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Discontinuous FEM for elliptic problems

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MOTIVATIONS

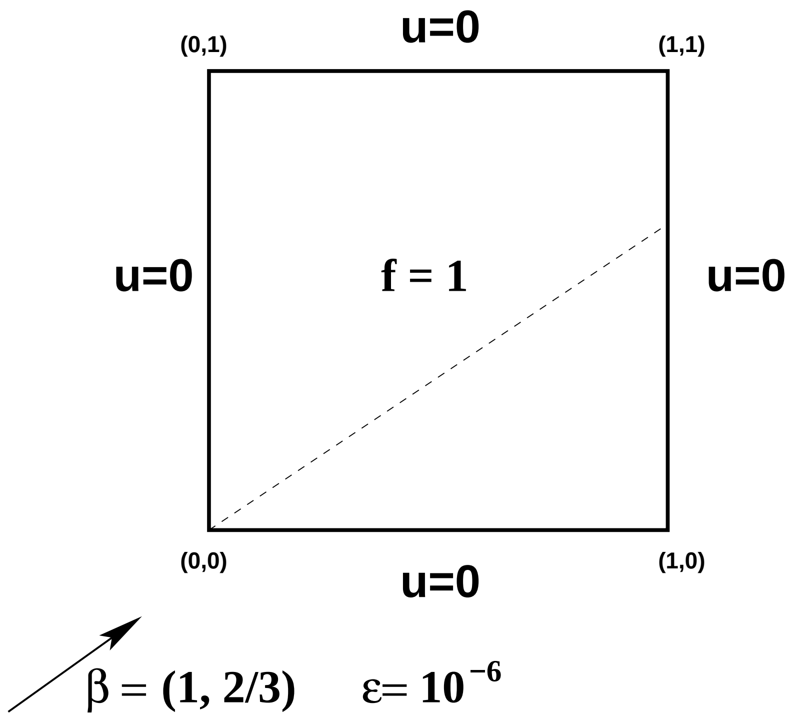
- Are discontinuous finite elements needed for purely elliptic problems?

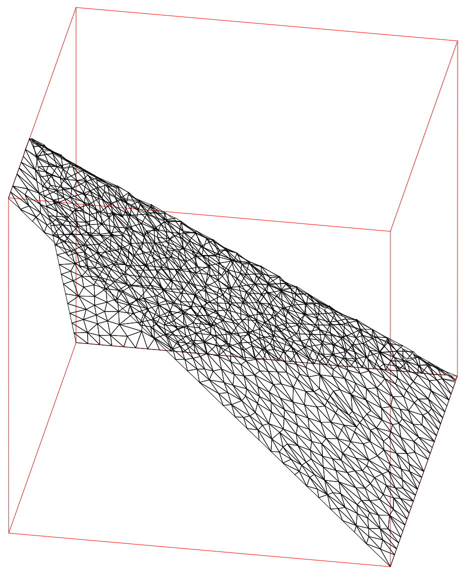
NOT REALLY

- They work very well for purely hyperbolic problems.
- It is worth analysing their behaviour on problems where the elliptic part is present but it is not dominant.

  (example: strongly advection-dominated equations, very thin Reissner-Mindlin plates)

ADVECTION-DOMINATED PROBLEM





PLAN:

- The flux formulation
- First choice of fluxes: IP and its variants
- Symmetry and adjoint consistency
- Second choice of fluxes: B-R and its variants
- Error estimates for various DG methods

MODEL (TOY) PROBLEM

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

$$\boldsymbol{\sigma} = \nabla u$$

$$\begin{cases} \boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega \\ -\operatorname{div} \boldsymbol{\sigma} = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

STARTING POINT

$\boldsymbol{\sigma} = \nabla u \qquad -\operatorname{div} \boldsymbol{\sigma} = f$

$\mathscr{T}_h$: decomposition of Ω into triangles E

- $\displaystyle\int_E \boldsymbol{\sigma} \cdot \boldsymbol{\tau} = -\int_E u \operatorname{div} \boldsymbol{\tau} + \int_{\partial E} u \, \boldsymbol{\tau} \cdot \mathbf{n} \quad \forall \boldsymbol{\tau}$
- $\displaystyle\int_E \boldsymbol{\sigma} \cdot \nabla v = \int_E f v + \int_{\partial E} v \, \boldsymbol{\sigma} \cdot \mathbf{n} \qquad \forall v$

To fix the ideas, we set for $k \geq 1$

$$V_h = \left\{ v \in L^2(\Omega) \text{ such that } v_{|E} \in \mathbb{P}_k(E) \ \forall E \in \mathscr{T}_h \right\}$$

$$\boldsymbol{\Sigma}_h = \left\{ \boldsymbol{\tau} \in \left[L^2(\Omega) \right]^2 \text{ such that } \boldsymbol{\tau}_{|E} \in [\mathbb{P}_k(E)]^2 \ \forall E \in \mathscr{T}_h \right\}$$

FLUX FORMULATION - 1

$$
\begin{cases}
\text{find } u_h \in V_h,\ \boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h \text{ such that:} \\
\displaystyle\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} = -\sum_E \int_E u_h \operatorname{div} \boldsymbol{\tau} + \sum_E \int_{\partial E} \hat{u}\, \boldsymbol{\tau} \cdot \mathbf{n} \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \\
\displaystyle\sum_E \int_E \boldsymbol{\sigma}_h \cdot \nabla v = \int_\Omega f v + \sum_E \int_{\partial E} v\, \hat{\boldsymbol{\sigma}} \cdot \mathbf{n} \quad \forall v \in V_h
\end{cases}
$$

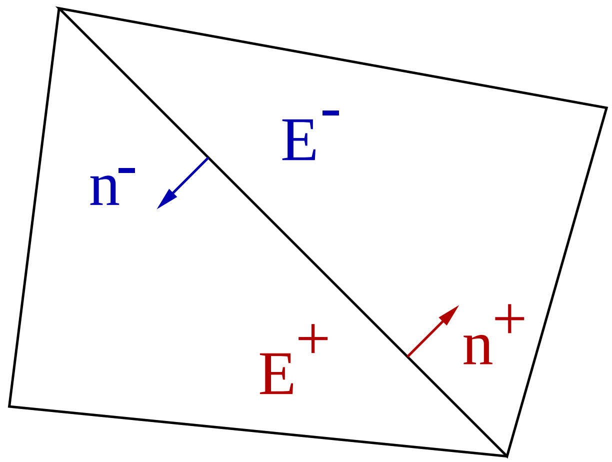
$\hat{u} = \hat{u}(u_h),\ \hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}(u_h, \boldsymbol{\sigma}_h)$: numerical fluxes $\simeq u_{|\partial E},\ \nabla u_{|\partial E}$

$\mathcal{E}_h$: set of all edges, $\mathcal{E}'_h$: internal edges

Consistent: if $\hat{u}(v) = v_{|\mathcal{E}_h}$ and $\hat{\boldsymbol{\sigma}}(v, \nabla v) = \nabla v_{|\mathcal{E}_h}$, $\forall v$ regular

Conservative: if fluxes (mostly $\hat{\boldsymbol{\sigma}}$) are single-valued on each $e \in \mathcal{E}_h$

AVERAGES AND JUMPS



Definition of average and jump on an internal edge:

$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h'$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+\mathbf{n}^+ + \boldsymbol{\tau}^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h'$$

On the boundary edges: $[\![v]\!] = v\mathbf{n}; \qquad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}$

A CRUCIAL FORMULA

q $\boldsymbol{\tau}$

$[\![\cdot]\!]$ $\{\cdot\}$

$$\sum_E \int_{\partial E} q\,\boldsymbol{\tau}\cdot\mathbf{n} = \sum_e \int_e [\![q]\!]\cdot\{\boldsymbol{\tau}\} + \sum_{e'} \int_{e'} \{q\}[\![\boldsymbol{\tau}]\!]$$

where e ranges over all edges and e' ranges over internal edges.

FLUX FORMULATION - 2

$$-\int_\Omega u_h \mathrm{div}_h \boldsymbol{\tau} = \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau} - \sum_E \int_{\partial E} u_h \boldsymbol{\tau} \cdot \mathbf{n}$$

and the previous crucial formula, we can rewrite the discrete problem as

$$\begin{cases} \text{find } u_h \in V_h, \ \boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h \text{ such that } \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \ \forall v \in V_h \\ \displaystyle\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} = \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau} + \sum_e \int_e [\![\hat{u} - u_h]\!] \cdot \boldsymbol{\tau} + \sum_{e'} \int_{e'} \{\hat{u} - u_h\} [\![\boldsymbol{\tau}]\!] \\ \displaystyle\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v = \int_\Omega f v + \sum_e \int_e [\![v]\!] \cdot \{\hat{\boldsymbol{\sigma}}\} + \sum_{e'} \int_{e'} \{\hat{v}\} [\![\hat{\boldsymbol{\sigma}}]\!] \end{cases}$$

ELIMINATING $\boldsymbol{\sigma}_h$ - 1

Important assumption (verified with our choice of V_h and $\boldsymbol{\Sigma}_h$):

$$\nabla(V_h) \subset \boldsymbol{\Sigma}_h$$

Then we can take $\boldsymbol{\tau} = \nabla_h v$ in the first equation, and substitute in the second equation

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} = \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau} + \sum_e \int_e [\![\hat{u} - u_h]\!] \cdot \boldsymbol{\tau} + \sum_{e'} \int_{e'} \{\hat{u} - u_h\} [\![\boldsymbol{\tau}]\!]$$

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v = \int_\Omega f v + \sum_e \int_e [\![v]\!] \cdot \{\hat{\boldsymbol{\sigma}}\} + \sum_{e'} \int_{e'} \{\hat{v}\} [\![\hat{\boldsymbol{\sigma}}]\!]$$

ELIMINATING $\boldsymbol{\sigma}_h$ - 2

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v + \sum_e \int_e [\![\hat{u} - u_h]\!] \cdot \{\nabla_h v\} + \sum_{e'} \int_{e'} \{\hat{u} - u_h\}[\![\nabla_h v]\!]$$

$$\int_\Omega f v + \sum_e \int_e [\![v]\!] \cdot \{\hat{\boldsymbol{\sigma}}\} + \sum_{e'} \int_{e'} \{\hat{v}\}[\![\hat{\boldsymbol{\sigma}}]\!].$$

It remains to express $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ in terms of u_h. This will be easy if we take $\hat{u} = \hat{u}(u_h)$ and $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}(u_h, \nabla_h u_h)$ (*primal methods*).

Most *flux methods* however take $\hat{u} = \hat{u}(u_h)$ but $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}(u_h, \boldsymbol{\sigma}_h)$. In this case, we shall need some additional work.

A FIRST CHOICE OF FLUXES: INTERIOR PENALTY

- $\hat{u} = \{u_h\}$ on e', $\quad \hat{u} = 0$ on $e \subset \partial\Omega$

$$\Longrightarrow \quad [\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = 0$$

- $\hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\}$ on every edge

$$\Longrightarrow \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\}$$

Substituting in the previous equation and rearranging terms, we have

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\} - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\} = \int_\Omega f v,$$

where we recognize the nonstabilized version of IP method (Douglas-Dupont, Wheeler, Arnold).

Usually, one arrives (for instance) at the IP method with a simpler argument. Take the equation $-\Delta u = f$, multiply it by a piecewise polynomial function v, and integrate by parts. Rearranging terms you have

$$\int_\Omega \nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla u\} = \int_\Omega f v.$$

Then, taking into account the fact that, for the solution u, one has clearly $[\![u]\!] = 0$, one adds a term to restore *symmetry* (???)

$$\int_\Omega \nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla u\} - \sum_e \int_e [\![u]\!] \cdot \{\nabla_h v\} = \int_\Omega f v.$$

An additional term, penalizing the jumps of u_h, is usually added to stabilize the method.

SYMMETRY

whole

opposite sign

adjoint consistency

$$\int_\Omega A\nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{A\nabla u\} = \int_\Omega f v$$

$$\int_\Omega A\nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{A\nabla u\} - \sum_e \int_e [\![u]\!] \cdot \{A^* \nabla_h v\} = \int_\Omega f v$$

ADJOINT CONSISTENCY

consistency

adjoint consistency

- $\hat{u} = \{u_h\}$ on e', $\quad \hat{u} = 0$ on $e \subset \partial\Omega$

$$\implies \quad [\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = 0$$

- $\hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\} - c\,|e|^{-1}[\![u_h]\!]$ on every edge

$$\implies \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\} - c\,|e|^{-1}[\![u_h]\!]$$

Then we have

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\} - \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\}$$

$$+ \sum_e c\,|e|^{-1} \int_e [\![u_h]\!] \cdot [\![v]\!] = \int_\Omega f v,$$

and we obtain the stabilized IP method (Douglas-Dupont, Wheeler, Arnold).

β

- $\hat{u} = \{u_h\}_{(1-\beta)}$ on e', $\hat{u} = 0$ on $e \subset \partial\Omega$

  $$\Longrightarrow \quad [\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = \{u_h\}_{(1-\beta)} - \{u_h\}$$

- $\hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\}_\beta - c\,|e|^{-1}[\![u_h]\!]$ on every edge

  $$\Longrightarrow \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\}_\beta - c\,|e|^{-1}[\![u_h]\!]$$

Then we have

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\}_\beta - \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\}_\beta$$

$$+ \sum_e c\,|e|^{-1} \int_e [\![u_h]\!] \cdot [\![v]\!] = \int_\Omega f v,$$

and we obtain the method of Heinrich

- $\hat{u} = \{u_h\} + \mathbf{n}_E[\![u_h]\!]$ on e', $\quad \hat{u} = 0$ on $e \subset \partial\Omega$

$$\Longrightarrow \quad [\![\hat{u} - u_h]\!] = -[\![u_h]\!] + 2[\![u_h]\!], \quad \{\hat{u} - u_h\} = 0$$

- $\hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\}$ on every edge

$$\Longrightarrow \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\}$$

Then we have

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\} + \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\} = \int_\Omega f v,$$

and we obtain the method of Baumann-Oden.

- $\hat{u} = \{u_h\} + \mathbf{n}_E[\![u_h]\!]$ on e', $\quad \hat{u} = 0$ on $e \subset \partial\Omega$

$$\Longrightarrow \quad [\![\hat{u} - u_h]\!] = -[\![u_h]\!] + 2[\![u_h]\!], \quad \{\hat{u} - u_h\} = 0$$

- $\hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\} - c\,|e|^{-1}[\![u_h]\!]$ on every edge

$$\Longrightarrow \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\} - c\,|e|^{-1}[\![u_h]\!]$$

Then we have

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\} + \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\}$$

$$+ \sum_e c\,|e|^{-1} \int_e [\![u_h]\!] \cdot [\![v]\!] = \int_\Omega f v,$$

and we obtain the stabilized version of Baumann-Oden (NIPG: see Wheeler-Rivière-Girault) .

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} = \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau} + \sum_e \int_e [\![\hat{u} - u_h]\!] \cdot \{\boldsymbol{\tau}\} + \sum_{e'} \int_{e'} \{\hat{u} - u_h\} [\![\boldsymbol{\tau}]\!]$$

and its substitution into

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v = \int_\Omega f v + \sum_e \int_e [\![v]\!] \cdot \{\hat{\boldsymbol{\sigma}}\} + \sum_{e'} \int_{e'} \{\hat{v}\} [\![\hat{\boldsymbol{\sigma}}]\!]$$

requires some additional work. It is now time to do it.

INTRODUCING $\boldsymbol{R}$, l

$$\left(\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} = \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau} + \sum_e \int_e [\![\hat{u} - u_h]\!] \cdot \{\boldsymbol{\tau}\} + \sum_{e'} \int_{e'} \{\hat{u} - u_h\} [\![\boldsymbol{\tau}]\!]\right)$$

DEFINITION: $v \longrightarrow \boldsymbol{R}([\![v]\!]) \in \boldsymbol{\Sigma}_h$ is given by:

- $$\int_\Omega \boldsymbol{R}([\![v]\!]) \cdot \boldsymbol{\tau} = -\sum_e \int_e [\![v]\!] \cdot \{\boldsymbol{\tau}\} \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h$$

DEFINITION: $v \longrightarrow l(\{v\}) \in \boldsymbol{\Sigma}_h$ is given by:

- $$\int_\Omega l(\{v\}) \cdot \boldsymbol{\tau} = -\sum_{e'} \int_{e'} \{v\} [\![\boldsymbol{\tau}]\!] \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h$$

Hence:

$$\boldsymbol{\sigma}_h = \nabla_h u_h - \boldsymbol{R}([\![\hat{u} - u_h]\!]) - l(\{\hat{u} - u_h\})$$

Substituting $\boldsymbol{\sigma}_h = \nabla_h u_h - \boldsymbol{R}([\![\hat{u} - u_h]\!]) - l(\{\hat{u} - u_h\})$ in the second equation $(\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\hat{\boldsymbol{\sigma}}\} - \sum_{e'} \int_{e'} \{\hat{v}\}[\![\hat{\boldsymbol{\sigma}}]\!] = \int_\Omega f v)$ we have:

$$\text{find } u_h \in V_h : \quad B_h(u_h, v) = (f, v) \quad \forall v \in V_h$$

where:

$$\begin{aligned} B_h(u_h, v) := & \int_\Omega \nabla_h u_h \cdot \nabla_h v - \int_\Omega \boldsymbol{R}([\![\hat{u} - u_h]\!]) \cdot \nabla_h v \\ & - \int_\Omega l(\{\hat{u} - u_h\}) \cdot \nabla_h v - \sum_e \int_e \{\hat{\boldsymbol{\sigma}}\} \cdot [\![v]\!] - \sum_{e'} \int_{e'} [\![\hat{\boldsymbol{\sigma}}]\!]\{v\} \\ \equiv & \int_\Omega \nabla_h u_h \cdot \nabla_h v + \sum_e \int_e [\![\hat{u} - u_h]\!] \cdot \{\nabla_h v\} \\ & + \sum_{e'} \int_{e'} \{\hat{u} - u_h\}[\![\nabla_h v]\!] - \sum_e \int_e \{\hat{\boldsymbol{\sigma}}\} \cdot [\![v]\!] - \sum_{e'} \int_{e'} [\![\hat{\boldsymbol{\sigma}}]\!]\{v\} \end{aligned}$$

A NEW CHOICE OF FLUXES: FIRST BASSI-REBAY METHOD

- $\hat{u} = \{u_h\}$ on e', $\hat{u} = 0$ on $e \subset \partial\Omega$

$$\Longrightarrow \quad [\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = 0$$

- $\hat{\boldsymbol{\sigma}} = \{\boldsymbol{\sigma}_h\}$ on every edge

$$\Longrightarrow \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\boldsymbol{\sigma}_h\}$$

Consequently: $\boldsymbol{\sigma}_h = \nabla_h u_h + \boldsymbol{R}([\![u_h]\!])$ ($\boldsymbol{R}([\![u_h]\!])$ = strain correction)

Therefore

$$\sum_e \int_e \{\hat{\boldsymbol{\sigma}}\} \cdot [\![v]\!] = \sum_e \int_e \{\nabla_h u_h\} \cdot [\![v]\!] + \sum_e \int_e \{\boldsymbol{R}([\![u_h]\!])\} \cdot [\![v]\!]$$

$$= -\int_\Omega \nabla_h u_h \cdot \boldsymbol{R}([\![v]\!]) - \int_\Omega \boldsymbol{R}([\![u_h]\!]) \cdot \boldsymbol{R}([\![v]\!])$$

Substituting in $B_h(u_h, v)$ gives the first Bassi-Rebay formulation.

FIRST BASSI-REBAY FORMULATION

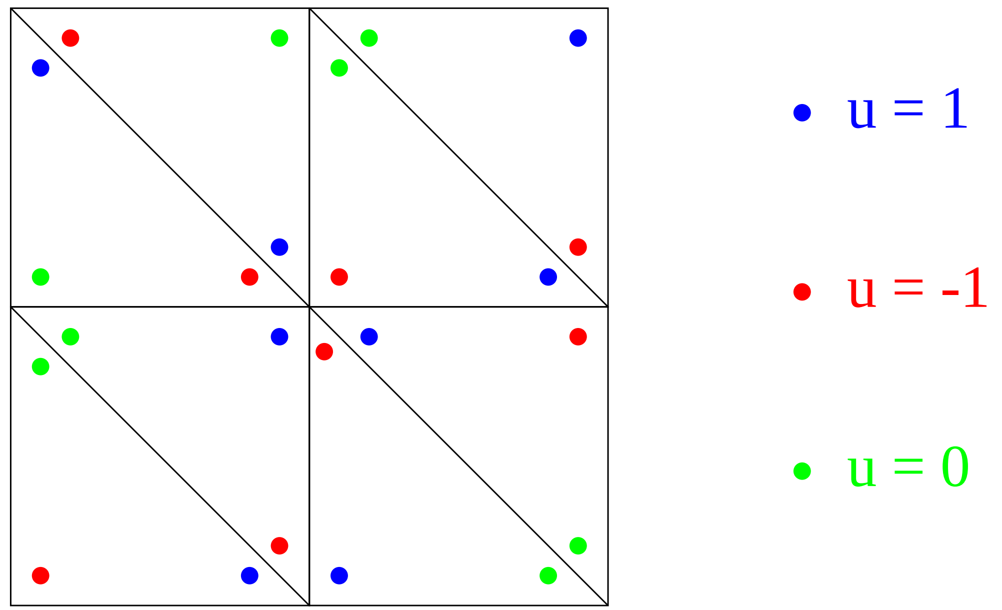
$$u_h \in V_h : \quad \int_\Omega [\nabla_h u_h + \boldsymbol{R}([\![u_h]\!])] \cdot [\nabla_h v + \boldsymbol{R}([\![v]\!])] = \int_\Omega f v \quad \forall v \in V_h$$

Equivalently:

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e \{\nabla_h u_h\} \cdot [\![v]\!]$$

$$- \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\} + \int_\Omega \boldsymbol{R}([\![u_h]\!]) \cdot \boldsymbol{R}([\![v]\!]) = \int_\Omega f v \quad \forall v \in V_h$$

INSTABILITY OF THE FIRST BASSI-REBAY-FORMULATION



Example of a NON constant u such that

$$\nabla u + \boldsymbol{R}([\![u]\!]) = 0 \text{ in every } E$$

STABILIZATION OF THE BASSI-REBAY-FORMULATION

DEFINITION: $v \longrightarrow \boldsymbol{r}_e([\![v]\!]) \in \boldsymbol{\Sigma}_h$ given by:

- $$\int_\Omega \boldsymbol{r}_e([\![v]\!]) \cdot \boldsymbol{\tau} + \int_e [\![v]\!] \cdot \{\boldsymbol{\tau}\} = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h$$

$$\Longrightarrow \sum_e \boldsymbol{r}_e([\![v]\!]) = \boldsymbol{R}([\![v]\!])$$

Second Bassi-Rebay formulation:

$$\int_\Omega [\nabla_h u_h + \boldsymbol{R}([\![u_h]\!])] \cdot [\nabla_h v + \boldsymbol{R}([\![v]\!])]$$

$$- \int_\Omega \boldsymbol{R}([\![u_h]\!]) \cdot \boldsymbol{R}([\![v]\!]) + c \sum_e \int_\Omega \boldsymbol{r}_e([\![u_h]\!]) \cdot \boldsymbol{r}_e([\![v]\!]) = \int_\Omega f v \quad \forall v \in V_h$$

- $\hat{u} = \{u_h\}$ on e', $\quad \hat{u} = 0$ on $e \subset \partial\Omega$

$$\Longrightarrow \quad [\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = 0$$

- $\hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\} - c\, \boldsymbol{r}_e([\![u_h]\!])$ on every edge

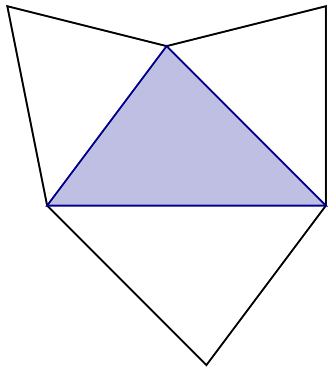
$$\Longrightarrow \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\} - c\,\{\boldsymbol{r}_e([\![u_h]\!])\}$$

Then we have

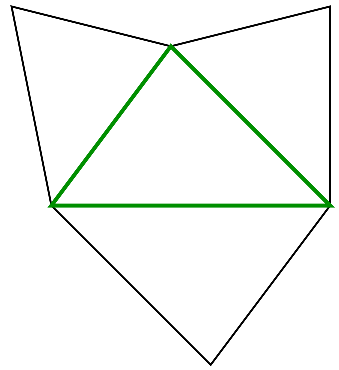
$$\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\} - \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\}$$
$$+ c \sum_e \int_\Omega \boldsymbol{r}_e([\![u_h]\!]) \cdot \boldsymbol{r}_e([\![v]\!]) = \int_\Omega f v,$$

and we see that the difference with IP is only in the choice of the stabilizing term.

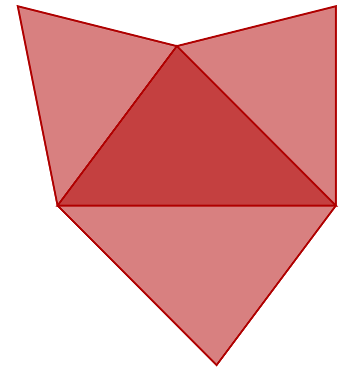
BANDWIDTH CONSIDERATIONS



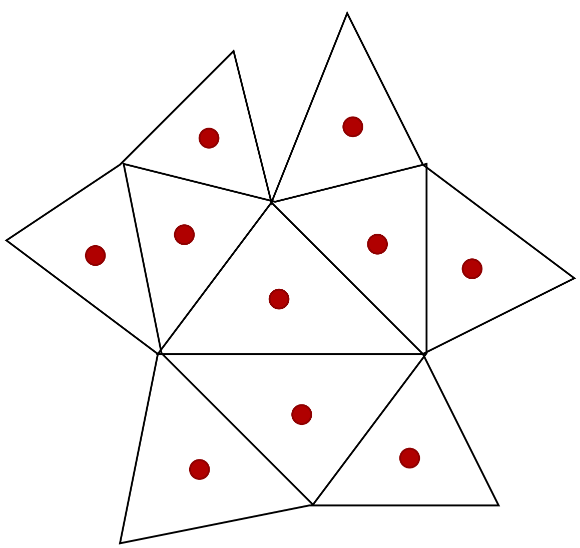
$u \neq 0$



$[\![u]\!] \neq 0$

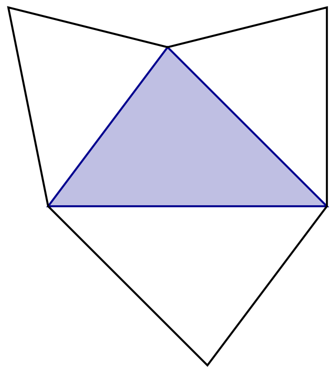


$\boldsymbol{R}([\![u]\!]) \neq 0$

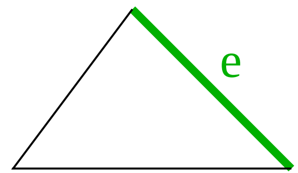


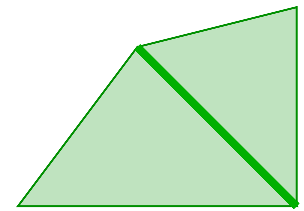
$$\int_\Omega \boldsymbol{R}([\![u]\!]) \cdot \boldsymbol{R}([\![v]\!]) \neq 0$$

BANDWIDTH CONSIDERATIONS

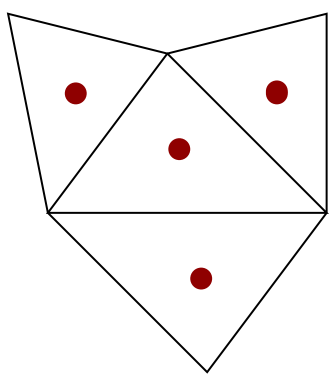


$u \neq 0$





$r_e([\![u]\!]) \neq 0$



$$\sum_e \int_\Omega \boldsymbol{r}_e([\![u]\!]) \cdot \boldsymbol{r}_e([\![v]\!]) \neq 0$$

VARIANTS OF THE B-R FORMULATION

As we have seen for the IP formulation, many variants are possible for the first Bassi-Rebay formulation. For instance, taking

- $\hat{u} = \{u_h\}_{(1-\beta)}$ on e', $\hat{u} = 0$ on $e \subset \partial\Omega$

$$\Longrightarrow \quad [\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = \{u_h\}_{(1-\beta)} - \{u_h\}$$

- $\hat{\boldsymbol{\sigma}} = \{\boldsymbol{\sigma}_h\}_\beta - c\,|e|^{-1}[\![u_h]\!]$ on every edge

$$\Longrightarrow \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\boldsymbol{\sigma}_h\}_\beta - c\,|e|^{-1}[\![u_h]\!]$$

where, as before, $\{v\}_\beta = \beta v^+ + (1-\beta)v^-$, we obtain the LDG method of Cockburn-Shu.

CONVERGENCE PROPERTIES

We need boundedness, stability and consistency in a suitable norm.

Define: $V(h) = V_h + H^2(\Omega) \cap H^1_0(\Omega) \subset H^2(\mathscr{T}_h)$

with the norm:

$$|||v|||^2 = |v|^2_{1,h} + \Sigma_E h^2_E |v|^2_{2,h} + \sum_e ||\boldsymbol{r}_e([\![v]\!])||^2_{0,\Omega} \quad v \in V(h)$$

Note: $\sum_e ||\boldsymbol{r}_e([\![v]\!])||^2_{0,\Omega} \simeq \sum_e h_e^{-1} ||[\![v]\!]||^2_{0,e} \quad v \in V(h)$

- Boundedness $\quad B_h(v,w) \leq C_b |||v|||\,|||w||| \qquad \forall v, w \in V(h)$
- Stability $\quad B_h(v,v) \geq C_s |||v|||^2 \qquad \forall v \in V_h$
- Consistency $\quad B_h(u,v) = (f,v) \qquad \forall v \in V_h$
- Approximation $\quad |||u - u_I||| \leq Ch^k |u|_{k+1,\Omega}$

HINT OF ERROR ESTIMATES

We assume the following approximation property:

$$|u - u_I|_{s,E} \leq C h_E^{k+1-s} |u|_{k+1,E} \quad s = 0, 1, 2$$

Then

$$C_s |||u_I - u_h|||^2 \leq B_h(u_I - u_h, u_I - u_h) = B_h(u_I - u, u_I - u_h)$$

$$\leq C_b |||u_I - u_h||| \, |||u_I - u||| \leq C h^k |u|_{k+1,\Omega} |||u_I - u_h|||$$

For optimal L^2 estimates we need "adjoint consistency":

$$\psi \in H_0^1(\Omega) : B^* \psi = g, \text{ then } B_h^*(\psi, v) \equiv B_h(v, \psi) = (g, v) \ \forall v \in V(h)$$

Taking $g = u - u_h$ he have then

$$||u - u_h||_0^2 = B_h(u - u_h, \psi) = B_h(u - u_h, \psi - \psi_I) \leq C h |\psi|_{2,\Omega} |||u - u_h|||$$

ESTIMATES FOR BASSI-REBAY 1

$$\boldsymbol{\sigma} = -\nabla u \qquad \boldsymbol{\sigma}_h = -\nabla_h u_h - \boldsymbol{R}([\![u_h]\!])$$

we have

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{0,\Omega} \leq C h^k |u|_{k+1,\Omega}$$

$$\|P_{k-1}(u - u_h)\|_{0,\Omega} \leq C h^{k+1} |u|_{k+1,\Omega}$$

where P_{k-1} is the projection operator (element by element) on the space of polynomials of degree $\leq k-1$.

For more detailed estimates on specific methods or classes of methods see for instance Castillo-Cockburn-Perugia-Schötzau, Cockburn-Shu, Dutra do Carmo-Duarte, Heinrich, Houston-Schwab-Süli, Wheeler-Rivière-Girault (and many many others).