# A fourth-order nonlinear PDE as gradient flow of the Fisher information in Wasserstein spaces

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# *Plan*

1. **The fourth order equation and its structure**
2. **Gradient flows and Wasserstein distance**
3. **Main results and ideas involved in the proof**

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2. **Gradient flows and Wasserstein distance**
3. **Main results and ideas involved in the proof**
    - Minimizing Movement algorithm
    - (Sub)differential calculus in Wasserstein spaces
    - “Second order logarithmic-Sobolev inequalities”
    - Heat flow and Entropy

# *1. The fourth order equation and its structure*

# Main problem

***Global existence of nonnegative solutions*** of the fourth order evolution PDE

$$\boxed{\partial_t u + \; 2\,\mathrm{div}\left(u\,\nabla\Big(\frac{\Delta\sqrt{u}}{\sqrt{u}}\Big)\right) \; = 0 \quad \text{in}\ \Omega_T := \Omega \times (0,T),}$$

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subject to the initial Cauchy condition

$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \quad \text{with} \quad u_0 \geq 0, \quad \int_\Omega u_0(x)\,dx < +\infty$$

and to the variational boundary conditions

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Here $\Omega$ is an open (possibly unbounded) *convex* domain of $\mathbb{R}^d$ with exterior unit normal $\boldsymbol{n}$.

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Divergence form, Neumann boundary condition:

*mass conservation* $\frac{d}{dt}\int_\Omega u(x,t)\,dx = 0;$ normalization $\int_\Omega u_0(x)\,dx = 1$

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**The differential operator:**

$$\boxed{2\,\mathrm{div}\left(u\,\nabla\Big(\frac{\Delta\sqrt{u}}{\sqrt{u}}\Big)\right)} = 2\sum_{i=1}^{d}\partial_i\left(u\,\partial_i\Big(\frac{\Delta\sqrt{u}}{\sqrt{u}}\Big)\right) \tag{I}$$

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$$= \Delta^2 u - \sum_{i,j}\partial_{ij}^2\left(\frac{\partial_i u\,\partial_j u}{u}\right) \tag{II}$$

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&= \Delta^2 u - \sum_{i,j} \partial_{ij}^2 \left(\frac{\partial_i u\,\partial_j u}{u}\right) && \text{(II)}\\
&= \sum_{i,j} \partial_{ij}^2 \Big(u\partial_{ij}^2 \log u\Big). && \text{(III)}
\end{aligned}$$

# *The second form of the equation (II)*

Formal calculations for regular strictly positive solutions show that (I) is equivalent to

$$\partial_t u + \Delta^2 u - \sum_{i,j=1}^{d} \partial_{ij}^2 \left( \frac{\partial_i u \, \partial_j u}{u} \right) = 0 \quad \text{in } \Omega_T, \tag{II}$$

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- **Technique:** Analytic semigroup generated by *the biharmonic operator* in Sobolev spaces continuously imbedded in $C^0(\Omega)$.
  *The singular term* is considered as a perturbation.

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$$\partial_t u + \boxed{\sum_{i,j=1}^{d} \partial_{ij}^2 \left( u \, \partial_{ij}^2 \log u \right)} = 0 \quad \text{in } \Omega_T. \qquad \text{(III)}$$

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- Global existence result for Dirichlet-Neumann boundary conditions in the bounded interval $\Omega = (a, b)$ of $\mathbb{R}$: the initial datum satisfies $\int_a^b \big( u_0(x) - \log u_0(x) \big)\, dx < +\infty$.

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- **Asymptotic decay:** CÁCERES, CARRILLO, JÜNGEL, TOSCANI ('02, '03)

# *Lyapunov functionals: Entropy*

Starting from the third form

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it is easy to see that ***the Entropy***

$$\mathscr{H}(u) := \int_\Omega u(x) \log u(x) \, dx = \int_\Omega u(x) \big( \log u(x) - 1 \big) \, dx$$

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The first form $$\partial_t u + 2\,\mathrm{div}\left(u\nabla\ \frac{\Delta\sqrt{u}}{\sqrt{u}}\ \right) = 0 \qquad \text{(I)}$$

is strictly related to the ***Fisher information functional***

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It is defined for

$u \geq 0,$ such that $s = \sqrt{u} \in H^1(\Omega)$

or, equivalently, for

$u \in W^{1,1}(\Omega)$ such that $\frac{1}{2}\int_\Omega \frac{|\nabla u(x)|^2}{u(x)}\,dx < +\infty$

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$$\frac{\delta\mathscr{I}}{\delta u} = L_z(u,\nabla u) - \sum_i \partial_i L_{p_i}(u,\nabla u) = -\frac{|\nabla u|^2}{2u^2} - \sum_i \partial_i\Big(\frac{\partial_i u}{u}\Big)$$

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$$= -\frac{|\nabla u|^2}{2u^2} + \frac{|\nabla u|^2}{u^2} - \frac{\Delta u}{u} = -2\Big(\frac{\Delta u}{2u} - \frac{|\nabla u|^2}{4u^2}\Big) = -2\boxed{\frac{\Delta\sqrt{u}}{\sqrt{u}}}$$

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Thus the *Fisher information* is another ***Lyapunov functional***, since

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*A priori estimates:*

$$\mathscr{I}(u_T) + 4\int_0^T\int_\Omega \left|\nabla\frac{\Delta\sqrt{u_t}}{\sqrt{u_t}}\right|^2 u_t(x)\,dx = \mathscr{I}(u_0)$$

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$$\mathscr{H}(u_T) + \int_0^T\int_\Omega \sum_{i,j}\left|\frac{\partial^2_{ij}u}{\sqrt{u}} - \frac{\partial_i u\partial_j u}{u\sqrt{u}}\right|^2 dx = \mathscr{H}(u_0)$$

# The structure of the equation

We are dealing (BRENIER, TOSCANI) with a parabolic equation with the following structure

$$\begin{cases} \partial_t u + \operatorname{div} \boldsymbol{q} = 0 & \text{in } \Omega_T, \quad \boldsymbol{q} \cdot \boldsymbol{n} = 0 \qquad \text{on } (\partial\Omega)_T \\ \boldsymbol{q} = u\boldsymbol{v} = u\boldsymbol{\nabla}\psi & \text{in } \Omega_T, \\ \psi = 2\dfrac{\Delta\sqrt{u}}{\sqrt{u}} = -\dfrac{\boldsymbol{\delta}\mathscr{I}}{\boldsymbol{\delta u}} & \text{in } \Omega_T, \quad \sum_i L_{p_i}(u, \nabla u)\boldsymbol{n}_i = 0 \quad \text{on } (\partial\Omega)_T. \end{cases}$$

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(***Continuity equation***)

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We are dealing (BRENIER, TOSCANI) with a parabolic equation with the following structure

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Applications: asymptotic behaviour of solutions, Logarithmic Sobolev Inequalities, approximation algorithms,...
(AGUEH, BRENIER, CARRILLO, MCCANN, GANGBO, GHOUSSOUB, OTTO, VILLANI,...)

# Examples

- $\mathscr{H}(u) := \int u \log u \, dx, \qquad \frac{\boldsymbol{\delta \mathscr{H}}}{\boldsymbol{\delta u}} = \log u + 1,$

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# *2. Gradient flows and Wasserstein distance*

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$u$ ***is a gradient flow iff equality holds,*** i.e.

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# Gradient flows w.r.t. the Wasserstein distance

**Ambient space:** $\mathscr{S} := \Big\{ w \geq 0 \quad \text{a.e. in } \Omega, \quad \int_\Omega w(x)\,dx = 1, \quad \int_\Omega |x|^2 w(x)\,dx < +\infty \Big\}$

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**Gradient flows**
**have finite energy:** $\displaystyle\int_0^T \|\boldsymbol{v_t}\|^{\boldsymbol{2}}_{\boldsymbol{u_t}}\,dt = \int_0^T\int_\Omega \Big|\boldsymbol{\nabla\frac{\delta\mathscr{I}}{\delta u}(u_t)}\Big|^{\boldsymbol{2}} u_t(x)\,dx\,dt \le \mathscr{F}(u_0) < +\infty.$

# The link with the Wasserstein distance

To each curve $u:(0,T)\to\mathscr{S}$ satisfying the continuity equation

$$\partial_t u+\operatorname{div}\big(u\boldsymbol{v}\big)=0\quad\text{in }\Omega_T,\qquad \|\boldsymbol{v}_t\|^2_{u_t}:=\int_\Omega|\boldsymbol{v}_t(x)|^2\,u_t(x)\,dx, \tag{$\star$}$$

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and therefore a “geodesic” distance between two element of $\mathscr{S}$:

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BENAMOU-BRENIER showed that $W$ coincides with the *Wasserstein distance*

$$W^2(\boldsymbol{u_1},u_2) := \inf\Big\{\int_\Omega |\boldsymbol{r}(\boldsymbol{x_1}) - \boldsymbol{x_1}|^2\,\boldsymbol{u_1}(\boldsymbol{x_1})\,d\boldsymbol{x_1} : \boldsymbol{r} : \Omega \to \Omega,$$

$$\int_\Omega \phi(\boldsymbol{x_2})\,\boldsymbol{u_2}(\boldsymbol{x_2})\,d\boldsymbol{x_2} = \int_\Omega \phi(\boldsymbol{r}(\boldsymbol{x_1}))\,\boldsymbol{u_1}(\boldsymbol{x_1})\,d\boldsymbol{x_1} \quad \forall\,\phi \in C^0_0(\Omega)\Big\}$$

# The Wasserstein distance and optimal transportation

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$$\boldsymbol{u_2}(\boldsymbol{x_2}) = \frac{\boldsymbol{u_1}(\boldsymbol{x_1})}{\det \nabla \boldsymbol{r}(\boldsymbol{x_1})}, \quad \boldsymbol{x_2} = \boldsymbol{r}(\boldsymbol{x_1})$$

if $\boldsymbol{r}$ is (essentially) injective and (approximately) differentiable.

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There exists a ***unique*** optimal transport map $\boldsymbol{r} = \mathsf{OptMap}(\boldsymbol{u_1}, \boldsymbol{u_2})$;
$\boldsymbol{r}$ is ***cyclically monotone.*** [BRENIER, KNOTT AND SMITH]

# The velocity vector

If $u : (0,T) \to \mathscr{S}$ is a curve with finite energy, i.e.

$\partial_t u + \mathrm{div}\,\big(u\boldsymbol{v}\big) = 0,$ for some vector field $\boldsymbol{v}_t$ satisfying $\displaystyle\int_{\Omega_T} |\boldsymbol{v}_t(x)|^2 u_t(x)\,dx\,dt < +\infty$

then there exists a unique (up to negligible sets) Borel vector field $\boldsymbol{v_t} \in L^2(\mu_t;\mathbb{R}^d)$, $\mu_t := u_t\,\mathcal{L}^d$, which realizes the minimum in the energy integral:

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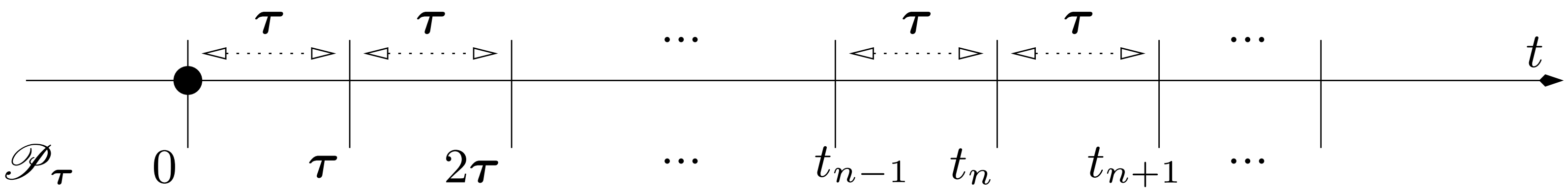
# Approximation of Gradient Flows by the Euler scheme

*In the euclidean case...*

- Choose a ***time step*** $\boldsymbol{\tau} > 0$ and

  a ***partition*** $\mathscr{P}_{\boldsymbol{\tau}} := \{t_1,\ t_2,\ ...,t_n,\ ...\}$ of the time interval $(0, +\infty)$, $\quad t_n := n\boldsymbol{\tau}$,



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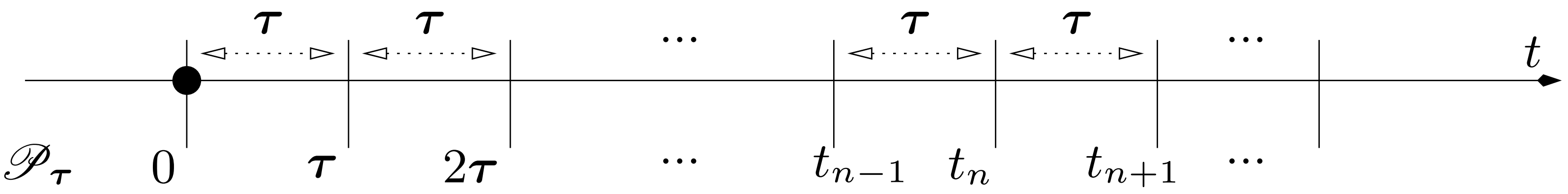

- Starting from the initial datum $\boldsymbol{U^0_{\tau}} := u_0$ *solve recursively the equation w.r.t.* $\boldsymbol{U^n_{\tau}}$, $n = 1, 2, \ldots,$

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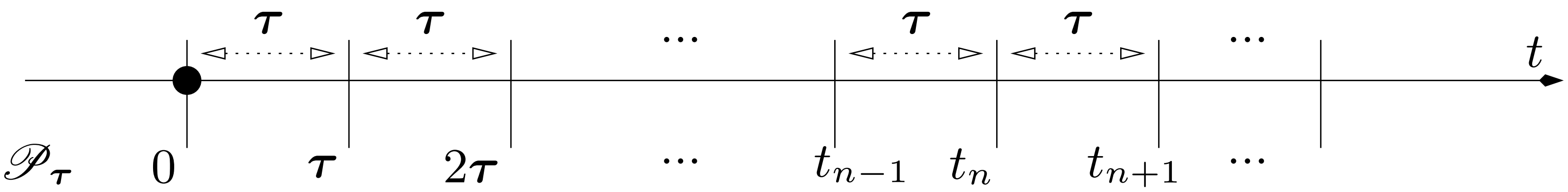

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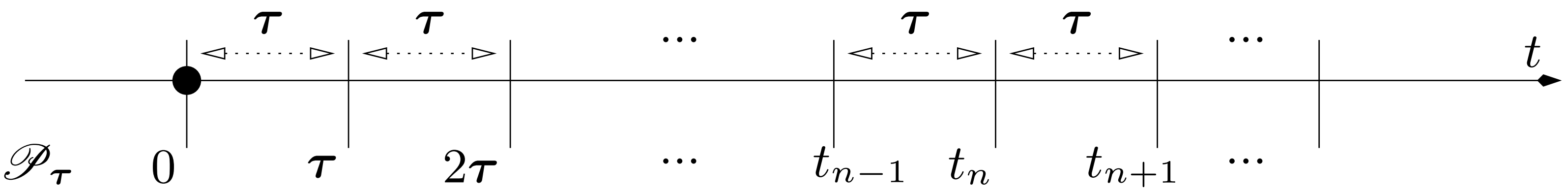

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- ***Resolvent map*** and ***Discrete exponential formula:***

$$\boxed{\boldsymbol{J}_{\boldsymbol{\tau}} : U_{\boldsymbol{\tau}}^{n-1} \to \boldsymbol{U}_{\boldsymbol{\tau}}^{\boldsymbol{n}}, \qquad \boldsymbol{U}_{\boldsymbol{\tau}}^{\boldsymbol{n}} = (\boldsymbol{J}_{\boldsymbol{\tau}})^n[u_0].}$$

# The “Discrete solution”

Approximation algorithm: $\boldsymbol{U_\tau^n}, \; n = 1, 2, \cdots$, solves

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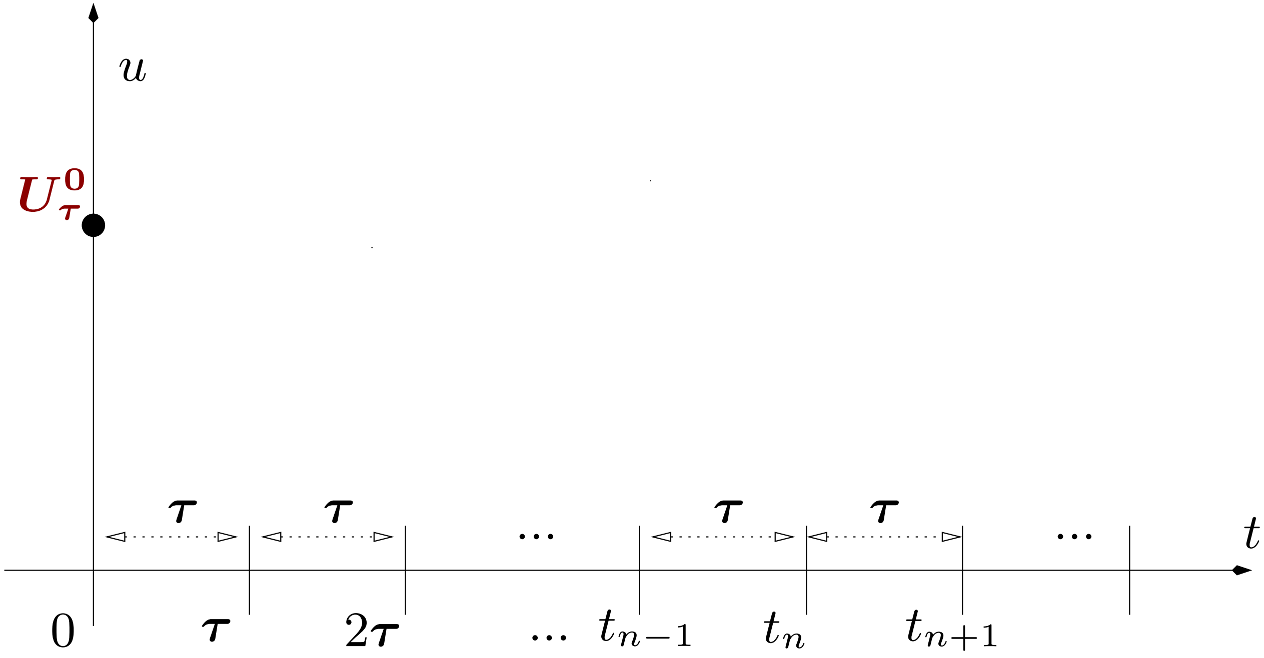

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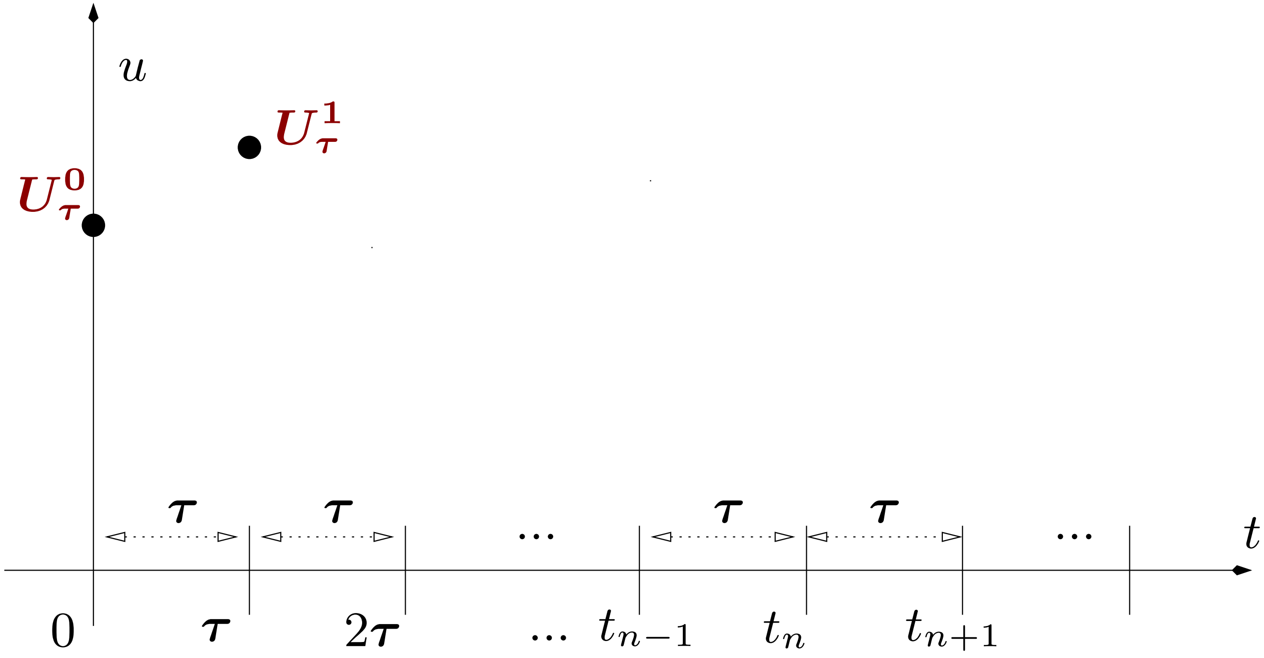

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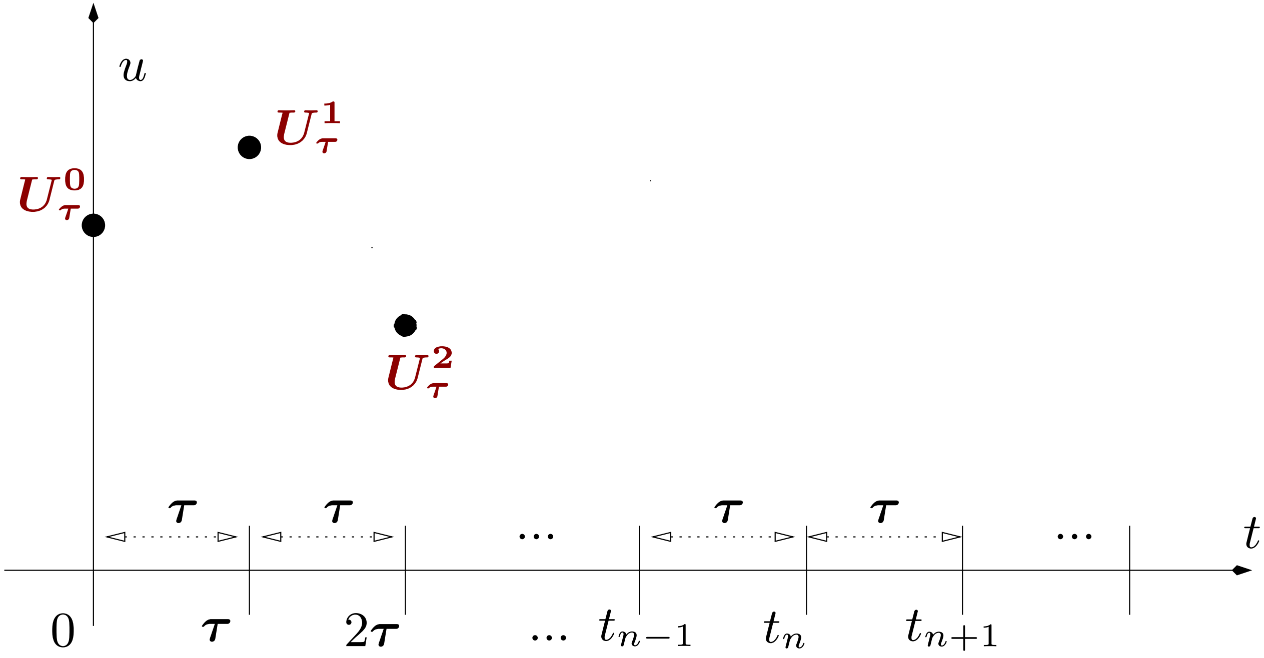

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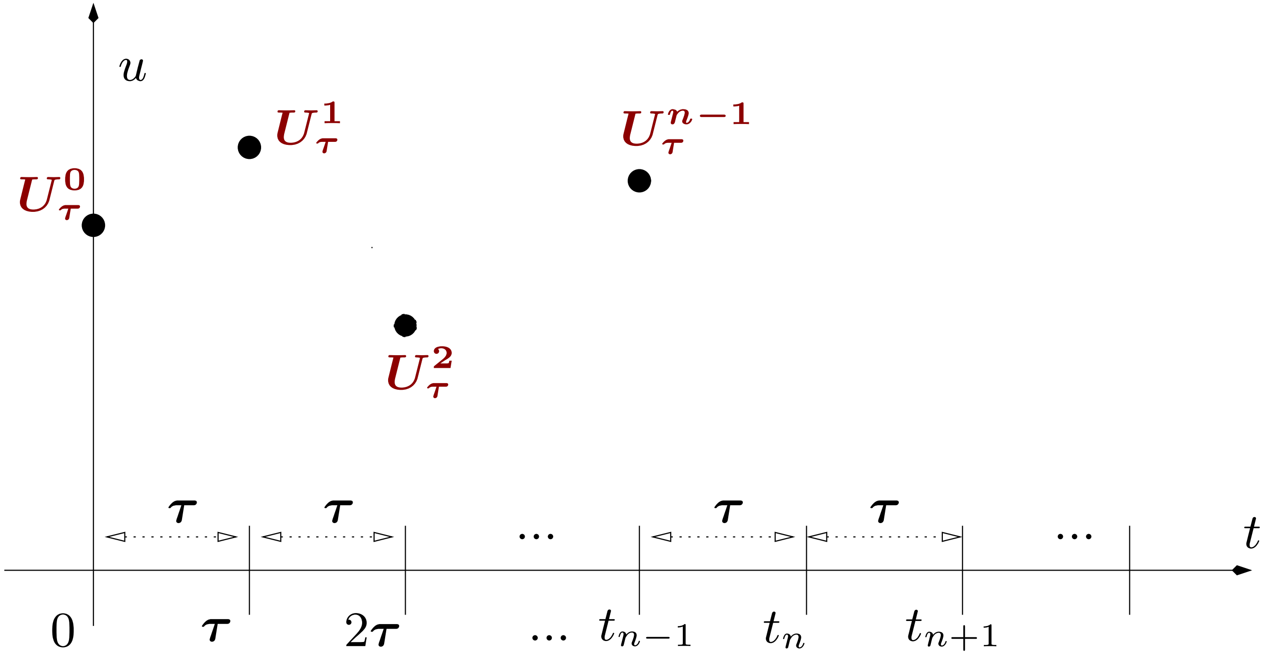

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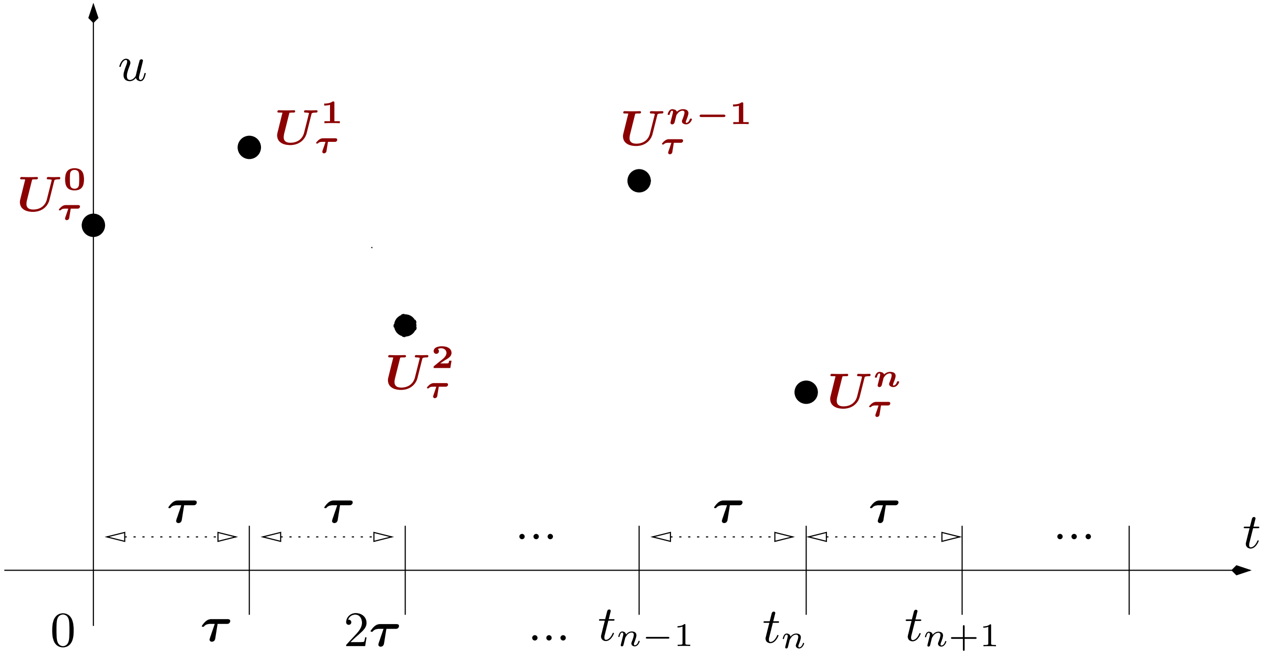

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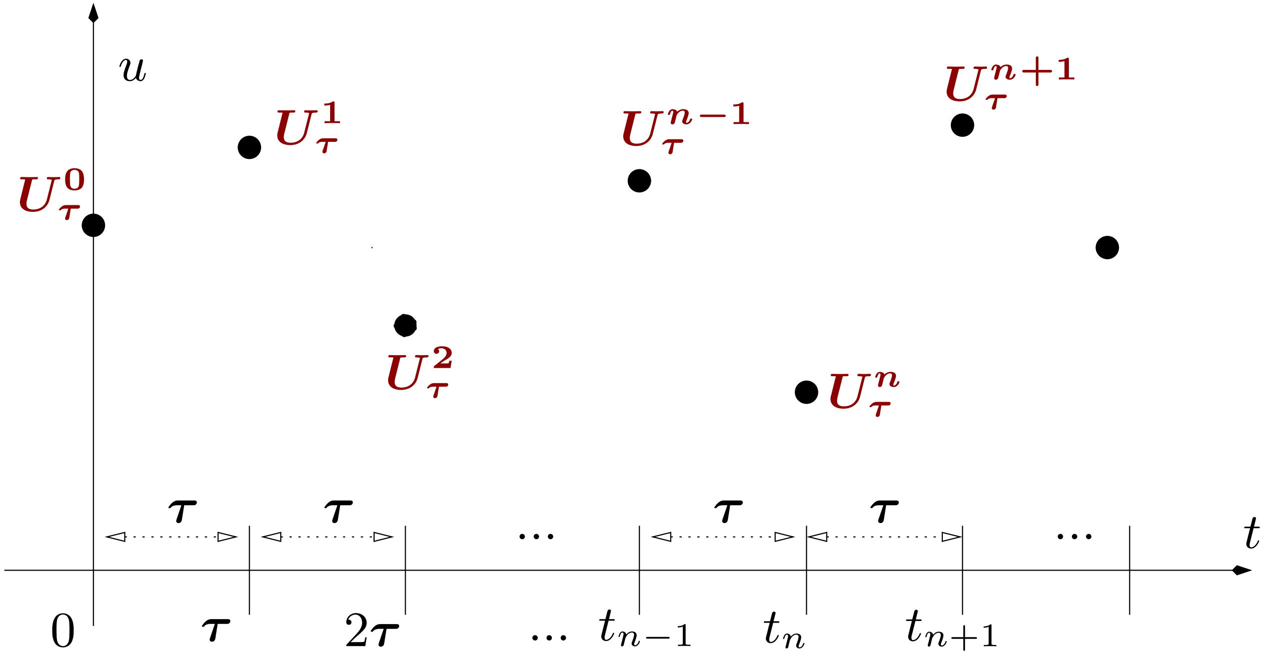

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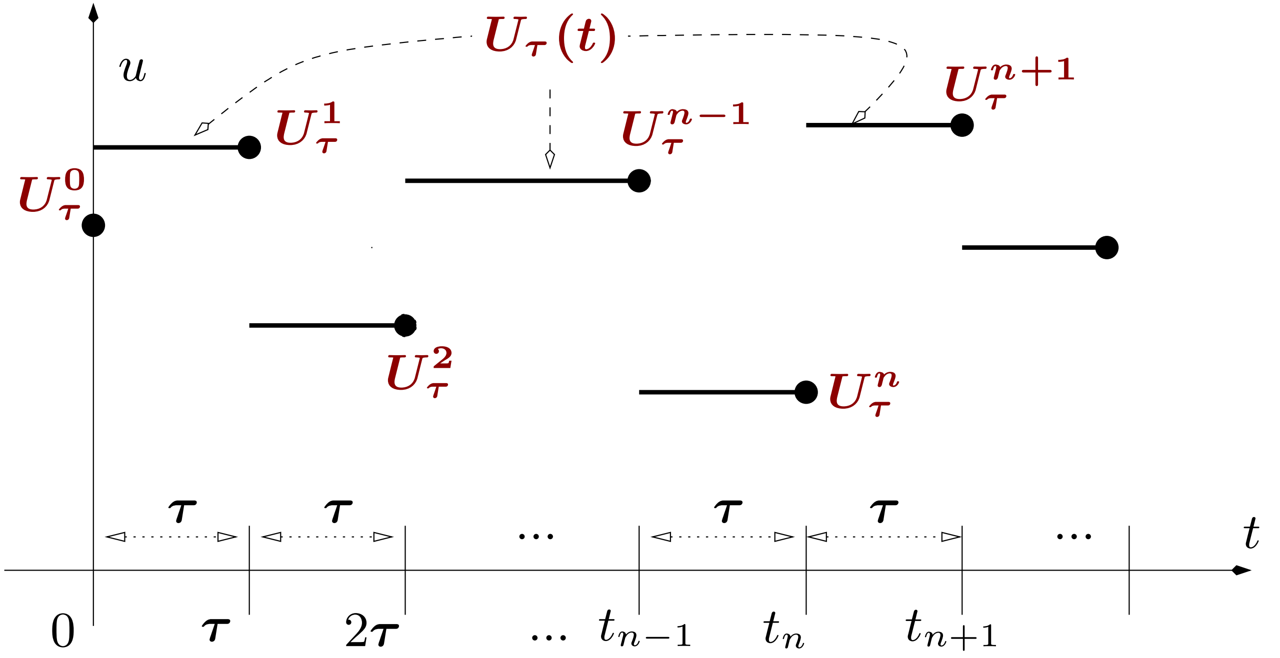

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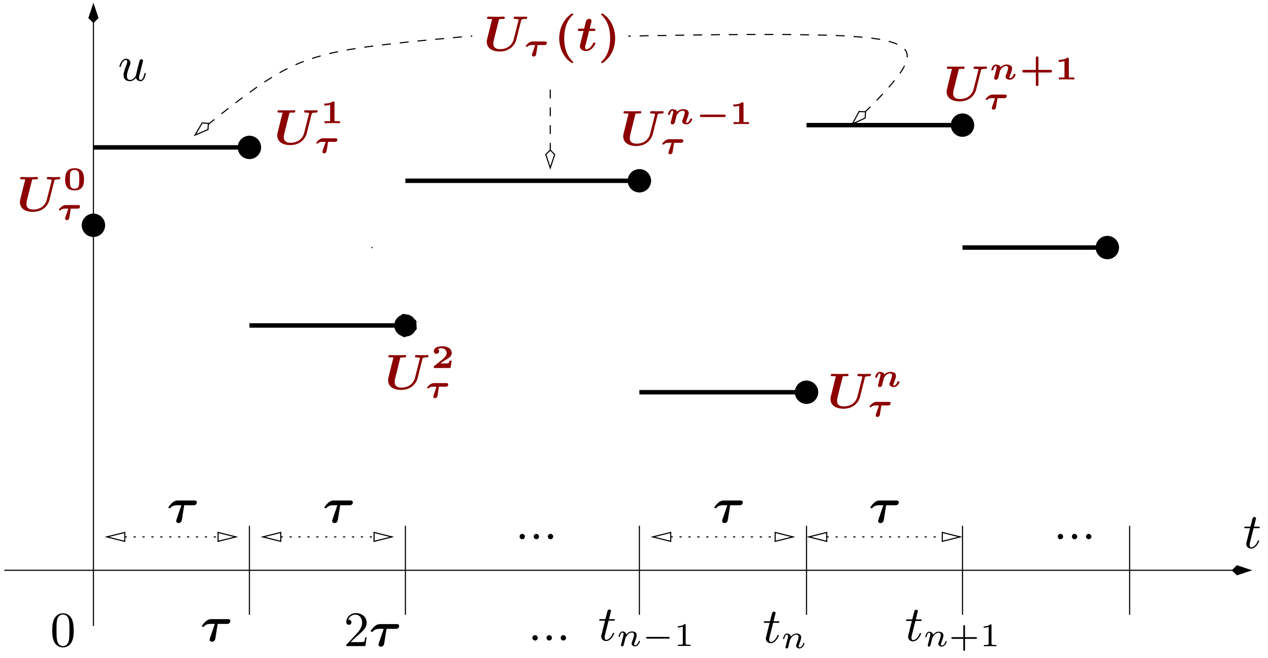

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We look for convergence results (up to subsequence) of $\boldsymbol{U_\tau}$ to a continuous solution

$u$ as $\boldsymbol{\tau} \downarrow 0$.

# The algorithm makes sense even in a metric framework

## Variational algorithm

Find $\boldsymbol{U_\tau^n} = \boldsymbol{J_\tau}(U_{\boldsymbol{\tau}}^{n-1})$ which solves $\dfrac{\boldsymbol{U_\tau^n} - U_{\boldsymbol{\tau}}^{n-1}}{\boldsymbol{\tau}} + \nabla\mathscr{F}(\boldsymbol{U_\tau^n}) = 0$

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DE GIORGI, MARINO, TOSQUES, DEGIOVANNI, AMBROSIO... ’80∼’90

*Abstract theory of minimizing movements and curves of maximal slope*

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| Luckhaus, Visintin, Mielke-Theil-Levitas ... ’90∼’00 | *Phase transitions, hystheresis* |
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| (Ambrosio-Gigli-S. ’04) | *Gradient flows in Metric and Wasserstein spaces* |

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The most difficult point is the third one: we shall need ***extra regularity properties*** and ***a priori estimates*** to pass to the limit.

# The notion of variational solution

$$\begin{cases} \partial_t u + \operatorname{div} \boldsymbol{q} = 0 & \text{in } \Omega_T, \quad \boldsymbol{q}\cdot\boldsymbol{n} = 0 \quad \text{on } (\partial\Omega)_T \\ \boldsymbol{q} = u\boldsymbol{v} = u\boldsymbol{\nabla}\boldsymbol{\psi} & \text{in } \Omega_T, \\ \boldsymbol{\psi} = 2\dfrac{\Delta\sqrt{u}}{\sqrt{u}} = -\dfrac{\boldsymbol{\delta}\mathscr{I}}{\boldsymbol{\delta u}} & \text{in } \Omega_T, \quad \partial_{\boldsymbol{n}} u = 0 \quad \text{on } (\partial\Omega)_T. \end{cases}$$

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***In the sense of distributions in*** $\mathscr{D}'(\mathbb{R}^d \times (0,T))$

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**Finite energy:** $$\int_{\Omega_T} |\boldsymbol{v_t(x)}|^2 u_t(x)\, dx\, dt = \int_{\Omega_T} \frac{|\boldsymbol{q_t(x)}|^2}{u_t(x)}\, dx\, dt < +\infty$$

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(***Integration by parts***)

*The meaning of* $\boldsymbol{q} = 2u\nabla\dfrac{\Delta\sqrt{u}}{\sqrt{u}}$:

$$u\nabla\frac{\Delta\sqrt{u}}{\sqrt{u}} = \nabla\Big(\sqrt{u}\Delta\sqrt{u}\Big) - \nabla u\frac{\Delta\sqrt{u}}{\sqrt{u}} = \nabla\Big(\sqrt{u}\Delta\sqrt{u}\Big) - 2\nabla\sqrt{u}\,\Delta\sqrt{u}.$$

Thus the above expression makes sense if $s := \sqrt{u} \in H^2(\Omega),\ s\Delta s \in W^{1,1}(\Omega)$ and defines $\boldsymbol{q}$ as

$$\boxed{\boldsymbol{q} := 2\nabla\Big(s\Delta s\Big) - 4\nabla s\,\Delta s}$$

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**Definition.** A nonnegative function $u = s^2 \in C^0([0,T]; L^1(\Omega))$ is a ***variational solution*** if

$$s \in L^2(0,T;H^2(\Omega)), \quad \partial_{\boldsymbol{n}} s = 0 \quad \text{on } \big(\partial\Omega\big)_T, \quad s\Delta s \in L^1(0,T;W^{1,1}(\Omega)),$$

and, defining $\boxed{\boldsymbol{q} := 2\nabla\Big(s\Delta s\Big) - 4\nabla s\,\Delta s}$ we have

$$\iint_{\Omega_T} \frac{|\boldsymbol{q}(x,t)|^2}{u(x,t)}\,dx\,dt < +\infty$$

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$$\iint_{\Omega_T} \frac{|\boldsymbol{q}(x,t)|^2}{u(x,t)}\,dx\,dt < +\infty \quad \rightsquigarrow \quad \boldsymbol{v_t} = \frac{\boldsymbol{q_t}}{u_t} \in L^2(\Omega, u_t\,dx; \mathbb{R}^d) \text{ for a.e. } t \in (0,T)$$

$$\iint_{\Omega_T} \Big( - u\partial_t\phi - \boldsymbol{q}\cdot\nabla\phi\Big)\,dx\,dt = \int_{\mathbb{R}^d} u_0(x)\phi(x,0)\,dx \quad \forall\,\phi \in C^\infty_0(\mathbb{R}^d \times [0,T)).$$

# Comparison

The notion of variational solution gives sense to the other forms of the equation, since $s = \sqrt{u} \in H^2(\Omega), \quad s\Delta s \in W^{1,1}(\Omega)$ yield

$$\Delta u \in W^{1,1}(\Omega),\ \Delta^2 u \in W^{-1,1}(\Omega),\ \partial^2_{ij}\frac{\partial_i u \partial_j u}{u} = 4\partial^2_{ij}\big(\partial_i s \partial_j s\big) \in W^{-1,1}(\Omega)$$

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so that

$$\int_\Omega \big|\partial^2_{ij}\log u\big|^2 u\, dx = \int_\Omega \frac{\big|\mathsf{L}_{ij}\big|^2}{u}\, dx$$

# 3. Main results and ideas involved in the proof

# Main result

**Existence of variational solutions:** If

$$u_0 \in \mathscr{S}, \quad \mathscr{I}(u_0) < +\infty$$

then each element $u \in GMM(u_0, \mathscr{I})$ (which is not empty) is a variational solution.

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$$\frac{d}{dt}\mathscr{I}(u_t) \le -\|\boldsymbol{v_t}\|^{\boldsymbol{2}}_{\boldsymbol{u_t}} = -\int_\Omega \left| \frac{\nabla\big(2s_t\Delta s_t\big) - 4\nabla s_t\Delta s_t}{s_t} \right|^2 dx,$$

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**Asymptotic decay:**

$$\mathscr{I}(u_t) \le \frac{C}{t} \quad \text{for } t > 0.$$

# *Proof*

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- General convergence result for the Minimizing Movement approach
- Extra estimates to prove a **closure property of the (sub)gradient of $\mathscr{I}$ under weak convergence:**
    - New second order inequality of Logarithmic-Sobolev type.
    - First variation along the Heat flow

# Wasserstein (Sub)gradient

In the case of a (smooth) functional $\mathscr{F}$ defined in the Euclidean spaces, a vector $\boldsymbol{v} = \nabla \mathscr{F}(u)$

$$\mathscr{F}(w) - \mathscr{F}(u) - \langle \boldsymbol{v}, w - u \rangle = o(|w-u|) \quad \text{as } w \to u$$

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For minimum problems, it is sufficient to consider the *subgradient* of $\mathscr{F}$:

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With this convention, if $u \in \mathscr{S},\ \mathscr{F}(u) < +\infty,\ \mu := u\,\mathcal{L}^d$, a vector $\boldsymbol{v} \in L^2(\mu;\mathbb{R}^d)$ is a (strong) ***subgradient*** for $\mathscr{F}$ at $u$ if

$$\boxed{\mathscr{F}(\boldsymbol{r}_{\#}u) - \mathscr{F}(u) - \langle \boldsymbol{v}, \boldsymbol{r}-\boldsymbol{i}\rangle_\mu \geq o(\|\boldsymbol{r}-\boldsymbol{i}\|_\mu)} \quad \text{as } \boldsymbol{r} \to \boldsymbol{i} \text{ in } L^2(\mu;\mathbb{R}^d).$$

# *Subgradients of Entropy and Fisher information*

As introduced by OTTO, we choose $\boldsymbol{r}^\varepsilon := \boldsymbol{i} + \varepsilon\boldsymbol{\xi}$ for a smooth vector field $\boldsymbol{\xi} \in C_c^\infty(\Omega;\mathbb{R}^d)$ and we observe that if $\boldsymbol{v} \in \partial\mathscr{F}(u)$ then

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In the case of the **Entropy** $\mathscr{H}$ we have

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In the case of the **Fisher Information** $\mathscr{I}$ we have

$$\boldsymbol{v} \in \partial\mathscr{I}(u) \quad \Rightarrow \quad \langle \boldsymbol{v}, \boldsymbol{\xi}\rangle_u = \sum_{i,j}\int_\Omega \Big(\partial^2_{ij}\xi_j\partial_i u + 4\partial_i\xi_j\partial_i s\partial_j s\Big)\,dx, \quad \forall\,\boldsymbol{\xi} \in C^\infty(\Omega;\mathbb{R}^d)$$

# *Subgradients of Entropy and Fisher information*

As introduced by OTTO, we choose $\boldsymbol{r}^\varepsilon := \boldsymbol{i} + \varepsilon\boldsymbol{\xi}$ for a smooth vector field $\boldsymbol{\xi} \in C^\infty_c(\Omega;\mathbb{R}^d)$ and we observe that if $\boldsymbol{v} \in \partial\mathscr{F}(u)$ then

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In the case of the **Entropy** $\mathscr{H}$ we have

$$\boldsymbol{v} \in \partial\mathscr{H}(u) \quad \Rightarrow \quad \begin{cases} \langle \boldsymbol{v}, \boldsymbol{\xi}\rangle_u = -\displaystyle\int_\Omega u(x)\,\mathrm{div}\,\boldsymbol{\xi}(x)\,dx, \quad \forall\,\boldsymbol{\xi} \in C^\infty(\Omega;\mathbb{R}^d) \\ \text{i.e.} \quad \mathscr{I}(u) < +\infty, \quad \boldsymbol{v} = \dfrac{\nabla u}{u} \end{cases}$$

In the case of the **Fisher Information** $\mathscr{I}$ we have

$$\boldsymbol{v} \in \partial\mathscr{I}(u) \quad \Rightarrow \quad \langle \boldsymbol{v}, \boldsymbol{\xi}\rangle_u = \sum_{i,j}\int_\Omega \Big(\partial^2_{ij}\xi_j\partial_i u + 4\partial_i\xi_j\partial_i s\partial_j s\Big)\,dx, \quad \forall\,\boldsymbol{\xi} \in C^\infty(\Omega;\mathbb{R}^d)$$

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# Subgradient and Chain rule

The previous definition of subgradient is well adapted to evaluate the time derivative of functionals along curve $u : (0, T) \to \mathscr{S}$. We consider the case of the Entropy $\mathscr{H}$ and we suppose that

- $u$ has finite energy with velocity vector $\boldsymbol{v_t} \in \mathrm{Tan}_{u_t}\mathscr{S}$
- $\int_0^T \mathscr{I}(u_t)\,dt < +\infty$.

(Recall that $\partial\mathscr{H}(u) = \frac{\nabla u}{u},\ \|\partial\mathscr{H}(u)\|_u^2 = 2\mathscr{I}(u)$:
thus you can formally expect $\frac{d}{dt}\mathscr{H}(u_t) = \langle \boldsymbol{v_t}, \partial\mathscr{H}(u_t)\rangle_{u_t}$)

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Then the map $t \mapsto \mathscr{H}(u_t)$ is absolutely continuous and

$$\boxed{\tfrac{d}{dt}\mathscr{H}(u_t) = \int_\Omega \boldsymbol{v_t(x)} \cdot \nabla u_t(x)\,dx = \langle \boldsymbol{v_t}, \frac{\nabla u_t}{u_t}\rangle_{u_t} \quad \text{a.e. in } (0,T).}$$

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Moreover, if $u$ is the gradient flow of the Entropy, i.e. the solution of the Heat equation, the (right) derivative exists at each time $t$:

$$\boxed{\frac{d}{dt_+}\mathscr{H}(u_t) = \int_\Omega \boldsymbol{v_t(x)} \cdot \nabla u_t(x)\,dx = -\langle \frac{\nabla u_t}{u_t}, \frac{\nabla u_t}{u_t}\rangle_{u_t} = -2\mathscr{I}(u_t)}$$

since $\boldsymbol{v_t} = -\partial\mathscr{H}(u_t) = -\frac{\nabla u_t}{u_t}$.

# Euler equation for the minimization scheme

In the case of a functional $\mathscr{F}$ defined in the Euclidean spaces, if $\boldsymbol{U_\tau^n}$ minimizes

$$\boldsymbol{U} \mapsto \Phi_\tau(\boldsymbol{U}; U_\tau^{n-1}) = \frac{1}{2\tau}|\boldsymbol{U} - U_\tau^{n-1}|^2 + \mathscr{F}(\boldsymbol{U})$$

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$$\frac{U_\tau^{n-1} - \boldsymbol{U_\tau^n}}{\tau} \in \partial\mathscr{F}(\boldsymbol{U_\tau^n})$$

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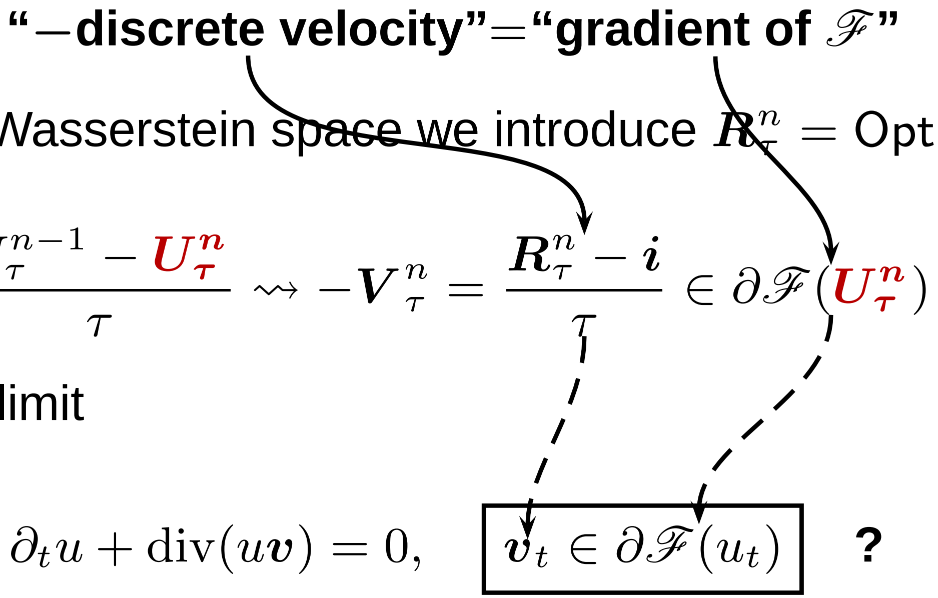

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Observe that the vectors $\boldsymbol{V}_\tau^n$ belongs to different spaces.

# A general convergence result

A functional $\mathscr{F} : \mathscr{S} \to (-\infty, +\infty]$ is ***regular*** if for every $u \in \mathscr{S}$ with and sequences $\boldsymbol{V}_n^i \in \partial\mathscr{F}(U_n^i)$, $i = 1, 2$, such that as $n \to +\infty$

$$U_n^i \rightharpoonup u \quad \text{in } \mathscr{D}'(\Omega), \quad \sup_n \int_\Omega \Big(|x|^2 + |\boldsymbol{V}_n(x)|^2\Big) U_n(x)\, dx < +\infty,$$

$$\mathscr{F}(U_n^i) \to F^i \quad \text{in } \mathbb{R}, \quad \boldsymbol{Q}_n^i := U_n^i \boldsymbol{V}_n^i \rightharpoonup \boldsymbol{q}^i = u\boldsymbol{v}^i \quad \text{in } \mathscr{D}'(\Omega; \mathbb{R}^d)$$

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**Theorem.** If $\mathscr{F}$ is regular then for each initial value $u_0\in\mathscr{S}$ with $\mathscr{F}(u_0)<+\infty$ every $u\in GMM(u_0;\mathscr{F})$ is a continuous curve satisfying the system

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# Main problem: closure of $\partial\mathscr{I}$

We know that if $\boldsymbol{V}_n \in \partial\mathscr{I}(U_n)$, then $\boldsymbol{Q}_n := U_n\boldsymbol{V}_n$ satisfies

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For the Fisher information we try to derive new *a priori* estimates.

# The main estimate

**Theorem.** If $\boldsymbol{V} = \partial\mathscr{I}(U)$ then $S := \sqrt{U} \in H^2(\Omega)$, $\partial_{\boldsymbol{n}} S = 0$ on $\partial\Omega$, $S\Delta S \in W^{1,1}(\Omega)$ and there exists a constant $c = c_d > 0$ such that

$$c\sum_{i,j} \|\partial^2_{ij} S\|^2_{L^2(\Omega)} \leq \sum_{i,j} \int_\Omega \frac{|\mathsf{L}_{ij}(x)|^2}{U(x)}\,dx \leq \int_\Omega \boldsymbol{V}(x)\cdot\nabla U(x)\,dx.$$

where $\mathsf{L}_{\mathsf{i,j}} =$ "$U\partial^2_{ij}\log U$" $= \partial^2_{ij}U - 4\partial_i S\,\partial_j S$.

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$$c\sum_{i,j} \|\partial^2_{ij} S\|^2_{L^2(\Omega)} \le 2\mathscr{I}(U)\Big(\int_\Omega |\boldsymbol{V}(x)|^2 U(x)\,dx\Big)^{1/2}.$$

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This estimate also holds for the limiting subdifferential of $\mathscr{I}$ and it shows that $\mathscr{I}$ is *regular*; an integration by parts yields

$$\boldsymbol{v} \in \partial_\ell\mathscr{I}(u) \quad\Rightarrow\quad \int_\Omega \boldsymbol{v}\cdot\boldsymbol{\xi}\,u\,dx = \int_\Omega \Big(2\nabla(s\Delta s) - 4\nabla s\Delta s\Big)\cdot\boldsymbol{\xi}\,dx, \quad s = \sqrt{u}.$$

i.e.

$$\boldsymbol{v} \in \partial_\ell\mathscr{I}(u) \quad\Rightarrow\quad \boldsymbol{q} = u\boldsymbol{v} = 2\nabla(s\Delta s) - 4\nabla s\Delta s.$$

Therefore, the abstract convergence result guarantees that any limit curve $u \in GMM(u_0;\mathscr{I})$ is a variational solution.

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where $\mathsf{L}_{\mathsf{i,j}} =$ "$U\partial^2_{ij}\log U$" $= \partial^2_{ij}U - 4\partial_i S\,\partial_j S$. In particular

$$c\sum_{i,j} \|\partial^2_{ij} S\|^2_{L^2(\Omega)} \le 2\mathscr{I}(U)\Big(\int_\Omega |\boldsymbol{V}(x)|^2 U(x)\,dx\Big)^{1/2}.$$

Finally, if $u \in GMM(u_0;\mathscr{I})$ is a variational solution with velocity vector $\boldsymbol{v_t}$, the chain rule for the Entropy yields

$$\frac{d}{dt}\mathscr{H}(u_t) = \int_\Omega \boldsymbol{v_t}\cdot\frac{\nabla u_t}{u_t}\,u_t\,dx = -\int_\Omega (-\boldsymbol{v_t})\cdot\nabla u_t\,dx \le -\sum_{i,j}\int_\Omega \frac{|\mathsf{L}_{ij}(x)|^2}{u_t(x)}\,dx$$

since $-\boldsymbol{v_t} \in \partial_\ell\mathscr{I}(u_t)$.

# *The main estimate*

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Two ingredients:

- A sort of "Second order logarithmic Sobolev inequality"
- First variation of $\mathscr{I}$ along the heat flow.

# Second order logarithmic functionals

There are at least three “secoond order” functionals analogous to the Fisher information, which in the case of $u \in C^2(\Omega)$, $u > 0$, read as

$$\mathscr{K}_1(u) := \sum_{i,j} \int_\Omega \left|\partial_{ij}^2 \sqrt{u}\right|^2 \, dx,$$

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It is known (P.L. LIONS, G. TOSCANI ’95) that they are equivalent, up to constants, if we control the $4$-Fisher information

$$\mathscr{I}_4(u) = \int_\Omega \left|\frac{\nabla u}{u}\right|^4 u\, dx = 4^4 \int_\Omega |\nabla \sqrt[4]{u}|^4 \, dx.$$

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E.g. we have

$$\mathscr{K}_3(u) = \sum_{i,j} \int_\Omega \Big(\frac{\partial_{ij}^2 u}{u} - \frac{\partial_i u\, \partial_j u}{u^2}\Big)^2 u\, dx$$

$$= \mathscr{K}_2(u) + \mathscr{I}_4(u) - 2 \int_\Omega \frac{\partial_{ij}^2 u}{u} \frac{\partial_i u\, \partial_j u}{u} \, dx \leq 2\mathscr{K}_2(u) + 2\mathscr{I}_4(u)$$

# *Second order logarithmic Sobolev inequality*

**Theorem.**

$$s = \sqrt{u} \in H^2(\Omega), \quad \text{with} \quad \partial_{\boldsymbol{n}} s = 0 \quad \text{on } \partial\Omega$$

if and only if

$$u \in W^{2,1}(\Omega), \quad \partial_{\boldsymbol{n}} u = 0 \quad \text{on } \partial\Omega, \quad \sum_{i,j} \int_\Omega \frac{|\partial^2_{ij} u|^2}{u} \, dx < +\infty.$$

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$$4 \int_\Omega |\nabla z|^4 \, dx = \int_\Omega \Big( 2 \sum_{i,j} \partial^2_{ij} s \partial_i z \partial_j z + \Delta s |\nabla z|^2 \Big)\, dx, \quad z := \sqrt[4]{u}$$

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# Second order functionals and the Heat flow

Here we use a smoothing technique which, in connection with Entropy and Fisher information, was introduced by BLACHMAN ('65) AND MCKEAN ('66). Recall that $\Omega$ is *convex*, but not necessarily smooth.

Let $u \in \mathscr{S}$ with $\mathscr{I}(u) < +\infty$. We consider the (smooth, positive) solution $s \mapsto \rho_s(\cdot)$ of the heat equation

$$\partial_s \rho - \Delta \rho = 0 \quad \text{in } \Omega \times (0, +\infty), \quad \partial_{\boldsymbol{n}} \rho = 0 \quad \text{on } \partial\Omega \times (0, +\infty)$$

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**Theorem.** For every $h > 0$ we have

$$\mathscr{I}(\rho_h) + \int_0^h \int_\Omega \Big( \sum_{i,j} \big|\partial^2_{ij} \log \rho_s(x)\big|^2 \rho_s(x) \Big)\, dx\, ds = \mathscr{I}(\rho_h) + \int_0^h \mathscr{K}_3(\rho_s)\, ds \le \mathscr{I}(u)$$

# First variation along the heat flow

We want to prove that if $\boldsymbol{v} = \partial \mathscr{I}(u)$ then $$\boxed{\mathscr{K}_3(u) \le \int_\Omega \boldsymbol{v}(x) \cdot \nabla u(x)\, dx.}$$

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$$\frac{\mathscr{I}(u)-\mathscr{I}(\rho_h)}{h} \le \langle \boldsymbol{v}, \frac{\boldsymbol{i}-\boldsymbol{r}_h}{h}\rangle_u + \frac{1}{h} o(\|\boldsymbol{r}_h - \boldsymbol{i}\|_u)$$

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so that, as $h \downarrow 0$,

$$\mathscr{K}_3(u) \le \liminf_{h \downarrow 0} \frac{1}{h}\int_0^h \mathscr{K}_3(\rho_s)\, ds \le \langle \boldsymbol{v}, \frac{\nabla u}{u} \rangle_u = \int_\Omega \boldsymbol{v} \cdot \nabla u\, dx$$

# *Extension: the relative Fisher information*

Gradient flow of the ***Relative Fisher information***

$$\mathscr{I}(u|\,g) := \frac{1}{2}\int_{\Omega} |\nabla \log(u/g)|^2 u\,dx$$

where $g := e^{-F}$ is a strictly positive function which is induced by a

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This leads to the equation

$$\partial_t u + 2\sum_{i=1}^{d} \partial_i \left( u\,\partial_i \Big( \frac{\Delta\sqrt{u}}{\sqrt{u}} - \frac{\Delta\sqrt{g}}{\sqrt{g}} \Big) \right) = 0 \quad \text{in } \Omega_T := \Omega \times (0,T),$$

where $2\frac{\Delta\sqrt{g}}{\sqrt{g}} = \frac{1}{2}|\nabla F|^2 - \Delta F = f.$

$g$ represent the density of the *invariant measure* $\gamma := g\mathcal{L}^d$ associated to the equation.

# Exponential decay

In the case of a uniformly convex potential $F$, i.e. satisfying

$$\partial^2_{ij}F(x)\xi_i\xi_j \geq \lambda|\boldsymbol{\xi}|^2 \quad \forall\, x \in \Omega, \boldsymbol{\xi} \in \mathbb{R}^d,$$

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In particular, this holds for the Gaussian $g(x) := \frac{1}{(2\pi\sigma)^{d/2}}e^{-|x|^2/2\sigma^2}$ with $\lambda = \sigma^{-2}$, as formally obtained by Carrillo-Toscani ('02).

# *Open problems*

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