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Discontinuous finite elements for Reissner-Mindlin plates

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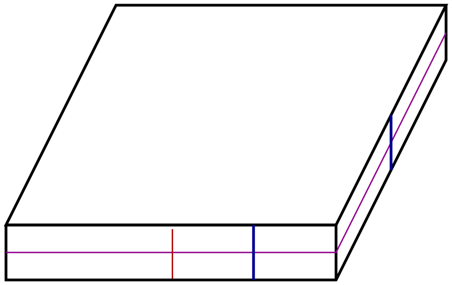
PLAN OF THE TALK

- The Plate Model and its scaling
- The Locking Phenomenon and its cure
- Some typical Reissner-Mindlin elements
- Discontinuous Galerkin Approach
- A new family of Reissner-Mindlin elements

BASIC REISSNER MINDLIN MODEL

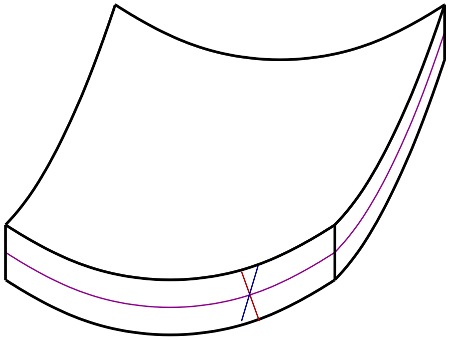


UNDEFORMED

MIDPLANE

FIBER

NORMAL TO MIDPLANE



DEFORMED

MIDPLANE

FIBER

NORMAL TO MIDPLANE

$w :=$ vertical displacement of (deformed) midplane

$\boldsymbol{\theta} :=$ angle between (deformed) fiber and vertical

$\nabla w \equiv \boldsymbol{\theta} \Leftrightarrow$ (deformed) fiber=normal to (deformed) midplane

REISSNER-MINDLIN EQUATIONS

Let f in, say, $L^2(\Omega)$ be the load per unit surface acting on the midplane, and let t be the thickness of the plate. The Reissner Mindlin equations require to find $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ such that

$$-t^3 \operatorname{div} C\, \varepsilon(\boldsymbol{\theta}) - t\boldsymbol{\gamma} = 0,$$

$$-t \operatorname{div} \boldsymbol{\gamma} = t\, f,$$

$$\boldsymbol{\gamma} = \lambda(\boldsymbol{\nabla} w - \boldsymbol{\theta}) \text{ in } \Omega,$$

C : tensor of bending moduli
$\boldsymbol{\theta}$: rotations
ε : usual symmetric gradient operator

$\boldsymbol{\gamma}$: shear stresses
w : transversal displacement
λ : shear correction factor

For simplicity, we take clamped boundary conditions:

$$\boldsymbol{\theta} = \mathbf{0}, \qquad w = 0.$$

The difficulties arise when $t << diam(\Omega)$.

SCALING THE REISSNER-MINDLIN EQUATIONS

The above equations correspond to the minimization of the functional

$$J^t(\boldsymbol{\theta}, w) = \underbrace{\frac{t^3}{2} a(\boldsymbol{\theta}, \boldsymbol{\theta})}_{\text{bending energy}} + \underbrace{\frac{t}{2} ||\nabla w - \boldsymbol{\theta}||^2_{0,\Omega}}_{\text{shear energy}} - \underbrace{t(f, w)}_{\text{ext. energy}} \, ,$$

under constraint of the kinematic boundary conditions, having set

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) := \int_\Omega C\, \varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\eta}) \mathrm{d}x$$

$(\cdot, \cdot)$ and $|| \cdot ||_{0,\Omega}$: inner-product and norm in $L^2(\Omega)$

To analyze the limit behavior as $t \to 0$ it is convenient to scale the load f to $f = t^2 g$. The scaled functional becomes:

$$J^t_s(\boldsymbol{\theta}, w) = \frac{1}{2} a(\boldsymbol{\theta}, \boldsymbol{\theta}) + \frac{t^{-2}}{2} ||\nabla w - \boldsymbol{\theta}||^2_{0,\Omega} - (g, w)$$

FROM REISSNER-MINDLIN TO KIRCHHOFF

We recall that we have to minimize the functional:

$$J_s^t(\boldsymbol{\eta}, v) = \frac{1}{2}a(\boldsymbol{\eta}, \boldsymbol{\eta}) + \frac{t^{-2}}{2}||\nabla v - \boldsymbol{\eta}||_{0,\Omega}^2 - (g, v).$$

Keeping g fixed, and letting $t \to 0$, the minimizing argument $(\boldsymbol{\theta}^t, w^t)$ of $J_s^t(\boldsymbol{\eta}, v)$ tends to a finite limit $(\boldsymbol{\theta}^0, w^0)$ such that

$$\boldsymbol{\theta}^0 = \nabla w^0,$$

and w^0 is the minimizing argument of

$$J^0(v) := \frac{1}{2}a(\nabla v, \nabla v) - (g, v)$$

with the boundary conditions $v = \partial v/\partial n = 0$ (that is, the solution of the Kirchhoff model).

THE LOCKING PHENOMENON

We take FE spaces $\Theta_h \subset (H^1_0(\Omega))^2$ and $W_h \subset H^1_0(\Omega)$, and consider

$$(\boldsymbol{\theta}^t_h, w^t_h) := arg\, min\, \{J^t_s(\boldsymbol{\eta}_h, v_h)\} \text{ over } \Theta_h \times W_h.$$

Q: will the limit $(\boldsymbol{\theta}^0_h, w^0_h)$, for $t \to 0$, be close to $(\boldsymbol{\theta}^0, w^0)$ for small h? Indeed, if this is *not* the case, then the convergence (in h) of $(\boldsymbol{\theta}^t_h, w^t_h)$ to $(\boldsymbol{\theta}^t, w^t)$ cannot be *uniform in t*, and we are bound for troubles when $t << diam(\Omega)$.

On the other hand, it is clear that we must have

$$(*) \qquad\qquad \boldsymbol{\theta}^0_h = \nabla w^0_h.$$

For simple-minded discretizations, it can occur that the set of pairs $(\boldsymbol{\theta}_h, w_h) \in \Theta_h \times W_h$ satisfying $(*)$ is very small. For instance, if both Θ_h and W_h are made of piecewise linear continuous functions, then $(*)$ implies $\boldsymbol{\theta}^0_h = \nabla w^0_h = 0$. This is the *locking phenomenon*.

The locking phenomenon

$$\begin{tikzcd}
(\theta_h^t, w_h^t) \arrow[r, "h \to 0"] \arrow[d, "t \to 0"'] & (\theta^t, w^t) \arrow[d, "t \to 0"] \\
(\theta_h^0, w_h^0) \arrow[r, "h \to 0"] & (\theta^0, w^0)
\end{tikzcd}$$

$$\theta_h^0 = \nabla w_h^0 \qquad\qquad \theta^0 = \nabla w^0$$

If the set $\{\theta_h^0 = \nabla w_h^0\}$ is too small

the convergence cannot be uniform in t

A COMMON REMEDY

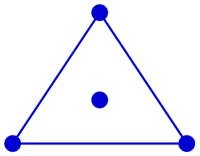
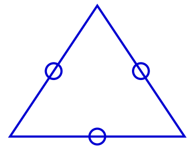
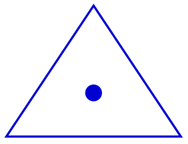
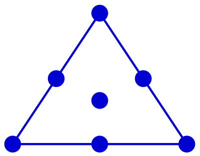
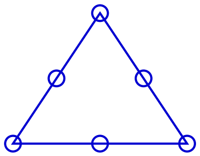
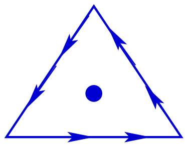
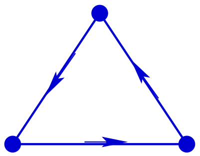
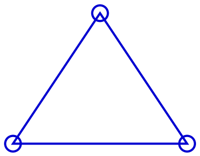
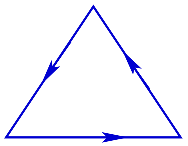
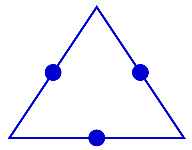
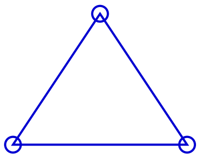
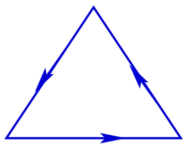
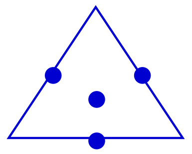
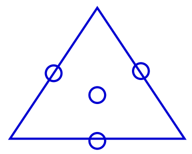
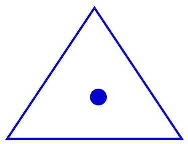
In order to avoid locking, a typical remedy is to change J^t into

$$J^t_h(\boldsymbol{\eta}, v) := \frac{1}{2} a(\boldsymbol{\eta}, \boldsymbol{\eta}) + \frac{t^{-2}}{2} ||P_h(\nabla v - \boldsymbol{\eta})||_{0,\Omega} - (g, v),$$

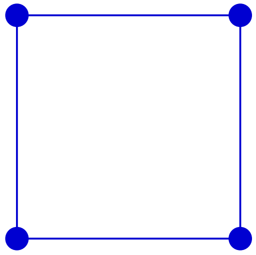
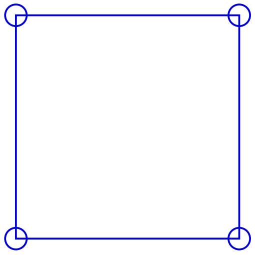
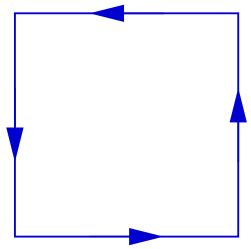
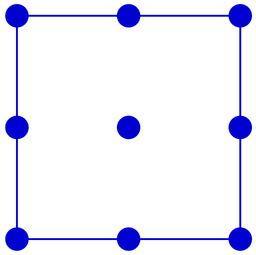
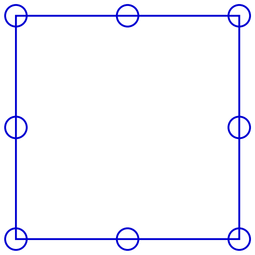
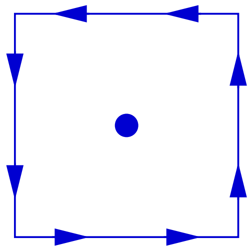
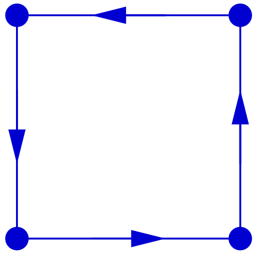
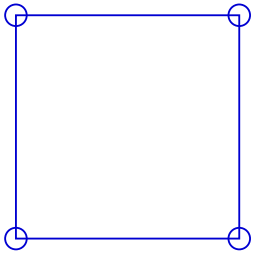
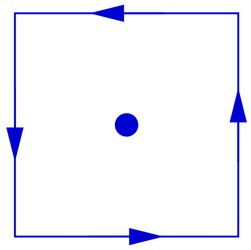
where P_h is a suitable projection operator (in general, on some lower degree polynomials). In the Engineering practice, the *reduction* corresponding to the use of P_h is actually realized by using a *reduced integration formula* in the shear term.

However, if (by bad luck) the space W_h contains functions v_h such that $||P_h(\nabla v_h)|| << ||\nabla v_h||$ (for $h \rightarrow 0$) then the method loses stability, as these modes can be enormously amplified in the discrete solution (*hourglass modes*). In particular, if, for some $v_h \in W_h$, different from 0, we have $P_h(\nabla v_h) = 0$, then the linear system giving the stationarity of J^t_h becomes singular.

Some elements for Reissner-Mindlin

$\boldsymbol{\theta}$	w	$\boldsymbol{\gamma}$	Reference
			Arnold−Falk
			MITC7 Brezzi−Bathe−Fortin
			Duran−Liberman
			Oñate et als
			Brezzi−Marini

Some elements for Reissner-Mindlin

$\boldsymbol{\theta}$	w	$\boldsymbol{\gamma}$	Reference
			T1 Hughes−Tezduyar 81 MITC4 Bathe−Dvorkin 85
			MITC9 Bathe−Brezzi−Fortin 89
			Duran−Liberman et als 02

DISCONTINUOUS GALERKIN APPROXIMATION

Let $\mathcal{T}_h$ be a decomposition of Ω in triangles

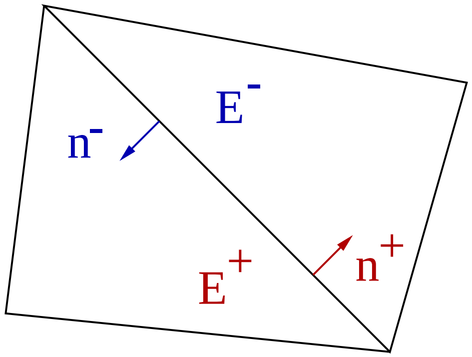
Let $\mathcal{E}_h$ be the set of all the edges

and let $\mathcal{E}_h'$ be the set of the internal edges.

We need to define jumps and averages at the boundary of each element for discontinuous functions (and vectors) the

AVERAGES AND JUMPS

(Reminder: DG treatment of Laplace operator)



Definition of average and jump on an internal edge:

$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [v] = v^+\mathbf{n}^+ + v^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h'$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \qquad [\boldsymbol{\tau}] = \boldsymbol{\tau}^+\mathbf{n}^+ + \boldsymbol{\tau}^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h'$$

On the boundary edges: $[v] = v\mathbf{n}; \qquad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}$

DG TREATMENT OF LAPLACE OPERATOR (IP)

(Reminder: DG treatment of Laplace operator)

Take the equation $-\Delta u = f$, multiply it by a piecewise polynomial ("discontinuous") function v, and integrate by parts. We obtain:

$$\sum_T \int_T \nabla u \cdot \nabla v - \sum_e \int_e [v] \cdot \{\nabla u\} = \int_\Omega f v.$$

Then, taking into account the fact that, for the solution u, one has clearly $[u] = 0$, one adds a term to restore symmetry;

$$\sum_T \int_T \nabla u \cdot \nabla v - \sum_e \int_e [v] \cdot \{\nabla u\} - \sum_e \int_e [u] \cdot \{\nabla v\} = \int_\Omega f v.$$

An additional term penalizing the jumps is then added to stabilize:

$$p(u, v) = \sum_e \frac{\kappa}{|e|} \int_e [u] \cdot [v].$$

DISCONTINUOUS APPROXIMATIONS OF R-M EQUATIONS

When using discontinuous approximations for Reissner-Mindlin, analogous modifications have to be introduced on each equation!

Example: Bilinear form associated with the bending energy:

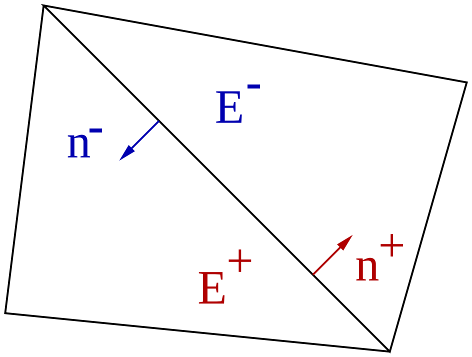
$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) := (C\,\varepsilon(\boldsymbol{\theta}), \varepsilon(\boldsymbol{\eta}))$$

$\mathcal{T}_h$: decomposition in triangles, $\mathcal{E}_h$: edges.

Discrete bilinear form (discontinuous trial and test functions):

$$\begin{aligned} a_h(\boldsymbol{\theta}, \boldsymbol{\eta}) = & \sum_{T \in \mathcal{T}_h} (C\varepsilon(\boldsymbol{\theta}), \varepsilon(\boldsymbol{\eta}))_T \\ & + \sum_{e \in \mathcal{E}_h} \int_e (-\{C\varepsilon_h(\boldsymbol{\theta})\} : [\boldsymbol{\eta}] - \{C\varepsilon_h(\boldsymbol{\eta})\} : [\boldsymbol{\theta}])\, ds \\ & + p_\Theta([\boldsymbol{\theta}], [\boldsymbol{\eta}]). \end{aligned}$$

AVERAGES AND JUMPS



Definition of average and jump on an internal edge:

$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [v] = v^+\mathbf{n}^+ + v^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h'$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \qquad [\boldsymbol{\tau}] = (\boldsymbol{\tau}^+ \otimes \mathbf{n}^+)_S + (\boldsymbol{\tau}^- \otimes \mathbf{n}^-)_S \quad \forall e \in \mathcal{E}_h'$$

On the boundary edges: $[v] = v\mathbf{n}; \qquad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}$

Examples of stabilizing (penalty) terms:

$$p_\Theta([\boldsymbol{\theta}],[\boldsymbol{\eta}]) = \sum_{e\in\mathcal{E}_h}\int_e \frac{\kappa^\Theta}{|e|}[\boldsymbol{\theta}]:[\boldsymbol{\eta}]ds,$$

with $\kappa^\Theta > 0$ (to be chosen), or

$$p_\Theta([\boldsymbol{\theta}],[\boldsymbol{\eta}]) = \sum_{e\in\mathcal{E}_h}\int_e \frac{\kappa^\Theta}{|e|}[Q_e^r\boldsymbol{\theta}]:[Q_e^r\boldsymbol{\eta}]ds$$

$r \geq 0$, $Q_e^r =$ projection edge by edge on $P_r(e)$.

For instance, for $r = 0$ this choice penalizes the jump of the meanvalues only.

DISCRETE FORMULATION (discontinuous trial and test functions)

$\mathcal{T}_h$: decomposition in triangles,

$\Theta_h,\ W_h,\ \Gamma_h$: piecewise polynomial spaces

$$a_T(\boldsymbol{\theta}_h,\boldsymbol{\eta}) := (C\,\varepsilon(\boldsymbol{\theta}_h),\varepsilon(\boldsymbol{\eta}))_T.$$

Find $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h) \in \Theta_h \times W_h \times \Gamma_h$:

$$\sum_T a_T(\boldsymbol{\theta}_h,\boldsymbol{\eta}) + c_\Theta(\boldsymbol{\theta}_h,\boldsymbol{\eta}) + p_\Theta(\boldsymbol{\theta}_h,\boldsymbol{\eta}) - (\boldsymbol{\gamma}_h, P_h\boldsymbol{\eta}) = 0 \qquad \boldsymbol{\eta}\in\Theta_h$$

$$\sum_T (\boldsymbol{\gamma}_h,\boldsymbol{\nabla} v)_T + c_W(\boldsymbol{\gamma}_h, v) + p_W(w_h, v) = (g, v) \qquad v\in W_h$$

$$t^2(\boldsymbol{\gamma}_h,\boldsymbol{\tau}) - \sum_T(\boldsymbol{\nabla} w_h,\boldsymbol{\tau})_T - c_W(\boldsymbol{\tau}, w_h) + (P_h\boldsymbol{\theta}_h,\boldsymbol{\tau}) = 0 \qquad \boldsymbol{\tau}\in\Gamma_h$$

where $c_\Theta,\ c_W$ are consistency terms,

and $p_\Theta,\ p_W$ are penalty terms.

$$\boxed{\text{Def.: } a_h(\boldsymbol{\theta},\boldsymbol{\eta}) = \textstyle\sum_T a_T(\boldsymbol{\theta},\boldsymbol{\eta}) + c_\Theta(\boldsymbol{\theta},\boldsymbol{\eta}) + p_\Theta(\boldsymbol{\theta},\boldsymbol{\eta})}$$

Note: if $\boldsymbol{\theta} \in H^2(\Omega)$ then $(-\operatorname{div} C\,\varepsilon(\boldsymbol{\theta}),\boldsymbol{\eta}) = a_h(\boldsymbol{\theta},\boldsymbol{\eta}) \quad \forall \boldsymbol{\eta} \in H^1(\mathcal{T}_h)$ regardless of the choice of κ^Θ and r.

$\Big(Eq.\ (1) \qquad -\operatorname{div} C\,\varepsilon(\boldsymbol{\theta}) - \boldsymbol{\gamma} = 0\Big)$

Multiplying by $\boldsymbol{\eta} \in H^1(\mathcal{T}_h)$, and integrating over Ω

$$a_h(\boldsymbol{\theta},\boldsymbol{\eta}) - (\boldsymbol{\gamma},\boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in H^1(\mathcal{T}_h)$$

$$\Downarrow$$

$$a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\eta}) - (\boldsymbol{\gamma},\boldsymbol{\eta}) + (\boldsymbol{\gamma}_h, P_h\boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in H^1(\mathcal{T}_h)$$

$$\boxed{c_W(\boldsymbol{\tau}, v) := \sum_{e\in\mathcal{E}_h} \int_e \{\boldsymbol{\tau}\}\cdot[v]\,ds \quad \boldsymbol{\tau}\in H^1(\mathcal{T}_h),\ v\in H^1(\mathcal{T}_h)}$$

$\big(Eq.\ (2) \qquad -\operatorname{div}\boldsymbol{\gamma} = g\big)$

Multiplying by $v \in H^1(\mathcal{T}_h)$ and integrating by parts

$$\begin{aligned}(\boldsymbol{\gamma}, \boldsymbol{\nabla}_h v) &= (g, v) + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\boldsymbol{\gamma}\cdot n_T\, v\, ds,\\ &= (g, v) + c_W(\boldsymbol{\gamma}, v) \quad v \in H^1(\mathcal{T}_h).\end{aligned}$$

Hence,

$$(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h, \boldsymbol{\nabla}_h v) - c_W(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h, v) + p_W(w - w_h, v) = 0 \quad v \in H^1(\mathcal{T}_h)$$

$$c_W(\boldsymbol{\tau}, v) := \sum_{e\in\mathcal{E}_h} \int_e \{\boldsymbol{\tau}\} \cdot [v]\, ds \quad \boldsymbol{\tau} \in H^1(\mathcal{T}_h),\ v \in H^1(\mathcal{T}_h)$$

$\big(Eq.\ (3) \quad t^2\boldsymbol{\gamma} = \boldsymbol{\nabla} w - \boldsymbol{\theta}\big)$

Multiplying by $\boldsymbol{\tau} \in H^1(\mathcal{T}_h)$ we have

$$t^2(\boldsymbol{\gamma}, \boldsymbol{\tau}) - \sum_T (\boldsymbol{\nabla} w, \boldsymbol{\tau})_T + (\boldsymbol{\theta}, \boldsymbol{\tau}) = 0 \quad v \in H^1(\mathcal{T}_h).$$

Hence, since $c_W(\boldsymbol{\tau}, w) = 0$

$$t^2(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\tau}) - c_W(\boldsymbol{\tau}, w - w_h) - \sum_T (\boldsymbol{\nabla}(w - w_h), \boldsymbol{\tau})_T + (\boldsymbol{\theta} - P_h\boldsymbol{\theta}_h, \boldsymbol{\tau}) = 0$$

A NEW CHOICE (Arnold-Brezzi-M)

We consider the following family of elements (k odd ≥ 1):

$$\mathbf{\Theta}_h = \boldsymbol{\mathcal{L}}_k^0(\mathcal{T}_h), \quad W_h = \mathcal{L}_k^0(\mathcal{T}_h), \quad \mathbf{\Gamma}_h = \boldsymbol{\mathcal{L}}_{k-1}^0(\mathcal{T}_h).$$

The choice for the interior penalty term p_Θ is

$$p_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}) = \sum_{e \in \mathcal{E}_h} \frac{\kappa^\Theta}{|e|} \int_e [\boldsymbol{\theta}] : [\boldsymbol{\eta}] \, ds,$$

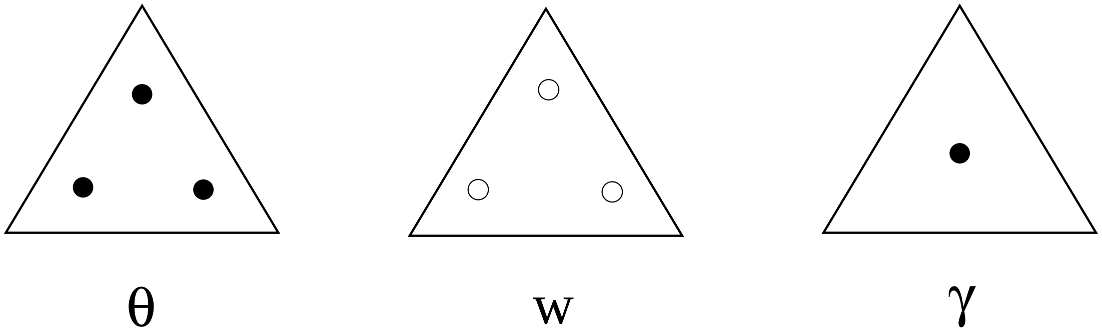
while for p_W we use a weaker penalization:

$$p_W(w, v) = \sum_{e \in \mathcal{E}_h} \frac{\kappa^W}{|e|} \int_e \boldsymbol{Q}_e[w] \cdot \boldsymbol{Q}_e[v] \, ds.$$

The parameters κ^Θ and κ^W are positive constants to be chosen; they must be sufficiently large to ensure stability.

THE SIMPLEST CHOICE: LINEAR ELEMENTS

It has to be noted that, for the simplest case $k = 1$, we can use linear elements for both rotations and transversal displacements and *we get away with that*.



Clearly, for the lowest order case, the reduction operator $\boldsymbol{P}_h$ is simply the L^2 projection onto the piecewise constant space $\mathcal{L}^0_0(\mathcal{T}_h)$ (that, for linear elements, is just the value at the barycenter). For $k > 1$ the choice is more complicated and will not be detailed here.

ERROR ESTIMATES

Definition of norms:

$$\begin{aligned}
|||\boldsymbol{\eta}|||_{\Theta}^{2} &:= \|\boldsymbol{\eta}\|_{1,h}^{2} + \textstyle\sum_{e} \left(\frac{1}{|e|} \|[\boldsymbol{\eta}]\|_{0,e}^{2} + |e| \, \|\{(C\,\varepsilon(\boldsymbol{\eta})\}\|_{0,e}^{2} \right) \\
|||v|||_{W}^{2} &:= |v|_{1,h}^{2} + \textstyle\sum_{e} \frac{1}{|e|} \|\boldsymbol{Q}_{e}[v]\|_{0,e}^{2} \\
|||\boldsymbol{\tau}|||_{\Gamma}^{2} &:= \|\boldsymbol{\tau}\|_{0}^{2} + \textstyle\sum_{e} |e| \|\{\boldsymbol{\tau}\}\|_{0,e}^{2}
\end{aligned}$$

We have the following result (for odd $k \geq 1$)

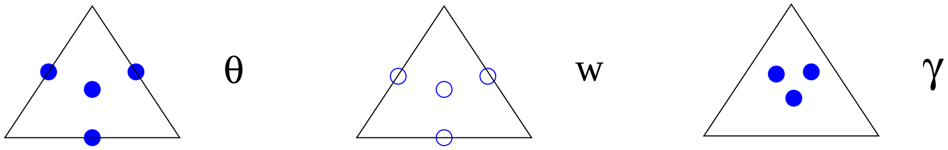
$$\begin{aligned}
|||\boldsymbol{\theta} - \boldsymbol{\theta}_{h}|||_{\Theta} + \quad & |||w - w_{h}|||_{W} + t|||\boldsymbol{\gamma} - \boldsymbol{\gamma}_{h}|||_{\Gamma} \\
& \leq C\,h^{k} \left(\|\boldsymbol{\theta}\|_{k+1,\Omega} + \|w\|_{k+1,\Omega} + t\|\boldsymbol{\gamma}\|_{k,\Omega} + \|\boldsymbol{\gamma}\|_{k-1,\Omega}\right).
\end{aligned}$$

An alternative nonconforming ”odd element”

$$P_\theta(T) \ = \ (P_1^{NC}(T))^2 \bigoplus (b_2^{NC}(T))^2 \qquad \text{8 dof}$$

$$P_w(T) \ = \ P_1^{NC}(T) \bigoplus b_2^{NC}(T) \qquad \text{4 dof}$$

$$P_\gamma(T) \ = \ (P_0(T))^2 \bigoplus \boldsymbol{\nabla} b_2^{NC}(T) \qquad \text{3 dof}$$



Same nodes for $\boldsymbol{\theta}$ and w. Convergence: $O(h)$.

CONCLUSIONS

- For Reissner Mindlin plates, the *Locking phenomenon* forbids the use of simple minded elements. However, several methods are now on the market that allow to overcome the difficulty.
- DG methods are nowadays very fashionable. Their interest in structural problems, as plate and shell problems, *has still to be proved in practice*. But, for instance, they allow the use of very simple finite elements as *linear-linear-constant* elements that might have some practical interest.
- The use of DG techniques might be useful for deriving new elements; for instance, the *linear-linear-nonconforming* element was found using typical DG instruments.