

Preconditioning the Mortar Method by Substructuring: The High Order Case.

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We analyze a class of preconditioners for the mortar method, based on substructuring. After splitting in a suitable way the degrees of freedom in *interior*, *edge* and *vertex*, we study the performance of a block Jacobi type preconditioner for which the condition number of the preconditioned matrix only grows polylogarithmically. Unlike the previous work by Achdou, Maday and Widlund [1], which is restricted to the case of first order finite element, this paper relies on abstract assumptions and therefore applies to finite element of any order. Moreover, the use of a suitable coarse preconditioner (whose effect we analyze) makes this technique more efficient.

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1 Introduction

Domain decomposition methods are becoming increasingly popular as a tool to solve problems arising in many different applications. The possibility of using different discretizations and/or methods in different subdomains, which is the main feature of the nonconforming version of such methods, adds a further advantage. Indeed, since there is no need for imposing strong matching conditions, we can employ without any adaptation the technologies developed for treating the problem with a single-domain approach. As an example, in an adaptive strategy, refinement can be carried out in each subdomain independently. Moreover, it is possible to use in each subdomain the best suited method: spectral methods where the solution is expected to be very smooth, finite elements where a complicated geometry requires it, wavelets where isolated singularities in a regular background are expected.

In order to make such techniques more competitive for real life applications, one has to deal with the problem of efficient implementation. As it happens with all domain decomposition methods (both conforming and non-conforming) the efficient implementation relies on parallelizing the solution process by assigning each subdomain to a processor and employing the preferred iterative scheme.

The design of an efficient scheme will then require a preconditioner for the linear system arising from such discretization. The approach that we will follow is the substructuring one, proposed by J.H. Bramble, J.E. Pasciak and A.H. Schatz [6] in the framework of conforming domain decomposition. This consists in considering a suitable splitting of the nonconforming discretization space in terms of “interior”, “edge” and “vertex” degrees of freedom and then using the related block-Jacobi type preconditioners. Such an approach was already applied to the mortar method by Y. Achdou, Y. Maday and O. Widlund in [1] for the case of order one finite elements. The results of [1] was extended to a general class of finite elements of any order in [4], showing that also for high order finite elements, the condition number of the preconditioned matrix grows at most polylogarithmically with the number of degrees of freedom per subdomain, analogously to what happens for the order one case.

In this paper, in order for the implementation of the preconditioner to be as efficient as possible, we modify the vertex related block of the preconditioner (which in the Achdou, Maday, Widlund paper and in [4] is simply

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a block of the Schur complement matrix), by constructing it on a suitable fixed very coarse mesh (in our tests 3×3 elements per subdomains). We analyze the resulting method and we present numerical tests showing the scalability of the method for Q1 and Q2 finite elements.

The paper is organized as follows. In section 2 we recall the mortar method and its main properties. In section 3 we introduce the class of substructuring preconditioners for the linear system arising from the discretization by the mortar method. In the same section we propose a coarse preconditioner obtained by considering an efficient variant of the vertex block preconditioner and we present the main theorem of the paper (Theorem 3.3) stating the convergence of the method. The proof of Theorem 3.3 is given in Section 4. Finally, numerical experiments showing the scalability of the method for Q_1 and Q_2 finite elements are presented in section 5 and our conclusions are summarized in section 6.

2 The Mortar Method

Let us at first recall the definition of the mortar method. For simplicity we will consider the following simple model problem (though the results that we present here will very easily extend to a more general situation): letting $\Omega \subset \mathbb{R}^2$ be a polygonal domain, and given $f \in L^2(\Omega)$ find u satisfying

$$-\sum_{i,j=1}^2 \frac{\partial}{\partial \mathbf{x}_j} \left(a_{ij}(\mathbf{x}) \frac{\partial u}{\partial \mathbf{x}_i} \right) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

We assume that for almost all $\mathbf{x} \in \Omega$ the matrix $a(\mathbf{x}) = (a_{ij}(\mathbf{x}))_{i,j=1,2}$ is symmetric positive definite, with smallest eigenvalue $\geq \alpha > 0$ and largest eigenvalue $\leq \alpha'$, α, α' independent of \mathbf{x} .

In order to discretize the above problem we start by considering a decomposition of Ω as the union of L subdomains Ω_ℓ ,

$$\Omega = \bigcup_{\ell=1, \dots, L} \Omega_\ell, \tag{2}$$

which, for the sake of simplicity we will assume to be quadrilateral (in general the constants in the inequalities will depend on the number of edges of the subdomains as well as on their aspect ratio). We set

$$\Gamma_{\ell n} = \partial\Omega_n \cap \partial\Omega_\ell, \quad S = \cup \Gamma_{\ell n}. \tag{3}$$

We denote by $\gamma_\ell^{(i)}$ ($i = 1, \dots, 4$) the i -th side of the ℓ -th domain:

$$\partial\Omega_\ell = \bigcup_{i=1}^4 \gamma_\ell^{(i)}.$$

Remark 2.1 The decomposition is said to be *geometrically conforming* if each edge $\gamma_\ell^{(i)}$ coincides with $\Gamma_{\ell n}$ for some n . If the decomposition is not geometrically conforming, then each interior edge $\gamma_\ell^{(i)}$ will be in general split as the union of several segments $\Gamma_{\ell n}$:

$$\gamma_\ell^{(i)} = \bigcup_{n \in I_\ell^{(i)}} \Gamma_{\ell n}, \tag{4}$$

where

$$I_\ell^{(i)} = \{n : |\partial\Omega_n \cap \gamma_\ell^{(i)}| \neq 0\}. \tag{5}$$

We will make the following regularity assumptions on the subdomains Ω_ℓ :

(G1) the subdomains are regular in shape and the geometrical decomposition is graded, that is

- (a) there exists a positive constant c_0 such that, for all k , Ω_ℓ contains a ball of diameter $c_0 H_k$, it is contained in a ball of diameter H_k , and the length of each side is bounded from below by $c_0 H_k$; moreover any interior angle ω satisfies $0 < c_1 < \omega < c_2 < \pi$ (c_0 , c_1 , and c_2 independent of k);
- (b) there exists a positive constant c_3 such that, if ℓ, k are such that $|\Gamma_{\ell k} \cap \partial\Omega_\ell| > 0$, then it holds

$$H_k/H_\ell \leq c_3;$$

(G2) the following bound holds

$$\max_{(\ell, i)} \left(\frac{|\gamma_\ell^{(i)}|}{\min_{n \in \mathcal{I}_\ell^{(i)}} |\Gamma_{\ell n}|} \right) \leq \rho. \quad (6)$$

The constants appearing in the estimates of the following sections will in generally depend on the bound ρ .

Given such splitting of the domain Ω , we will consider a nonconforming domain decomposition method for the solution of such a problem. Let us at first introduce the corresponding functional setting. We set

$$\begin{aligned} \|u\|_{H^1(\Omega_\ell)}^2 &= H_\ell^{-2} \int_{\Omega_\ell} |u|^2 d\mathbf{x} + \int_{\Omega_\ell} |\nabla u|^2 d\mathbf{x}, & |u|_{H^1(\Omega_\ell)}^2 &= \int_{\Omega_\ell} |\nabla u|^2 d\mathbf{x}, \\ |\eta|_{s, \partial\Omega_\ell}^2 &= H_\ell^{2s-1} \int_{\partial\Omega_\ell} \int_{\partial\Omega_\ell} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{2s+1}} dx dy, & s &\in (0, 1) \\ \|\eta\|_{s, \partial\Omega_\ell}^2 &= |\eta|_{s, \partial\Omega_\ell}^2 + H_\ell^{-1} \int_{\partial\Omega_\ell} |\eta|^2 ds, & s &\in (0, 1). \end{aligned}$$

Remark that the above norms are defined in such a way that they are scaling invariant, that is they are preserved when Ω_ℓ is rescaled to the reference domain $]0, 1]^2$.

In the following for $\gamma_\ell^{(i)}$ edge of Ω_ℓ we will also make explicit use of the spaces $H_0^s(\gamma_\ell^{(i)})$ and $H_{00}^{1/2}(\gamma_\ell^{(i)})$, which are defined as the subspaces of those functions η of $H^s(\gamma_\ell^{(i)})$ (resp. $H^{1/2}(\gamma_\ell^{(i)})$) such that the function $\bar{\eta}$ defined as $\bar{\eta} = \eta$ on $\gamma_\ell^{(i)}$ and $\bar{\eta} = 0$ on $\partial\Omega \setminus \gamma_\ell^{(i)}$ belongs to $H^s(\partial\Omega)$ (resp. to $H^{1/2}(\partial\Omega)$). The spaces $H_0^s(\gamma_\ell^{(i)})$ and $H_{00}^{1/2}(\gamma_\ell^{(i)})$ are endowed with the norms

$$\|\eta\|_{H_0^s(\gamma_\ell^{(i)})} = \|\bar{\eta}\|_{H^s(\gamma_\ell^{(i)})} \quad \|\eta\|_{H_{00}^{1/2}(\gamma_\ell^{(i)})} = \|\bar{\eta}\|_{H^{1/2}(\gamma_\ell^{(i)})}.$$

We recall that for $s < 1/2$ the spaces $H^s(\gamma_\ell^{(i)})$ and $H_0^s(\gamma_\ell^{(i)})$ coincide as sets and have equivalent norms. However, the constants in the norm equivalence goes to infinity as s tends to $1/2$.

Let the spaces X and T be defined

$$X = \prod_{\ell} \{u_\ell \in H^1(\Omega_\ell) \mid u_\ell = 0 \text{ on } \partial\Omega \cap \partial\Omega_\ell\}, \quad T = \prod_{\ell} H_*^{1/2}(\partial\Omega_\ell), \quad (7)$$

where $H_*^{1/2}(\Omega_\ell)$ is defined by

$$H_*^{1/2}(\partial\Omega_\ell) = H^{1/2}(\partial\Omega_\ell) \quad \text{if } \partial\Omega_\ell \cap \partial\Omega = \emptyset$$

and

$$H_*^{1/2}(\partial\Omega_\ell) = \{\eta \in H^{1/2}(\partial\Omega_\ell), \eta|_{\partial\Omega_\ell \cap \partial\Omega} \equiv 0\} \sim H_{00}^{1/2}(\partial\Omega_\ell \setminus \partial\Omega),$$

otherwise. We will denote by $\|\cdot\|_{1/2,\partial\Omega_\ell}$ the related norm, and by $\|\cdot\|_{-1/2,\ell}$ the norm of the corresponding dual space.

On X we introduce the following broken norm and semi-norm:

$$\|u\|_X = \left(\sum_\ell \|u\|_{1,\Omega_\ell}^2 \right)^{\frac{1}{2}}, \quad |u|_X = \left(\sum_\ell |u|_{1,\Omega_\ell}^2 \right)^{\frac{1}{2}}. \tag{8}$$

In the following it will also be convenient to introduce the following norm on T :

$$\|\eta\|_T = \left(\sum_\ell \|\eta_\ell\|_{1/2,\partial\Omega_\ell}^2 \right)^{1/2} \tag{9}$$

For each ℓ let now \mathcal{V}_h^ℓ be a family of finite dimensional subspaces of $H^1(\Omega_\ell) \cap C^0(\bar{\Omega}_\ell)$, depending on a parameter $h = h_\ell > 0$ and satisfying an homogeneous boundary condition on $\partial\Omega \cap \partial\Omega_\ell$. Set

$$T_h^\ell = \mathcal{V}_h^\ell|_{\partial\Omega_\ell}, \tag{10}$$

and, for each edge $\gamma_\ell^{(i)}$ of the subdomain Ω_ℓ let

$$T_{\ell,i} = \{ \eta : \eta \text{ is the trace on } \gamma_\ell^{(i)} \text{ of some } u_\ell \in \mathcal{V}_h^\ell \} \tag{11}$$

$$T_{\ell,i}^0 = \{ \eta \in T_{\ell,i} : \eta = 0 \text{ at the vertices of } \gamma_\ell^{(i)} \}. \tag{12}$$

We set

$$X_h = \prod_{\ell=1}^L \mathcal{V}_h^\ell \subset X, \quad T_h = \prod_{\ell=1}^L T_h^\ell \subset T. \tag{13}$$

Let now a composite bilinear form $a_X : X \times X \rightarrow \mathbb{R}$ be defined as follows:

$$a_X(u, v) = \sum_\ell \int_{\Omega_\ell} \sum_{i,j} a_{ij}(\mathbf{x}) \frac{\partial u_\ell}{\partial \mathbf{x}_i} \frac{\partial v_\ell}{\partial \mathbf{x}_j} d\mathbf{x}. \tag{14}$$

The bilinear form a_X is clearly not coercive on X . In order to obtain a well posed problem we will then consider proper subspaces of X , consisting of functions satisfying a suitable *weak continuity* constraint.

For defining such weak continuity constraint according to the mortar method we start by choosing a splitting of the skeleton S as the disjoint union of a certain number of subdomain sides $\gamma_\ell^{(i)}$, which we will call “multiplier sides” (in the usual terminology these are called “non mortars” or “slave sides”). More precisely, we choose an index set $I \subset \{1, \dots, L\} \times \{1, \dots, 4\}$ such that,

$$S = \bigcup_{(\ell,i) \in I} \gamma_\ell^{(i)}, \quad \begin{aligned} & (\ell_1, i_1), (\ell_2, i_2) \in I, \\ & (\ell_1, i_1) \neq (\ell_2, i_2) \end{aligned} \Rightarrow \gamma_{\ell_1}^{(i_1)} \cap \gamma_{\ell_2}^{(i_2)} = \emptyset. \tag{15}$$

Furthermore we will denote by $I^* \subset \{1, \dots, L\} \times \{1, \dots, 4\}$ the index-set corresponding to “trace sides” (“mortars” or “master sides” in the usual terminology), which is defined in such a way that $I^* \cap I = \emptyset$ and $S = \cup_{(\ell,i) \in I^*} \gamma_\ell^{(i)}$.

Remark 2.2 In the geometrically conforming case this reduces to say that for each segment $\Gamma_{\ell,\ell'} = \gamma_\ell^{(i)} = \gamma_{\ell'}^{(i')}$ we will chose one side (let us say ℓ) to be the master side, while the other side will be the slave side.

For each $m = (\ell, i) \in I$ let a finite dimensional multiplier space M_h^m (also depending on the parameter h) on γ_m ,

$$M_h^m \subset L^2(\gamma_m), \quad \dim(M_h^m) = \dim(T_h^{m,0}), \quad (16)$$

be given. We set:

$$M_h = \{\eta \in H^{-1/2}(S), \forall m \in I \eta|_{\gamma_m} \in M_h^m\} \sim \prod_{m \in I} M_m. \quad (17)$$

The *constrained* approximation and trace spaces \mathcal{X}_h and \mathcal{T}_h are then defined as follows:

$$\mathcal{X}_h = \{v_h \in X_h, \int_S [v_h] \lambda ds = 0, \forall \lambda \in M_h\} \quad (18)$$

$$\mathcal{T}_h = \{\eta \in T_h, \int_S [\eta] \lambda ds = 0, \forall \lambda \in M_h\}. \quad (19)$$

We can now introduce the following discrete problem:

Problem 2.1 Find $u_h \in \mathcal{X}_h$ such that for all $v_h \in \mathcal{X}_h$

$$a_X(u_h, v_h) = \int_{\Omega} f v_h dx. \quad (20)$$

We make the assumption that the class M_h of multipliers is chosen in such a way to guarantee ellipticity uniformly with respect to the mesh-size parameter h and to the number L of subdomains. More precisely we assume that there exists a constant \bar{C} such that for all functions $u \in X$ verifying

$$\int_S [u] \lambda = 0 ds, \quad \forall \lambda \in M_h$$

it holds (uniformly on h)

$$|u|_X \gtrsim \|u\|_{0,\Omega}, \quad (21)$$

which, setting $\eta = (\eta^\ell)_\ell$ with $\eta^\ell = u^\ell|_{\Gamma_\ell}$, implies

$$|\eta|_T \simeq \|\eta\|_T \quad (22)$$

It is out of the scope of this paper to discuss the conditions under which such inequality holds, we refer to [2] for further discussion. Under such an assumption for all $h > 0$, problem P_h admits a unique solution u_h which satisfies the following error estimate:

$$\|u - u_h\|_X \lesssim \left(\inf_{v_h \in \mathcal{X}_h} \|u - v_h\|_X + \inf_{\lambda \in M_h} \left\| \frac{\partial u}{\partial \nu} - \lambda \right\|_{-1/2,S} \right), \quad (23)$$

$\|\cdot\|_{-1/2,S}$ denoting the norm of T' , dual of T .

We will make the following quite typical assumptions on the spaces considered.

(A1): $\forall m = (\ell, i) \in I$ ($\gamma_\ell^{(i)}$ multiplier side), there exists a bounded projection $\pi_h^m : L^2(\gamma_m) \rightarrow T_h^{m,0}$, such that for all $\eta \in L^2(\gamma_m)$ and for all $\lambda \in M_h^m$

$$\int_{\gamma_m} (\eta - \pi_h^m \eta) \lambda = 0, \quad (24)$$

and for all $\eta \in H_{00}^{1/2}(\gamma_m)$

$$\|\pi_h^m \eta\|_{H_{00}^{1/2}(\gamma_m)} \lesssim \|\eta\|_{H_{00}^{1/2}(\gamma_m)}; \quad (25)$$

(A2): for all $\ell = 1, \dots, L$, the following inverse inequalities hold: for all elements $\eta \in T_h^\ell$ and for all s, r $0 \leq s < r \leq 1$

$$|\eta|_{r, \Gamma_\ell} \lesssim h_\ell^{s-r} |\eta|_{s, \Gamma_\ell}, \tag{26}$$

$$|\eta|_{r, \gamma_\ell^{(i)}} \lesssim h_\ell^{s-r} |\eta|_{r, \gamma_\ell^{(i)}} \quad i = 1, \dots, 4; \tag{27}$$

(A3): $\forall \ell$ and $\forall \eta \in T_h^\ell$ there exist a function $w_h \in \mathcal{V}_h^\ell$ such that

$$w_h = \eta \quad \text{on } \Gamma_\ell, \quad \|w_h\|_{1, \Omega_\ell} \lesssim \|\eta\|_{H^{1/2}(\Gamma_\ell)}. \tag{28}$$

Remark 2.3 We remark that by space interpolation, assumption (A.1) implies that the projection operator π_h^m verifies for all $s, 0 < s < 1/2$:

$$\|\pi_h^m \eta\|_{H_0^s(\gamma_m)} \lesssim \|\eta\|_{H_0^s(\gamma_m)}, \tag{29}$$

uniformly in s .

Following [5], we define a global linear operator $\pi_h : \prod_{\ell=1}^L L^2(\partial\Omega_\ell) \rightarrow \prod_{\ell=1}^L L^2(\partial\Omega_\ell)$ that we will need in the following: more precisely, for $\eta = (\eta_\ell)_{\ell=1, \dots, L}$, $\pi_h(\eta) = (\eta_\ell^*)_{\ell=1, \dots, L}$ is defined on multiplier sides as π_h^m applied to the jump of η , while it is set identically zero on trace sides and on the external boundary $\partial\Omega$:

$$\eta_\ell^*|_{\gamma_m} = \pi_h^m([\eta]|_{\gamma_m}), \text{ for } m = (\ell, i) \in I \tag{30}$$

$$\eta_\ell^*|_{\gamma_m} = 0, \text{ for } m = (\ell, i) \in I^*,$$

$$\eta_\ell^* \equiv 0 \text{ on } \partial\Omega_\ell \cap \partial\Omega.$$

Writing conventionally

$$\frac{H}{h} = \min_\ell \left\{ \frac{H_\ell}{h_\ell} \right\}, \tag{31}$$

the following Lemma, of which we recall the proof (see [5]) for the sake of completeness, holds.

Lemma 2.4 *If assumptions (A1–A3) hold, then for any $\eta = (\eta_\ell)_{\ell=1, \dots, L}$ in the trace space T it holds:*

$$\|\pi_h(\eta)\|_T \lesssim \left(1 + \log \left(\frac{H}{h} \right) \right) \|\eta\|_T. \tag{32}$$

If in addition η is linear on each $\gamma_\ell^{(i)}$ then the bound can be improved to

$$\|\pi_h(\eta)\|_T \lesssim \left(1 + \log \left(\frac{H}{h} \right) \right)^{1/2} \|\eta\|_T. \tag{33}$$

Proof. We start by recalling that the following inequalities hold (see [3]). Let $U = \cup U_n, n = 1, \dots, N$, U, U_n intervals. Assume that $|U|/|U_n| \leq \rho$. Then for all $\eta \in H^{1/2}(U)$ it holds for all $\varepsilon, 0 < \varepsilon \leq 1/2$,

$$\|\eta\|_{H^{1/2-\varepsilon}(U)}^2 \lesssim \sum_n \|\eta\|_{H_0^{1/2-\varepsilon}(U_n)}^2 \lesssim \frac{|U|^{2\varepsilon}}{\varepsilon^2} \|\eta\|_{H^{1/2}(U)}^2, \tag{34}$$

with constants dependent on ρ , and independent of ε (we recall that $H^{1/2-\varepsilon}$ and $H_0^{1/2-\varepsilon}$ are the same space, the respective norms being indeed equivalent, but the constants in the equivalence depending on ε).

Let now $\eta = (\eta_\ell)_{\ell=1,\dots,L}$ be any element of $T = \prod_{\ell=1,\dots,L} H_*^{1/2}(\partial\Omega_\ell)$. Thanks to (29), for any ε , $0 < \varepsilon < 1/2$ it holds:

$$\begin{aligned} \sum_{m=(\ell,i) \in I} \|\pi_h^m([\eta])\|_{H_0^{1/2}(\gamma_m)}^2 &\lesssim \sum_{m=(\ell,i) \in I} h_\ell^{-2\varepsilon} \|\pi_h^m([\eta])\|_{H_0^{1/2-\varepsilon}(\gamma_m)}^2 \lesssim \\ &\lesssim \sum_{m=(\ell,i) \in I} h_\ell^{-2\varepsilon} \sum_{n \in I_\ell^{(i)}} (\|\eta_\ell\|_{H_0^{1/2-\varepsilon}(\Gamma_{\ell n})}^2 + \|\eta_n\|_{H_0^{1/2-\varepsilon}(\Gamma_{\ell n})}^2) = \\ &= \sum_{\ell=1}^L h_\ell^{-2\varepsilon} \sum_{n: \Gamma_{\ell n} \neq \emptyset} \|\eta_\ell\|_{H_0^{1/2-\varepsilon}(\Gamma_{\ell n})}^2 \lesssim \frac{h_\ell^{-2\varepsilon} H_\ell^{2\varepsilon}}{\varepsilon^2} \sum_{\ell=1}^L \|\eta_\ell\|_{1/2, \partial\Omega_\ell}^2 \lesssim \\ &\lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \sum_{\ell=1}^L \|\eta_\ell\|_{1/2, \partial\Omega_\ell}^2, \end{aligned}$$

where the last bound derives by choosing $\varepsilon = \frac{1}{\lceil \log_2(H/h) \rceil}$.

The proof of (33) is essentially identical to the proof of (32), once we observed that property (34) can be enhanced as follows: let $U = \cup U_n$ and let η a function (not necessarily continuous) η linear on each subinterval U_n . Then by direct computation it is not difficult to check that:

$$\|\eta\|_{H^{1/2-\varepsilon}(U)}^2 \lesssim \sum_n \|\eta\|_{H_0^{1/2-\varepsilon}(U_n)}^2 \lesssim \frac{|U|^{2\varepsilon}}{\varepsilon} \|\eta\|_{H^{1/2}(U)}^2. \quad (35)$$

□

3 Substructuring Preconditioners for the Mortar Element Method

The aim of this section is to review the properties of a class of preconditioners for the linear system arising from the discretization described in the previous section. We will consider the “substructuring” approach first introduced in [6] and already studied in the case of the mortar Finite Element method in [1]. The principle of these preconditioners consists in distinguishing three types of degrees of freedom: *interior* degrees of freedom (corresponding to basis functions vanishing on the skeleton and supported on one sub-domain), *edge* degrees of freedom, and *vertex* degrees of freedom. This corresponds to splitting the functions $u \in \mathcal{X}_h$ as the sum of three suitably defined components: $u = u^0 + u^E + u^V$. When expressed in a basis related to such a splitting, substructuring preconditioners are of the block diagonal form

$$\hat{A} = \begin{pmatrix} \hat{A}^{00} & 0 & 0 \\ 0 & \hat{A}^{EE} & 0 \\ 0 & 0 & \hat{A}^{VV} \end{pmatrix}, \quad (36)$$

where the matrices A^{00} and A^{EE} are in turn block diagonal.

Let us describe how the splitting is constructed. Given any discrete function $w = (w_\ell)_{\ell=1,\dots,L} \in X_h$ we can split it in a unique way as the sum of an *interior* function $w^0 \in \mathcal{X}_h^0$ and a discrete lifting, performed subdomain-wise of its trace $\eta(w) = (w_\ell|_{\Omega_\ell})_{\ell=1,\dots,L}$ which by abuse of notation we will denote by $R_h(w)$ (rather than using the heavier notation $R_h(\eta(w))$):

$$w = w^0 + R_h(w), \quad w^0 \in \mathcal{X}_h^0, \quad (37)$$

with $R_h(w) = (R_h^\ell(w_\ell))_{\ell=1,\dots,K}$, $R_h^\ell(w_\ell)$ being the unique element in \mathcal{V}_h^ℓ satisfying

$$R_h^\ell(w_\ell) = w_\ell \text{ on } \Gamma_\ell, \quad \int_{\Omega_\ell} \sum_{i,j} a(\mathbf{x}) \frac{\partial}{\partial \mathbf{x}_i} \frac{\partial}{\partial \mathbf{x}_j} R_h^\ell(w_\ell) v_h^\ell d\mathbf{x} = 0, \quad \forall v_h \in \mathcal{V}_h^\ell \cap H_0^1(\Omega_\ell).$$

That is we can split respectively the spaces X_h of unconstrained functions and \mathcal{X}_h of constrained functions as direct sums of an interior and of a (respectively unconstrained or constrained) trace component:

$$X_h = X_h^0 \oplus R_h(T_h), \tag{38}$$

$$\mathcal{X}_h = \mathcal{X}_h^0 \oplus R_h(\mathcal{T}_h). \tag{39}$$

It is not difficult to verify that $a_X : X_h \times X_h \rightarrow \mathbb{R}$ verifies

$$a_X(w, v) = a_X(w^0, v^0) + a_X(R_h(w), R_h(v)) := a_X(w^0, v^0) + s(\eta(w), \eta(v)), \tag{40}$$

where the *discrete Steklov-Poincaré* operator $s : T_h \times T_h \rightarrow \mathbb{R}$ is defined by

$$s(\xi, \eta) := \sum_{\ell} \int_{\Omega_{\ell}} (a(\mathbf{x}) \nabla R_h^{\ell}(\xi)) \cdot \nabla R_h^{\ell}(\eta).$$

Remark 3.1 The preconditioner proposed can be generalized by replacing the lifting operator R_h^{ℓ} , chosen here in a way that is closely related to the equation to be solved, with any other operator \tilde{R}_h^{ℓ} verifying

$$|\tilde{R}_h^{\ell} \eta|_X \sim |\eta|_T.$$

Though the proof of theorem 3.3 is simplified by the choice we made, since this implies that (40) holds, it is not difficult to realize that the result still holds also for a more general choice of the lifting operator.

The problem of preconditioning the matrix A corresponding to the discretization of a_X , reduces to finding good preconditioners for the matrices A_0 and S corresponding respectively to the bilinear forms $a_X : \mathcal{X}_h^0 \times \mathcal{X}_h^0 \rightarrow \mathbb{R}$ and $s : T_h \times T_h \rightarrow \mathbb{R}$. Remark that, when restricted to the subspace \mathcal{X}_h^0 of functions vanishing on the boundary of each Ω_{ℓ} , the bilinear form a_X is block diagonal, the coupling between subdomains being taken into account only by the Steklov-Poincaré operator. We assume here to have good preconditioners for the stiffness matrices A_0^{ℓ} in each subdomain, and we concentrate on the preconditioning of the discrete Steklov-Poincaré operator S .

3.1 The splitting of the trace space

In order to construct a preconditioner for S , we start by observing that the space of constrained skeleton functions T_h can be further split as the sum of *vertex* and *edge* functions. If we denote by $\mathcal{L} \subset \prod_{\ell=1}^L H_*^{1/2}(\partial\Omega_{\ell})$ the space

$$\mathcal{L} = \{(\eta_{\ell})_{\ell=1, \dots, L}, \eta_{\ell} \text{ is linear on each edge of } \Omega_{\ell}\}, \tag{41}$$

then the space of constrained *vertex* functions can be defined by

$$\mathcal{T}_h^V = (Id - \pi_h)\mathcal{L}. \tag{42}$$

In the following we will make the (not restrictive) assumption $\mathcal{L} \subset T_h$, which yields $\mathcal{T}_h^V \subset T_h$.

We then introduce the space of constrained *edge* functions $\mathcal{T}_h^E \subset T_h$ defined by

$$\mathcal{T}_h^E = \{\eta = (\eta_{\ell})_{\ell=1, \dots, L} \in T_h, \eta_{\ell}(A) = 0, \forall \text{ vertex } A \text{ of } \Omega_{\ell}\}. \tag{43}$$

It is quite easy to see that

$$T_h = \mathcal{T}_h^V \oplus \mathcal{T}_h^E. \tag{44}$$

Moreover it is quite simple to check that a function in \mathcal{T}_h^E is uniquely defined by its value on trace edges, the value on multiplier edges being forced by the constraint.

3.1.1 The edge preconditioner

As we already anticipated, the preconditioner for S will be of block-Jacobi type. Let us start by introducing the blocks relative to the edges. For any trace side $\gamma_\ell^{(i)}$, $m = (\ell, i) \in I^*$, let $b_{\ell,i} : T_{\ell,i}^0 \times T_{\ell,i}^0 \rightarrow \mathbb{R}$ be a symmetric bilinear form satisfying for all $\eta \in T_{\ell,i}^0$:

$$b_{\ell,i}(\eta, \eta) \simeq \|\eta\|_{H_0^{1/2}(\gamma_\ell^{(i)})}. \quad (45)$$

Then, we define the edge block diagonal global bilinear form $b^E : \mathcal{T}_h^E \times \mathcal{T}_h^E \rightarrow \mathbb{R}$ by

$$b^E(\eta, \xi) = \sum_{(\ell,i) \in I^*} b_{\ell,i}(\eta_\ell, \xi_\ell). \quad (46)$$

3.1.2 The coarse preconditioner

Let us now consider the block corresponding to \mathcal{T}_h^V . Contrary to what happens for the conforming domain decomposition method presented in [6] the space \mathcal{T}_h^V does not only depend on the discretization of Ω into subdomains, but also (through the action of the constraint operator) on the meshes in the different subdomains. The proposal made in [1] for linear elements and already analyzed for high order elements in [5] is to build the preconditioner as $\hat{s} : \mathcal{T}_h \times \mathcal{T}_h \rightarrow \mathbb{R}$ be defined by

$$\hat{s}(\eta, \xi) = s(\eta^V, \xi^V) + b^E(\eta^E, \xi^E). \quad (47)$$

With such a choice, both for linear and higher order elements one can prove and estimate on the condition number of the preconditioned system of the form

$$\text{Cond}(\hat{A}^{-1}A) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2, \quad (48)$$

for the geometrically conforming case, and

$$\text{Cond}(\hat{A}^{-1}A) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^4, \quad (49)$$

for the geometrically nonconforming case.

However if the mesh size h is fine, assembling the vertex block of the preconditioner with such a choice could be quite expensive. We propose to derive a cheaper preconditioner by taking advantage of the observation that all the spaces \mathcal{T}_h^V for different values of h depend on the same set of degrees of freedom (the values at the vertices of the Ω_ℓ 's), and they are, therefore, isomorphic.

Let us then start by choosing for each ℓ an auxiliary discretization space $\mathcal{V}_\delta^\ell \subset H^1(\Omega_\ell) \cap C^0(\bar{\Omega}_\ell)$ with $\delta = \delta_\ell \geq h_\ell$, and for each $m = (\ell, i) \in I$ ($\gamma_m = \gamma_\ell^{(i)}$ multiplier edge) a corresponding auxiliary multiplier space $M_\delta^m \subset L^2(\gamma_m)$. The spaces X_δ , M_δ , and \mathcal{X}_δ , as well as T_δ^ℓ , T_δ and \mathcal{T}_δ are constructed starting from the \mathcal{V}_δ^ℓ 's and the M_δ^m 's in the same way as the spaces X_h , M_h , and \mathcal{X}_h , as well as T_h^ℓ , T_h and \mathcal{T}_h are built from the \mathcal{V}_h^ℓ 's and the M_h^m 's, according to definitions similar to (13) (17), (18), (12).

We make on the spaces \mathcal{V}_δ^ℓ and M_δ^m the same assumptions that we made on the spaces \mathcal{V}_h^ℓ and M_h^m .

(B1): $\forall m = (\ell, i) \in I$ ($\gamma_\ell^{(i)}$ multiplier side), there exists a bounded projection $\pi_\delta^m : L^2(\gamma_m) \rightarrow T_\delta^{m,0}$, such that for all $\eta \in L^2(\gamma_m)$ and for all $\lambda \in M_\delta^m$

$$\int_{\gamma_m} (\eta - \pi_\delta^m \eta) \lambda \, ds = 0, \quad (50)$$

and for all $\eta \in H_{00}^{1/2}(\gamma_m)$

$$\|\pi_\delta^m \eta\|_{H_{00}^{1/2}(\gamma_m)} \lesssim \|\eta\|_{H_{00}^{1/2}(\gamma_m)}. \tag{51}$$

(B2): For $\ell = 1, \dots, L$ the following inverse inequalities hold: for all elements $\eta \in T_\delta^m$ and for all s, r $0 \leq s < r \leq 1$

$$|\eta|_{r, \Gamma_\ell} \lesssim \delta^{s-r} |\eta|_{s, \Gamma_\ell}, \tag{52}$$

$$|\eta|_{r, \gamma_\ell^{(i)}} \lesssim \delta^{s-r} |\eta|_{r, \gamma_\ell^{(i)}} \quad i = 1, \dots, 4. \tag{53}$$

inequality: for all $w \in \mathcal{V}_h^\ell$ it holds $\|\cdot\|_{\infty, \Omega_\ell}$.

(B3): $\forall \ell$ and $\forall \eta \in T_\delta^\ell$ there exist a function $w_h \in \mathcal{V}_\delta^\ell$ such that

$$w_h = \eta \quad \text{on } \Gamma_\ell, \quad \|w_h\|_{1, \Omega_\ell} \lesssim \|\eta\|_{H^{1/2}(\Gamma_\ell)}. \tag{54}$$

Analogously to π_h we can define the operator $\pi_\delta : \prod_{\ell=1}^L L^2(\partial\Omega_\ell) \rightarrow \prod_{\ell=1}^L L^2(\partial\Omega_\ell)$: for $\eta = (\eta_\ell)_{\ell=1, \dots, L}$, $\pi_\delta(\eta) = (\eta_\ell^*)_{\ell=1, \dots, L}$ is defined on multiplier sides as π_δ^m applied to the jump of η , while it is set identically zero on trace sides and on the external boundary $\partial\Omega$:

$$\eta_\ell^*|_{\gamma_m} = \pi_\delta^m([\eta]|_{\gamma_m}), \text{ for } m = (\ell, i) \in I \tag{55}$$

$$\eta_\ell^*|_{\gamma_m} = 0, \text{ for } m = (\ell, i) \in I^*,$$

$$\eta_\ell^* \equiv 0 \text{ on } \partial\Omega_\ell \cap \partial\Omega.$$

$\eta|_{\gamma_\ell^{(i)}} \in H_{00}^{1/2}(\gamma_\ell^{(i)})$, ($i = 1, \dots, 4$) it holds $\|\cdot\|_{H_{00}^{1/2}(\gamma_\ell^{(i)})}$ Again, writing conventionally

$$\frac{H}{\delta} = \min_\ell \left\{ \frac{H_\ell}{\delta_\ell} \right\}, \tag{56}$$

we have, thanks to Lemma 2.4

$$\|\pi_\delta(\eta)\|_T \lesssim \left(1 + \log \left(\frac{H}{\delta} \right) \right) \|\eta\|_T. \tag{57}$$

If in addition η is linear on each $\gamma_\ell^{(i)}$ then the bound can be improved to

$$\|\pi_\delta(\eta)\|_T \lesssim \left(1 + \log \left(\frac{H}{\delta} \right) \right)^{1/2} \|\eta\|_T. \tag{58}$$

As already observed, letting

$$\mathcal{T}_\delta^V := (Id - \pi_\delta)\mathcal{L}$$

the spaces \mathcal{T}_δ^V and \mathcal{T}_h^V are isomorphic, though not necessarily uniformly with respect to h and δ . Let us write down the isomorphism explicitly. Let $L^\ell : C^0(\Gamma_\ell) \rightarrow \mathcal{L}^\ell = \{\eta_\ell : \eta_\ell \text{ linear on each edge of } \Omega_\ell\}$ be defined by $L^\ell(\eta)$ with $L^\ell \eta(A_i) = \eta(A_i)$ for all vertex A_i of Ω_ℓ . We can assemble $L : \prod_\ell C^0(\Gamma_\ell) \rightarrow \mathcal{L}$ as

$$L(\eta) = (L^\ell \eta_\ell)_\ell$$

and we then define $P_\delta : \prod_\ell C^0(\Gamma_\ell) \rightarrow \mathcal{T}_\delta^V$ as

$$P_\delta = (Id - \pi_\delta)L;$$

it is not difficult to check that $\mathcal{T}_\delta^V = P_\delta \mathcal{T}_h^V$.

We can now define the vertex block of the preconditioner. For each subdomain Ω_ℓ let $R_\delta^\ell : T_\delta^\ell \rightarrow \mathcal{V}_\delta^\ell$ denote the discrete harmonic lifting, that is $R_\delta^\ell(\eta)$ is the unique element in \mathcal{V}_δ^ℓ which verifies

$$R_\delta^\ell \eta|_{\partial\Omega_\ell} = \eta \quad \text{and} \quad (T_\delta^\ell(\eta), w_h) = 0, \quad \text{for all } w_h \in \mathcal{V}_\delta^\ell \cap H_0^1(\Omega_\ell).$$

It is well known that the following inequalities hold:

$$\begin{aligned} \|R_\delta^\ell \eta\|_{H^1(\Omega_\ell)} &\lesssim \|\eta\|_{1/2, \partial\Omega_\ell}, \\ |R_\delta^\ell \eta|_{H^1(\Omega_\ell)} &\lesssim |\eta|_{1/2, \partial\Omega_\ell}. \end{aligned} \quad (59)$$

We can easily assemble a global lifting operator $R_\delta : T_h \rightarrow X_h$; more precisely for $\eta = (\eta_\ell)_{\ell=1, \dots, L}$ we set

$$R_\delta(\eta) = (R_\delta^\ell \eta_\ell)_{\ell=1, \dots, L}.$$

Remark 3.2 Trivially, if η satisfies the jump constrain, so does $R_h(\eta)$. In other words, if $\eta \in \mathcal{T}_h$ then $R_h(\eta) \in \mathcal{X}_h$.

In view of (44) and using the classical trace Theorem [8], it is easy to see that for all $\eta \in T_h$ it holds:

$$|R_\delta(\eta)|_X \simeq |\eta|_T. \quad (60)$$

Now, we let $b^V : T_h^V \times T_h^V \rightarrow \mathbb{R}$ be defined by

$$b^V(\eta^V, \xi^V) := \sum_\ell \int_{\Omega_\ell} (a(\mathbf{x}) \nabla R_\delta^\ell(P_\delta \eta^V)) \cdot \nabla R_\delta^\ell(P_\delta \xi^V). \quad (61)$$

3.1.3 The preconditioner

Finally, we can assemble the preconditioner:

$\hat{s} : \mathcal{T}_h \times \mathcal{T}_h \rightarrow \mathbb{R}$ be defined by

$$\hat{s}(\eta, \xi) = b^V(\eta^V, \xi^V) + b^E(\eta^E, \xi^E) \quad (62)$$

and state the following theorem:

Theorem 3.3 Let $\eta \in \mathcal{T}_h$ then we have:

$$\left(1 + \log\left(\frac{H}{h}\right)\right)^{-2} s(\eta, \eta) \lesssim \hat{s}(\eta, \eta) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 s(\eta, \eta). \quad (63)$$

Moreover, if the decomposition is geometrically conforming then

$$\left(\left(1 + \log\left(\frac{H}{h}\right)\right)\left(1 + \log\left(\frac{H}{\delta}\right)\right)\right)^{-1} s(\eta, \eta) \lesssim \hat{s}(\eta, \eta) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 s(\eta, \eta). \quad (64)$$

The proof of theorem 3.3 follows essentially the guidelines of the proofs of the analogous results in [6, 1]. However, for the sake of completeness we will retrace in the next section the main steps in the framework of the abstract formulation here considered (see also [4]).

Corollary 3.4 Let A and \hat{A} be the matrices obtained by discretizing respectively the bilinear forms a and \hat{a} with $\hat{a}(u, v) = \tilde{a}_x(u_0, v_0) + \hat{s}(\eta_u, \eta_v)$. Then it holds

$$\text{Cond}(\hat{A}^{-1}A) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^4. \quad (65)$$

Moreover, if the decomposition is geometrically conforming then

$$\text{Cond}(\hat{A}^{-1}A) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^3. \quad (66)$$

4 Proof of Theorem 3.3

To start with, thank to (60) of the previous section we have for all $\xi \in T_h$

$$s(\xi, \xi) \simeq |\xi|_T^2, \tag{67}$$

while for all constrained functions $\xi \in \mathcal{T}_h$ it holds

$$s(\xi, \xi) \simeq \|\xi\|_T^2. \tag{68}$$

In order to prove Theorem 3.3 we will make use of the following two lemmata, whose proof can be found in [4].

Lemma 4.1 For all $\eta = (\eta_\ell)_{\ell=1, \dots, L} \in \mathcal{T}_h^E$ we have

$$\|\eta\|_T^2 \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \sum_{(\ell, i) \in I^*} \|\eta_\ell\|_{H_0^{1/2}(\gamma_\ell^{(i)})}^2. \tag{69}$$

Lemma 4.2 Let assumption (A2) hold, and let $\xi \in T_h^\ell$, $\xi(A) = 0$ for all A vertex of Ω_ℓ . Let $\zeta_L \in H^{1/2}(\partial\Omega_\ell)$, ζ_L linear on each edge of Ω_ℓ . Then it holds

$$\sum_{k=1}^4 \|\xi\|_{H_0^{1/2}(\gamma_\ell^{(i)})}^2 \lesssim \left(1 + \log\left(\frac{H_\ell}{h_\ell}\right)\right)^2 \|\xi + \zeta_L\|_{H^{1/2}(\partial\Omega_\ell)}^2. \tag{70}$$

The effect of replacing the coarse grain preconditioner by an analogous term computed on the coarser auxiliary grid with meshsize δ will be studied with the aid of the following lemma.

Lemma 4.3 Let $\eta = (\eta_\ell)_\ell \in T_h$. Then it holds

$$|L\eta|_T^2 \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) |\eta|_T^2. \tag{71}$$

Analogously, for $\eta = (\eta_\ell)_\ell \in T_\delta$ it holds that

$$|L\eta|_T^2 \lesssim \left(1 + \log\left(\frac{H}{\delta}\right)\right) |\eta|_T^2. \tag{72}$$

Proof. A direct computation using the linearity of $L^\ell \eta_\ell$ shows

$$|L^\ell \eta_\ell|_{H^{1/2}(\partial\Omega_\ell)}^2 \lesssim \sum_{i=1}^4 (\eta_\ell(A_i) - \eta_\ell(B_i))^2$$

A_i and B_i being the two vertices of $\gamma_\ell^{(i)}$. Now we recall that (see [3], Lemma 3.1(i)) for all $\xi \in T_h^\ell$ we have

$$(\xi(A_i) - \xi(B_i))^2 \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) |\xi|_{H^{1/2}(\gamma_\ell^{(i)})}^2. \tag{73}$$

Assembling all the contributions we easily conclude that the thesis holds. □

Let us consider at first the non geometrically conforming case. Let $\eta \in \mathcal{T}_h$, $\eta = \eta^V + \eta^E$. By applying (45) and (69) we get

$$s(\eta, \eta) \lesssim |\eta^E|_T^2 + |\eta^V|_T^2 \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 b^E(\eta^E, \eta^E) + |\eta^V|_T^2. \tag{74}$$

Now, using Lemma 4.3, since $(Id - \pi_\delta)L\eta \in \mathcal{T}_\delta$ we can bound

$$\begin{aligned}
|\eta^V|_T^2 &= |(Id - \pi_h)L\eta|_T^2 \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) |L\eta|_T^2 \\
&= \left(1 + \log\left(\frac{H}{h}\right)\right) |L(Id - \pi_\delta)L\eta|_T^2 \\
&\lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) \left(1 + \log\left(\frac{H}{\delta}\right)\right) |(Id - \pi_\delta)L\eta|_T^2.
\end{aligned}$$

Since $\delta_\ell \geq h_\ell$ for all ℓ , which yields $\log(H/\delta) \leq \log(H/h)$, in view of the definition (62) of \hat{s} , we get

$$s(\eta, \eta) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \hat{s}(\eta, \eta).$$

On the other hand we have

$$\begin{aligned}
b^V(\eta^V, \eta^V) &\lesssim |(Id - \pi_\delta)L\eta|_T^2 \\
&\lesssim \left(1 + \log\left(\frac{H}{\delta}\right)\right) |L\eta|_T^2 \\
&\lesssim \left(1 + \log\left(\frac{H}{\delta}\right)\right) \left(1 + \log\left(\frac{H}{h}\right)\right) |\eta|_T^2 \\
&\lesssim s(\eta, \eta).
\end{aligned}$$

Let us now consider the second term in the sum at the right hand side of (62).

We can write

$$b^E(\eta^E, \eta^E) \lesssim \sum_{m \in I^*} \|\eta^E\|_{H_{00}^{1/2}(\gamma_m)}^2 \lesssim \sum_{m \in I^*} \|\eta^E\|_{H_{00}^{1/2}(\gamma_m)}^2 + \sum_{m \in I} \|\eta - L\eta\|_{H_{00}^{1/2}(\gamma_m)}^2$$

We now observe that on “trace sides” ($m \in I^*$) we have $\eta^E = \eta - L\eta$. Then, we can apply lemma 4.2 and we get

$$b^E(\eta^E, \eta^E) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 s(\eta, \eta), \quad (75)$$

which, again in view of (62), concludes the proof of the first part of Theorem 3.3.

Let us now consider the geometrically conforming case. Again we have

$$s(\eta, \eta) \lesssim |\eta^E|_T^2 + |\eta^V|_T^2 \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right)^2 b(\eta^E, \eta^E) + |\eta^V|_T^2. \quad (76)$$

Now we observe that, in the geometrically conforming case, letting $\eta \in \mathcal{T}_h^E$ and letting $\gamma_m = \Gamma_{\ell, \ell'}$ with ℓ master side and ℓ' slave side we have

$$\|\eta^{\ell'}\|_{H_{00}^{1/2}(\gamma_m)} = \|\pi_m \eta^\ell\|_{H_{00}^{1/2}(\gamma_m)} \lesssim \|\eta^\ell\|_{H_{00}^{1/2}(\gamma_m)},$$

which implies

$$|\eta|_T = \sum_{\ell} |\eta^\ell|_{H^{1/2}(\Gamma_\ell)} = \sum_{(\ell, i) \in I \cup I^*} \|\eta^\ell\|_{H_{00}^{1/2}(\gamma_\ell^{(i)})} \lesssim \sum_{(\ell, i) \in I} \|\eta^\ell\|_{H_{00}^{1/2}(\gamma_\ell^{(i)})},$$

whence

$$|\eta^E|_T \lesssim b^E(\eta^E, \eta^E).$$

If we bound $|\eta^V|$ as in the geometrically non conforming case we obtain

$$s(\eta, \eta) \lesssim \left(1 + \log\left(\frac{H}{h}\right)\right) \left(1 + \log\left(\frac{H}{\delta}\right)\right) \hat{s}(\eta, \eta).$$

5 Numerical tests.

In this section, we present some numerical examples illustrating the convergence properties of the substructuring preconditioner when used with the conjugate gradient method. In particular we provide numerical tests showing the scalability of the method for Q1 and Q2 finite elements.

We test the preconditioners on the following model problem:

$$-\Delta u = f \quad \text{in } \Omega =]0, 1[\times]0, 1[, \quad u = 0 \quad \text{on } \partial\Omega$$

and for all tests we set $f = 1$. A uniform, geometrically conforming, domain decomposition of Ω in $K = N \times N$ equal square subdomains of size $H \times H$, with $H = 1/N$ is considered.

Let S be the matrix associated to the discrete Steklov–Poincaré operator $s(\cdot, \cdot)$, then, after applying the change of basis corresponding to switching from the standard nodal basis to the basis corresponding to the splitting (44), and after ordering of the indices as nodes lying on the edges and on the vertex, we can write S as:

$$S = \begin{pmatrix} S_{EE} & S_{EV} \\ S_{EV}^T & S_{VV} \end{pmatrix}.$$

The preconditioner for S will be of block-Jacobi type: we drop all couplings between different edges and between edges and vertex points.

First we consider the preconditioner introduced in [1]: S_{EE} is replaced by its block diagonal part

$$S_{EE}^{diag} = \begin{pmatrix} S_{E_1, E_1} & 0 & 0 & 0 \\ 0 & S_{E_2, E_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & S_{E_N, E_N} \end{pmatrix}$$

with one block for each mortar where N is the number of mortars). Moreover, we recall that to obtain a convenient and inexpensive preconditioner, each S_{E_i, E_i} as well as S_{VV} should be replaced by efficient approximations.

In order to do so, we approximate S_{EE}^{diag} with the matrix S_R built as the square root of the stiffness matrix associated on each edge to the discretization of the operator $-d^2/dx^2$ with homogeneous Dirichlet boundary conditions at the extrema. Concerning the block S_{VV} , we propose to introduce a coarse auxiliary mesh as described in section (3.1.2): more precisely we consider a uniform mesh in each subdomain made up of 3×3 elements. Let S_{VV_c} be the vertex block of the Schur complement matrix associated to this coarse mesh; then we approximate the vertex block S_{VV} by S_{VV_c} .

In summary, the numerical tests relate the following three preconditioners for the Schur complement system:

$$S_1 = \begin{pmatrix} S_{EE}^{diag} & 0 \\ 0 & S_{VV} \end{pmatrix} \quad S_2 = \begin{pmatrix} S_R & 0 \\ 0 & S_{VV} \end{pmatrix} \quad S_3 = \begin{pmatrix} S_R & 0 \\ 0 & S_{VV_c} \end{pmatrix}.$$

5.1 Linear FEM

First we consider the case of linear finite elements. In each subdomain Ω_k we take a uniform mesh \mathcal{T}^k composed by $n_k \times n_k$ equal square elements of size $h_k \times h_k$, $h_k = H/n_k = 1/(Nn_k)$. Then, we define V_h^k in each subdomain Ω_k , to be the space of Q_1 finite elements on the mesh \mathcal{T}^k :

$$V_h^k := \{u_h \in C^0(\Omega_k) : u_{h|\tau} \in Q_1(\tau), \forall \tau \in \mathcal{T}^k\}.$$

The multiplier space was chosen based on a dual basis, see [9].

Conforming decomposition. We start by considering the case of a conforming decomposition, that is, we set $n_k = n$ for all k ; in this case $h_k = h = H/n$. Hence we obtain the same solution that we would get using a conforming finite element method on a regular grid of $Nn \times Nn$ square elements of dimension $h \times h$.

In order to test the dependence on H (size of the subdomains) and on h we test the preconditioners for n in the range [5, 40] and N in the range [4, 32]. The preconditioners were studied for several combinations of N and n . Note that in this case $\max_k H_k/h_k = H/h = n$. Tables 1-2 and 3 show the number of conjugate gradient iterations for reducing the residual of a factor 10^{-5} with preconditioners given by S_1 , S_2 and S_3 respectively. For comparison and to show the efficiency of the preconditioners, in Table 4 we report also the number of conjugate gradient iterations obtained without preconditioning.

The results are in close agreement with the theory: the condition number of the preconditioned matrix grows at most polylogarithmically with the number of degrees of freedom per subdomain, as indicated by (66,48).

Table 1 Q_1 FEM. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} with the preconditioner S_1 , for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains).

$K = N^2$	n=5	n=10	n=20	n=40
	# iter.	# iter.	# iter.	# iter.
16	21	23	24	25
64	21	23	26	27
144	21	24	26	27
256	21	23	26	27
400	21	23	26	27
576	21	23	26	27
784	21	23	25	27
1024	21	23	25	27

Table 2 Q_1 FEM. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} with the preconditioner S_2 , for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains).

$K = N^2$	n=5	n=10	n=20	n=40
	# iter.	# iter.	# iter.	# iter.
16	23	25	26	27
64	24	26	27	28
144	24	26	27	29
256	24	26	27	29
400	24	26	27	28
576	24	26	27	28
784	24	26	27	28
1024	23	26	27	28

Table 3 Q_1 FEM. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} with the preconditioner S_3 , for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains).

$K = N^2$	n=5	n=10	n=20	n=40
	# iter.	# iter.	# iter.	# iter.
16	23	25	27	31
64	24	26	28	32
144	24	26	29	32
256	24	26	29	32
400	24	26	29	30
576	24	26	29	30
784	24	26	29	30
1024	24	26	29	30

Table 4 Q_1 FEM. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} , without preconditioner for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains). The method does not converge always within the maximum number of iteration (=100).

$K = N^2$	n=5	n=10	n=20	n=40
	# iter.	# iter.	# iter.	# iter.
16	23	30	40	55
64	40	54	74	-
144	57	78	-	-
256	75	-	-	-
400	93	-	-	-
576	-	-	-	-
784	-	-	-	-
1024	-	-	-	-

Nonconforming decomposition. Then we present results also for nonconforming decompositions and again we verify that the condition number only depends polylogarithmically on $\max_k H_k/h_k$ in agreement with the theory.

We consider two test configurations, both on a decomposition of Ω into 5×5 subdomains. In the first case (see figure 1 left), for all subdomains we set $n_k = 10$ except two (n. 13 and 14) where we set $n_k = n$ with n that varies in the set $[10, \dots, 52]$.

In the second configuration (see figure 1 right) we set again $n_k = 10$, this time except that on a 3×3 block of subdomains where $n_k = n$, with n varying again between 10 and 52.

The results are provided in Tables 5 and 6. For both configurations we test the three preconditioners S_1, S_2, S_3 and we report also the number of iterations required without preconditioning. Moreover, in order to study the influence of the coarse mesh used, we consider a new coarse auxiliary mesh: a uniform mesh in each subdomain made up of 10×10 elements. Let S_{VVc10} be the vertex block of the Schur complement matrix associated to this coarse mesh; then we consider the following preconditioner:

$$S_{10} = \begin{pmatrix} S_R & 0 \\ 0 & S_{VVc10} \end{pmatrix}.$$

The theoretical estimates on the condition number only depends on $\max H_k/h_k$ and take the same values for both configurations. No significant differences can be seen in Tables 5, 6 when using preconditioners S_1 and S_2 .

Similar results are obtained using both preconditioner S_3 or S_{10} . Thus, the use of the auxiliary mesh made up of 3×3 elements each subdomain instead of the auxiliary mesh made up of 10×10 elements does not significantly influence the convergence.

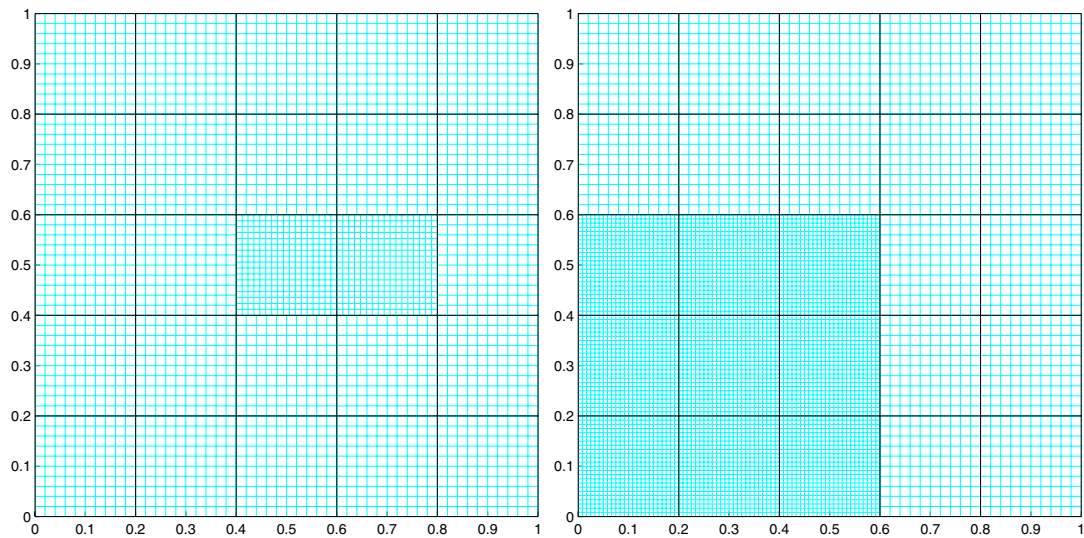


Fig. 1 Splitting of Ω into 5×5 subdomains. Configuration 1 (left): for all subdomains we set $n_k = 10$ except two (n. 13 and 14) where $n_k = n$ with n that varies in the set $[10, \dots, 52]$. Configuration 2 (right): we set again $n_k = 10$, this time except that on a 3×3 block of subdomains where $n_k = n$ and n in $[10, \dots, 52]$.

Table 5 Q_1 FEM. Configuration 1: decomposition of Figure 1 left. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} for four preconditioners and for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains).

	S_1	S_2	S_{10}	S_3	no prec.
n= 10	23	25	25	25	32
n= 17	27	30	30	30	52
n= 24	30	33	33	34	55
n= 31	31	35	35	35	57
n= 38	32	36	36	36	62
n= 45	33	37	37	37	66
n= 52	35	38	37	38	68

Table 6 Q_1 FEM. Configuration 2: decomposition of Figure 1 right. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} for four preconditioners and for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains).

	S_1	S_2	S_{10}	S_3	no prec.
n = 10	23	25	25	25	32
n = 17	26	31	31	32	59
n = 24	29	33	35	35	65
n = 31	31	36	37	39	73
n = 38	33	37	38	40	79
n = 45	34	38	39	41	86
n = 52	35	40	40	41	92

Hence, we may conclude that again the results are in close agreement with the theory: the condition number grows at most polylogarithmically with the number of degrees of freedom per subdomain, as indicated by (66,48).

Random mesh. Finally, to study a truly nonconforming decomposition, we test the preconditioner on 4 different splitting of Ω into respectively 4×4 , 8×8 , 12×12 and 16×16 subdomains (Figure 2 a),b),c),d) respectively). In each case we randomly assigned the values of n_k in such a way that for $\sim 1/3$ of the subdomains $n_k = 5$, for about another third $n_k = 10$, and for the remaining subdomains $n_k = 15$. The four configurations considered are shown in Figure 2.

In Table 7 we report the number of conjugate gradient iterations needed by preconditioners S_1, S_2, S_3 and without preconditioning. Each row of the table relates to a configuration of Figure 2 and shows the convergence of the different preconditioners.

Again our numerical tests confirm the theory; indeed Table 7 clearly shows that the convergence satisfies the theoretical estimates.

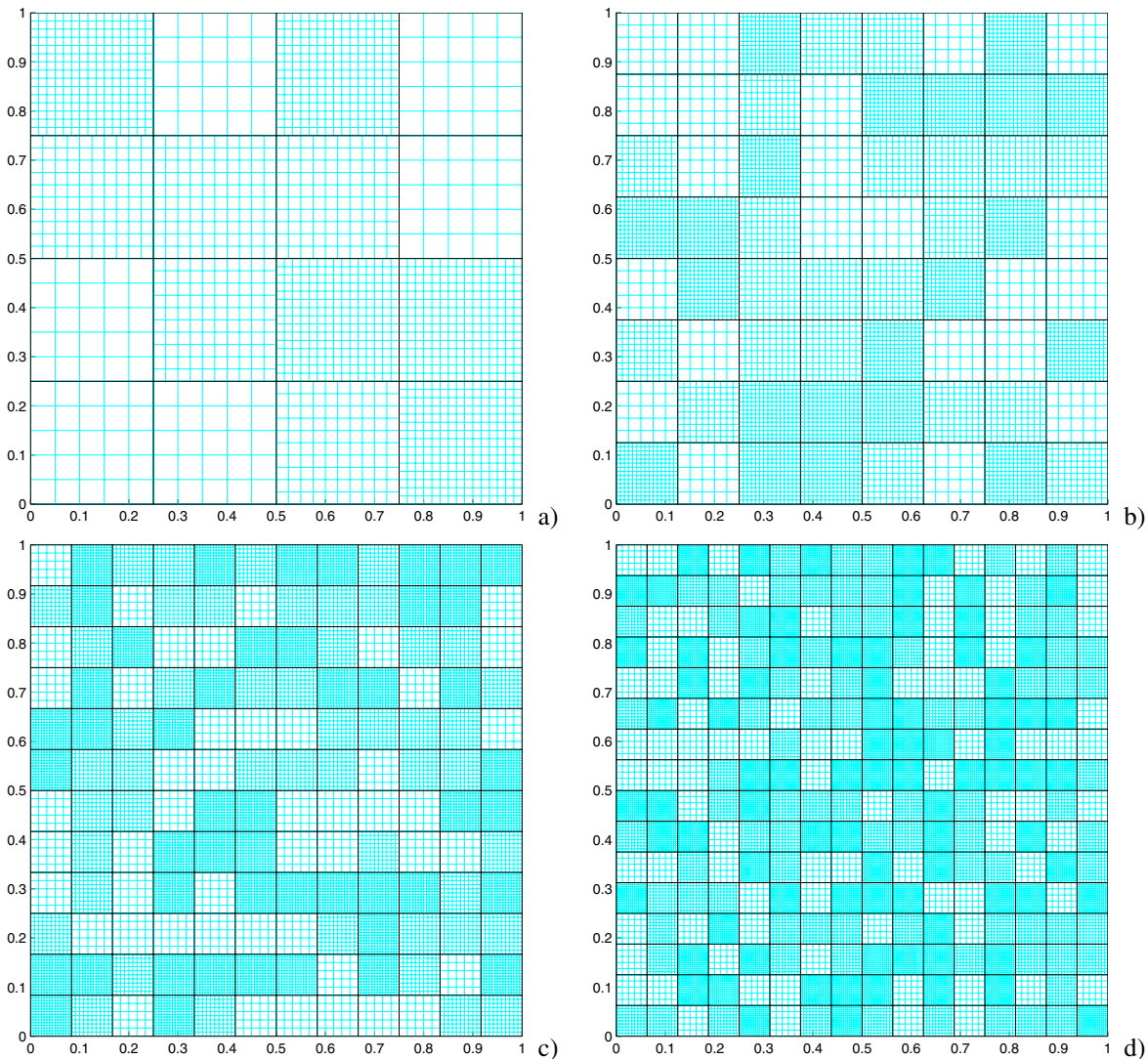


Fig. 2 Splitting of Ω into respectively 4×4 , 8×8 , 12×12 and 16×16 subdomains. The values of n_k is randomly assigned in such a way that for $\sim 1/3$ of the subdomains $n_k = 5$, for about another third $n_k = 10$, and for the remaining subdomains $n_k = 15$.

Table 7 Q_1 FEM. Random mesh: decomposition of Figure 2. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} for preconditioners S_1, S_2, S_3 and without preconditioning. Each row relates to a configuration of Figure 2.

	S_1	S_2	S_3	no prec.
(a)	27	30	30	44
(b)	30	33	34	84
(c)	31	34	36	-
(d)	31	34	35	-

5.2 Quadratic finite elements.

Finally we considered the case of quadratic finite elements. In each subdomain Ω_k we take a uniform mesh \mathcal{T}^k composed by $n_k \times n_k$ equal square elements of size $h_k \times h_k$, $h_k = H/n_k = 1/(Nn_k)$. Now, we define V_h^k in each subdomain Ω_k , to be the space of Q_2 finite elements on the mesh \mathcal{T}^k :

$$V_h^k := \{u_h \in C^0(\Omega_k) : u_{h|_\tau} \in Q_2(\tau), \forall \tau \in \mathcal{T}^k\}.$$

As before the multiplier space was chosen based on a dual basis, see [7, 9].

Conforming decomposition. Now we consider the same conforming decomposition introduced for the Q_1 FEM. In order to test the dependence on H (size of the subdomains) and on h we study the preconditioners for different values of n in the range [5, 40] and N in the range [4, 32]. Table 8 shows the number of conjugate gradient iterations for reducing the residual of the Schur complement system of a factor 10^{-5} with the preconditioner S_3 . For comparison purposes in Table 9 we report also the number of conjugate gradient iterations obtained without preconditioning.

As expected the number of iterations increases if compared with the linear FEM case due to the greater number of degrees of freedom. The convergence still remains in agreement with the theory, i.e. the condition number of the preconditioned matrix grows at most polylogarithmically with the number of degrees of freedom per subdomain, as indicated by (66,48).

Table 8 Q_2 FEM. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} , with S_3 preconditioner for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains).

$K = N^2$	n=5 # iter.	n=10 # iter.	n=20 # iter.	n=40 # iter.
16	25	26	29	32
64	27	29	31	35
144	27	29	32	35
256	27	29	32	35
400	27	29	32	34
576	27	29	32	34
784	27	29	32	34
1024	27	29	32	34

Nonconforming decomposition. Tables 10 and 11 provide the convergence of the preconditioners S_1, S_2, S_{10}, S_3 and without preconditioning when Q_2 FEM are used for the two configurations of Figure 1. The comparison of Tables 10 and 11 with the corresponding Tables 5 and 6 obtained using Q_1 FEM, shows only an increase in the number of iterations due to the greater number of degrees of freedom but the same convergence in agreement with the theoretical estimates of the condition number.

Table 9 Q_2 FEM. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} , without preconditioner for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains). The method does not converge always within the maximum number of iteration (=100).

$K = N^2$	n=5 # iter.	n=10 # iter.	n=20 # iter.	n=40 # iter.
16	25	30	43	60
64	41	56	78	-
144	59	80	-	-
256	78	-	-	-
400	95	-	-	-
576	-	-	-	-
784	-	-	-	-
1024	-	-	-	-

Table 10 Q_2 FEM. Configuration 1: decomposition of Figure 1 left. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} for four preconditioners and for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains).

	S_1	S_2	S_{10}	S_3	no prec.
n= 10	25	27	27	28	33
n= 17	30	33	33	33	55
n= 24	34	36	36	37	58
n= 31	35	38	38	38	61
n= 38	35	39	39	39	67
n= 45	36	40	40	40	71
n= 52	37	41	41	41	75

Table 11 Q_2 FEM. Configuration 2: decomposition of Figure 1 right. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} for four preconditioners and for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains). Without preconditioning the method does not always converge with the maximum number of iterations (100).

	S_1	S_2	S_{10}	S_3	no prec.
n= 10	25	27	27	28	33
n= 17	31	33	34	36	61
n= 24	34	36	36	38	66
n= 31	35	39	40	42	74
n= 38	37	40	41	43	80
n= 45	38	41	42	43	87
n= 52	38	42	42	43	95

Random mesh. Finally we test the preconditioners in the same truly nonconforming decomposition displayed in Figure 2. Table 12 provide the convergence of the preconditioners S_1, S_2, S_3 and without preconditioners when Q_2 FEM are used and corresponds to Table 7 where Q_1 FEM were considered. Also in this case we note only an increase in the number of iterations but the same convergence predicted by the theory.

Table 12 Q_2 FEM. Random mesh: decomposition of Figure 2. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} for preconditioners S_1, S_2, S_3 and without preconditioning. Each row relates to a configuration of Figure 2.

	S_1	S_2	S_3	no prec.
(a)	30	33	34	48
(b)	32	35	37	87
(c)	33	36	38	-
(d)	33	37	39	-

6 Conclusions

In this paper we considered a class of preconditioners for the linear system that arises using the mortar method. We focused on the substructuring approach already applied by Achdou, Maday, Widlund [1] to the mortar method for the case of finite elements of first order. We provided an estimate (Theorem 3.3) which relies on abstract assumptions so that the result holds for finite elements of any order.

Moreover we wanted to improve the efficiency of the algorithm without sacrificing the convergence rate. Hence we studied an efficient variant of the vertex block (61)-(62) yielding a cheaper and easier to implement preconditioner. As illustrated by Theorem 3.3 and Corollary 3.4, for finite elements of any order, this variant allows to obtain a preconditioned matrix whose condition number grows at most polylogarithmically with the number of degrees of freedom. The numerical results confirmed the convergence properties of the preconditioner when using the conjugate gradient method and showed the scalability of the method for Q_1 and Q_2 finite elements.

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