Esercizio 1. $\boxed{U}$ (Per le trasformate di Laplace: alla fine $z = e^s$)

Dalle tavole
$$\sum_{n\geq 0} n\,\delta(t-n) \xrightarrow{Z} \frac{z}{(z-1)^2}$$
$$\sum_{n\geq 0} \frac{n(n-1)}{2}\,\delta(t-n) \xrightarrow{Z} \frac{z}{(z-1)^3}$$

$n^2 = n(n-1)+n$ quindi $\sum_{n\geq 0} n^2\delta(t-n) \xrightarrow{Z} \frac{2z}{(z-1)^3} + \frac{z}{(z-1)^2} = \boxed{\frac{2z^2-z}{(z-1)^3}}$.

$\boxed{V}$

Cominciamo a trasformare $\tilde{V}^n := \frac{(2n-1)}{4^n} H(n)$
$$(2n-1)H(n) \xrightarrow{Z} \frac{2z}{(z-1)^2} - \frac{z}{(z-1)} = \frac{2z^2-3z}{(z-1)^2}$$
$$4^{-n}(2n-1)H(n) \xrightarrow{Z} \frac{32z^2-12z}{(4z-1)^2} = \tilde{V}(z)$$
$$V(z) = \tilde{V}(z^2) = \frac{32z^4-12z^2}{(4z^2-1)^2}$$

$\boxed{W}$

$$\cos^2(n\pi/3) = \left(\frac{e^{in\pi/3} + e^{-in\pi/3}}{2}\right)^2 = \left[\left(e^{2\pi/3\, i}\right)^n + \left(e^{-2\pi/3\, i}\right)^n + 2\right]\cdot\frac{1}{4}$$
$$W(z) = \left(\frac{z}{z-e^{2\pi/3\, i}} + \frac{z}{z-e^{-2\pi/3\, i}} + \frac{2z}{z-1}\right)\frac{1}{4} = \frac{1}{2}\left(\frac{z(z-\cos\frac{2\pi}{3})}{z^2-2(\cos\frac{2\pi}{3})z+1} + \frac{z}{z-1}\right)$$

$\boxed{X}$

$$\sum n\,\delta(t-n) \to \frac{z}{(z-1)^2} \qquad 2^n\cos n = \frac{1}{2}\left((2\cdot e^{i})^n + (2e^{-i})^n\right)$$

da cui
$$X(z) = \frac{1}{2}\left(\frac{z/2e^{i}}{(z/2e^{i}-1)^2} + \frac{z/2e^{-i}}{(z/2e^{-i}-1)^2}\right)$$
$$= \frac{ze^{i}}{(z-2e^{i})^2} + \frac{ze^{-i}}{(z-2e^{-i})^2}$$

$\boxed{Y}$

$$Y^n = H(n-2) \xrightarrow{\mathcal{Z}} \frac{z^{-1}}{(z-1)} = \frac{1}{z(z-1)}$$

$\boxed{A}$

$$\sum_{n\geq -3}^{+} \delta(t-n) \xrightarrow{\mathcal{Z}} \frac{z^4}{(z-1)} \qquad \sum_{n\geq -3}^{+} \delta(t-3n) \xrightarrow{\mathcal{Z}} \frac{(z^3)^4}{z^3-1} = \frac{z^{12}}{z^3-1}$$

$\boxed{B}$

$$\sum_n^{+} \delta(t-2n+1) = \sum^{+} \delta(t+1) * \sum_n^{+} \delta(t-2n) \xrightarrow{\mathcal{Z}} z \cdot \frac{z^2}{z^2-1} = \frac{z^3}{z^2-1}$$

Esercizio 2.

$$U * H \xrightarrow{\mathcal{Z}} \frac{(2z^2-z)\,z}{(z-1)^4} \qquad \Big|\ Y * H \xrightarrow{\mathcal{Z}} \frac{1}{(z-1)^2}$$

$$\delta(t-3) * A \longrightarrow z^{-3}\,\frac{z^{12}}{z^3-1} = \frac{z^9}{z^3-1} \qquad \Big|\ A * Y \xrightarrow{\mathcal{Z}} \frac{z^{12}}{z(z-1)(z^3-1)}$$

da cui, per antitrasformata,

$$(U * H)^n = \operatorname{Res}\left(\frac{(2z^2-z)\,z^n}{(z-1)^4};\ z=1\right) \cdot H(n) \qquad 1 := \text{polo di } 4^\circ \text{ ordine}$$

$$= \frac{1}{3!}\,\frac{d^3}{dz^3}\left(2z^{n+2} - z^{n+1}\right)\Big|_{z=1} = \frac{1}{6}\left[2(n+2)(n+1)\cdot n - (n+1)\cdot n\,(n-1)\right]$$

$$= \frac{1}{6}\left[(n+1)\,n\,[2n+4 - n+1]\right] = \frac{(n+1)\,n}{6}(n+5) \quad H(n)$$

$$(Y * H)^{**} = \sum_{n\geq 1}^{+} (n-1)\,\delta(t-n) \qquad (Y * H)^n = H(n-1)(n-1)$$

$$\delta(t-3) * A = \sum_{n\geq -3}^{+} \delta(t-3(n+1)) = \sum_{n\geq -2}^{+} \delta(t-3n) \quad \text{direttamente dalla def.}$$

Nel caso di $A*Y$ il segnale non è causale. Si comincia da

$$\frac{z^2}{z(z-1)^2(z^2+z+1)} = \frac{z}{(z-1)^2(z^2+z+1)}$$ che ha poli

in 1 (doppio) e in $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$\mathrm{Res}\left(\frac{z^n}{(z-1)^2(z^2+z+1)}\ ;\ z=1\right) = \frac{d}{dz}\frac{z^n}{(z^2+z+1)}\Big|_{z=1} = \frac{nz^{n-1}(z^2+z+1)-z^n(2z+1)}{(z^2+z+1)^2}\Big|_{z=1}$$

$$= \frac{3n-3}{3^2}$$

$$\mathrm{Res}\left(\frac{z^n}{(z-1)^2(z^2+z+1)}\ ;\ -\frac{1}{2}\pm i\frac{\sqrt{3}}{2}\right) = \frac{\left(-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}\right)^n}{\left(-\frac{3}{2}\pm i\frac{\sqrt{3}}{2}\right)^2\left(\pm i\frac{\sqrt{3}}{2}\right)}$$

$$(A*Y)^n = H(n+10)\left[\frac{1}{9}(3(n+10)-3) + \frac{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{n+10}}{i\frac{\sqrt{3}}{2}\left(i\frac{\sqrt{3}}{2}-\frac{3}{2}\right)} - \frac{\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^{n+10}}{i\frac{\sqrt{3}}{2}\left(-\frac{i\sqrt{3}}{2}-\frac{3}{2}\right)}\right.$$

Esercizio 3

Estendendo $v^n = 0$ per $n < 0$ si ha

$$v^{n+2} - \sqrt{2}\, v^{n+1} + v^n = \begin{cases} 0 & \text{se } n < -2 \\ 1 & \text{se } n = -2 \\ -\sqrt{2} & \text{se } n = -1 \\ 0 & \text{se } n \geq 0 \end{cases}$$

da cui $\left(z^2 - \sqrt{2} z + 1\right) \mathcal{U}(z) = z^2 - \sqrt{2} z$

$$\mathcal{U}(z) = \frac{z^2 - \sqrt{2} z}{z^2 - 2\frac{\sqrt{2}}{2} z + 1} = \frac{z^2 - \frac{\sqrt{2}}{2} z - \frac{\sqrt{2}}{2} z}{z^2 - 2\frac{\sqrt{2}}{2} z + 1} \xrightarrow{\mathcal{Z}^{-1}} \cos\left(\frac{\pi}{4} n\right) - \sin\left(\frac{\pi}{4} n\right) -$$

$$v^n = \left[\cos\left(\frac{\pi}{4} n\right) - \sin\left(\frac{\pi}{4} n\right)\right] H(n)$$

Esercizio 4: In trasformata $\mathcal{Z}$:

$$\left(z^2 + z + \omega^2\right) \mathcal{V}(z) = \left(z^2 + z\right) \mathcal{U}(z)$$

Funzione di trasferimento (razionale) $\dfrac{z^2 + z}{z^2 + z + \omega^2}$

Poli $z^2 + z + \omega^2 = 0 \longrightarrow z = \dfrac{-1 \pm \sqrt{1 - 4\omega^2}}{2}$

(Caso A) $4\omega^2 \leq 1$ radici reali $-\frac{1}{2} \pm \frac{\sqrt{1 - 4\omega^2}}{2}$

poiché $\sqrt{1 - 4\omega^2} \leq 1$ con uguale sse $\omega = 0$
le radici sono sempre in modulo < 1 tranne che per $\omega = 0$
in cui c'è il polo semplice in -1.

Caso B) $4\omega^2 > 1$ radici coniugate $-\frac{1}{2} \pm i \frac{\sqrt{4\omega^2 - 1}}{2}$

di modulo $\frac{1}{4} + \frac{4\omega^2 - 1}{4} = \omega^2$

In sintesi:

Se $\omega=0$ Filtro deb. stabile ma non stabile.

Se $0<\omega<1$ Filtro stabile.

Se $\omega=1$ Filtro deb. stabile ma non stabile

Se $\omega>1$ Filtro non stabile, neppure debolmente.

Se $\omega = 5/6$ le radici sono $-\frac{1}{2} \pm \frac{i}{2}\sqrt{\frac{100-36}{36}} = -\frac{1}{2} \pm \frac{i}{2}\frac{8}{6}$

$$= -\frac{1}{2} \pm \frac{2}{3}i$$

La risposta impulsiva e'

$$A^n = \operatorname{Res}\left(\frac{(z+1)z^n}{z^2+z+\omega^2}; -\frac{1}{2} \pm \frac{2}{3}i\right) = \frac{\left(\frac{1}{2}+\frac{2}{3}i\right)\left(-\frac{1}{2}+\frac{2}{3}i\right)^n}{\frac{4}{3}i} +$$

$$+ \frac{\left(\frac{1}{2}-\frac{2}{3}i\right)\left(-\frac{1}{2}-\frac{2}{3}i\right)^n}{-\frac{4}{3}i} \quad \text{(causale)}$$

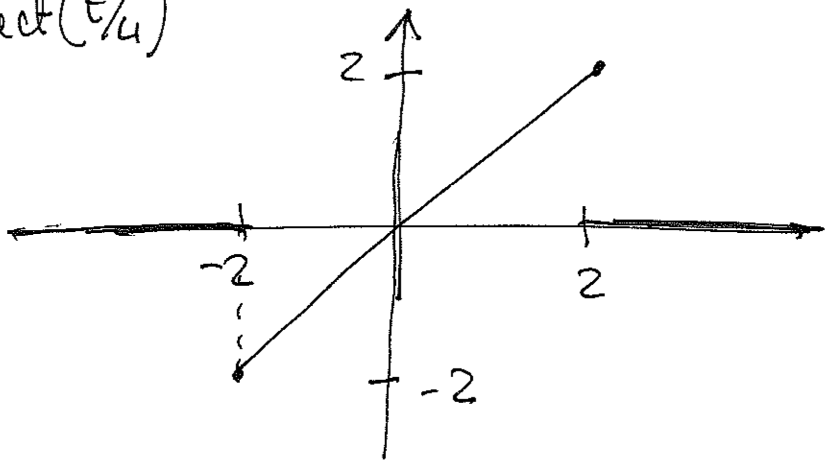
$U^n = A^{n-1} - 4A^n$ nel primo caso.

$$\mathcal{U}(z) = \frac{z}{z+1} \cdot \frac{z^2+z}{z^2+z+\omega^2} = \frac{z^2}{z^2+z+\omega^2}.$$ I poli sono sempre $-\frac{1}{2} \pm i\frac{2}{3}$

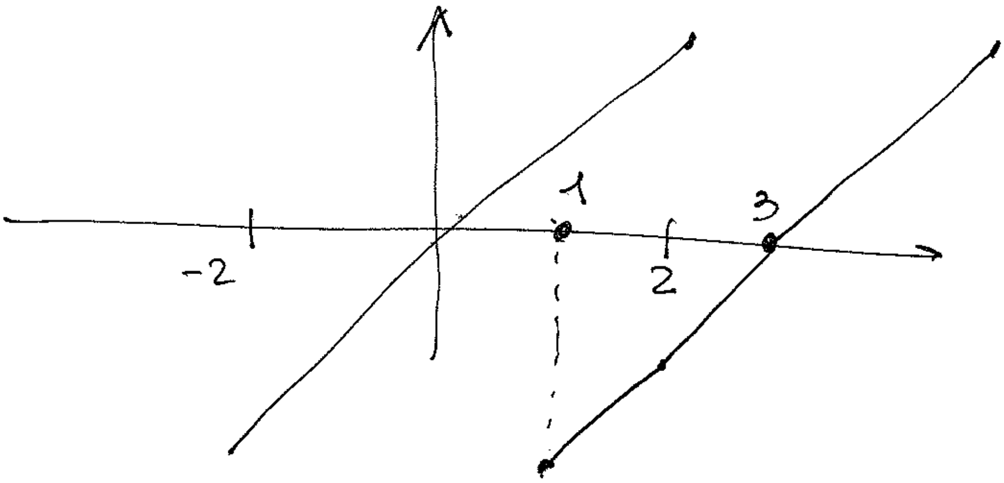
da cui $$U^n = H(n)\left[\frac{\left(-\frac{1}{2}+i\frac{2}{3}\right)^{n+1}}{\frac{4}{3}i} - \frac{\left(-\frac{1}{2}-i\frac{2}{3}\right)^{n+1}}{\frac{4}{3}i}\right]$$

$$= \frac{3}{2}H(n) \operatorname{Im}\left[\left(-\frac{1}{2}+\frac{2}{3}i\right)^{n+1}\right].$$

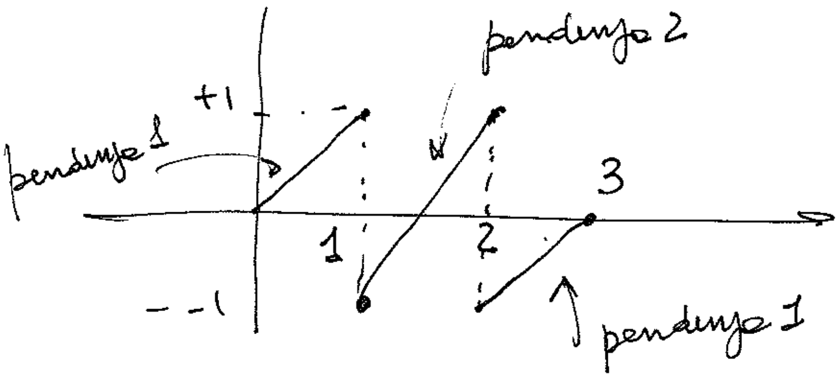
$t\,\mathrm{rect}(t/4)$



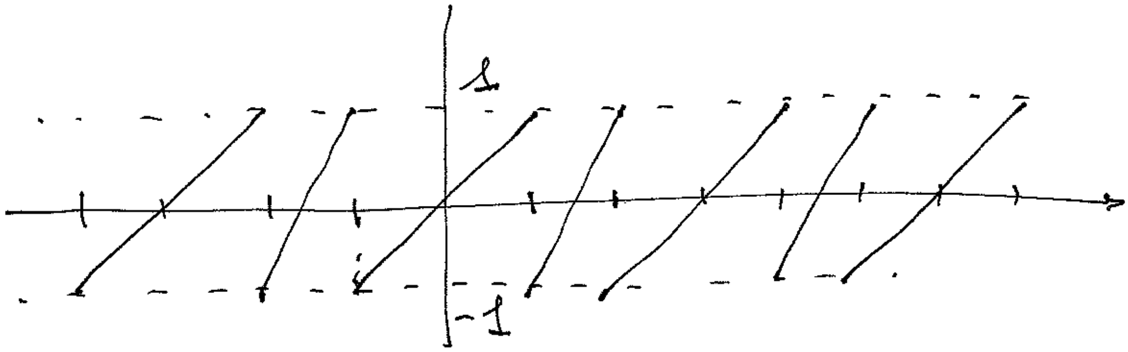
supporto di lunghezza ~~2~~ 4 : basta calcolare 1 traslazione



La somma in $[0,3)$ produce



$v(t) =$



Ultimo esercizio

Passando in trasformata di Fourier (poiché $u \in \mathcal{S}'$ e $v \in \mathcal{S}'$)

$$\left((2\pi i f)^2 + 2\pi i f + 1\right)\hat{v}(f) = \hat{u}(f)$$

$$\left(-4\pi^2 f^2 + 2\pi i f + 1\right)\hat{v}(f) = \hat{u}(f)$$

Ponendo $2\pi i f = s$ si ha $s^2 + s + 1 = 0$ da cui

$s = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ che non sono puramente immaginarie ed hanno parte reale negativa. Quindi il problema di divisione si risolve senza problemi

$$\hat{v}(f) = \frac{1}{-4\pi^2 f^2 + 2\pi i f + 1}\hat{u}(f)$$

Funzione di trasferimento: $\hat{a}(f) = \frac{1}{-4\pi^2 f^2 + 2\pi i f + 1} = \mathcal{A}(2\pi i f)$

dove $\mathcal{A}(s) = \frac{1}{s^2 + s + 1}$ - Poiché $\mathcal{A}$ ha ~~radici~~ poli con parte ~~immaginaria~~ reale negativa, il dominio di $\mathcal{A}$ contiene l'asse immaginario e quindi per invertire $\hat{a}(f)$ si può equivalentemente invertire $\mathcal{A}(s)$ con Laplace

$$\mathcal{A}(s) = \frac{1}{(s+\frac{1}{2})^2 + \frac{3}{4}} \longrightarrow A(t) = a(t) = \frac{2}{\sqrt{3}} H(t) e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) -$$

Essendo a causale, il filtro è causale -