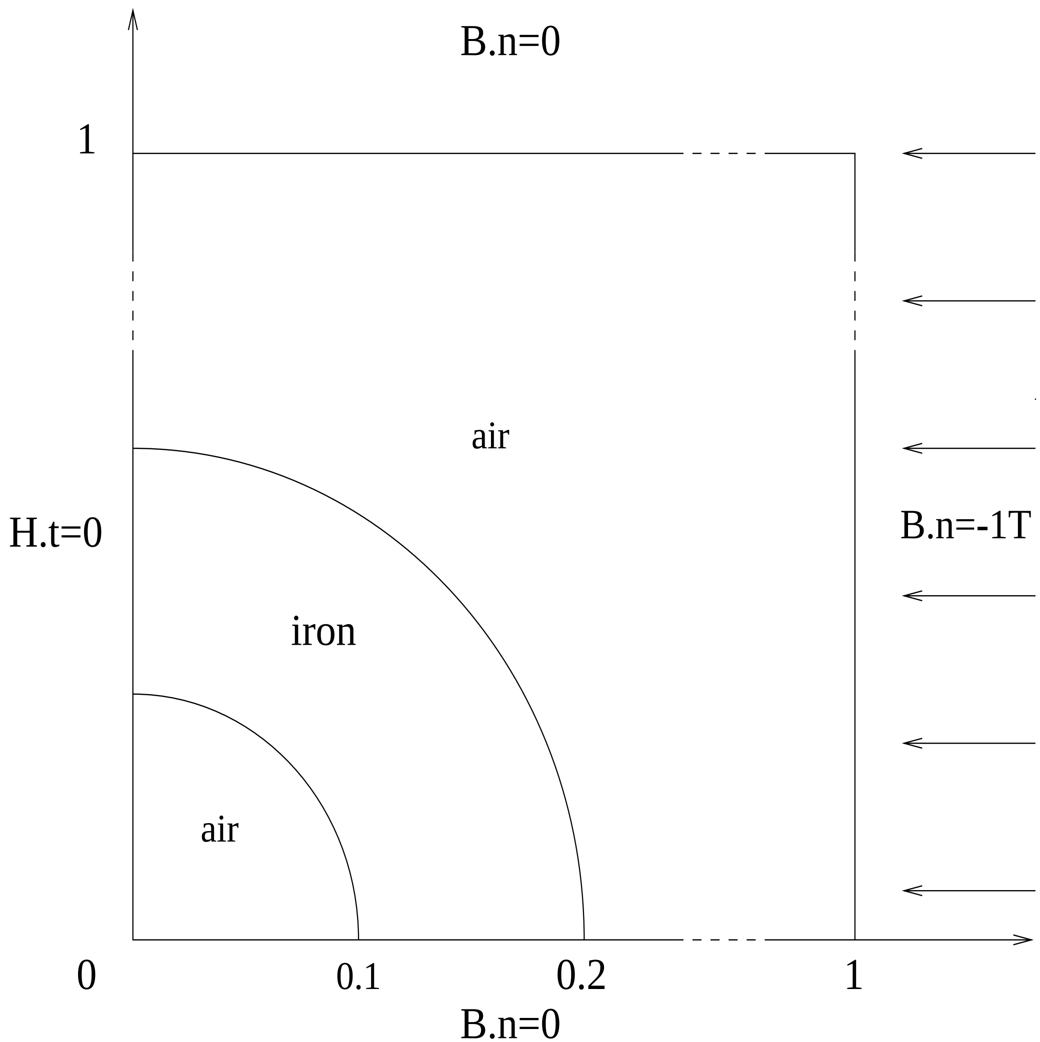
Concluding remarks

- Exploiting structure is rewarding
- Exact case is quite clear
- Inexact preconditioning:

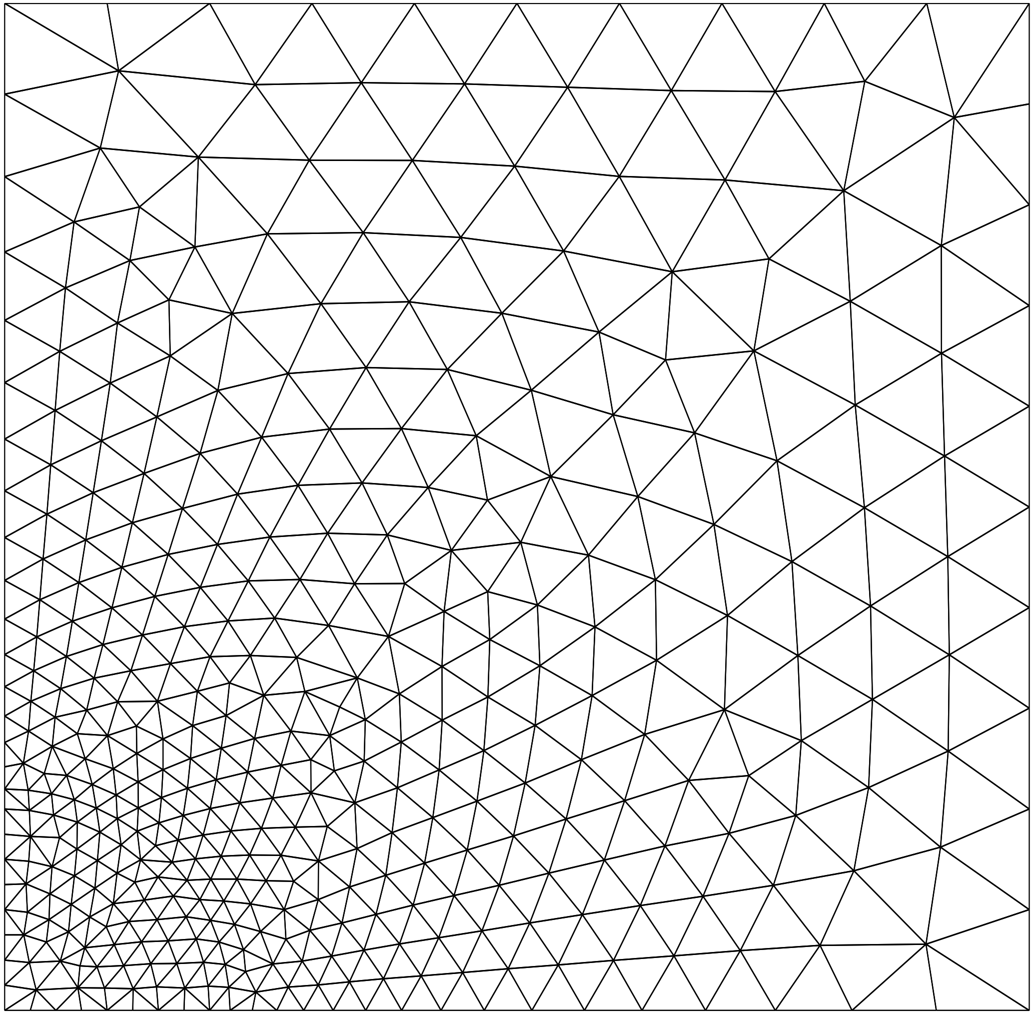
  Competitive on large pbs, Less problem-dependent than exact,

  Tools to monitor sensitivity of preconditioner
- Choice of solver also plays a role

2D problem. 2088×2088 Matrix



2D problem. 2088×2088 *Matrix*



Factorization

$$\mathcal{Q} = \begin{bmatrix} I & 0 \\ B^T & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -B^T B \end{bmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$$

$$\mathcal{Q}^{-1} = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -(B^T B)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -B^T & I \end{bmatrix}$$

+

Computational cost

+ 3D Magnetostatic problem

$\hat{H}$: Incomplete Cholesky fact. (ICT package, Saad & Chow) *

Elapsed Time

Size	MA27	QMR		QMR
		$\mathcal{Q}$	$\widehat{\mathcal{Q}}(2)(it)$	ILDLT(10)
1119	**0.6**	**3.0**	**1.7**(18)	**0.7**
2208	**2.3**	**11.7**	**3.1**(18)	**1.5**
4371	**10.2**	**64.6**	**8.4**(20)	**5.2**
8622	**83.4**	**466.0**	**18.3**(29)	**31.0**
22675	**753.5**	**3745.5**	**63.2**(45)	**246.0**