3. SOME SOLUTIONS

1) Determine the first four approximations of the actual root of the equation $x^3 - \frac{1}{2} = 0$ in the interval $[0,1]$ by the bisection method.

Solution - We have

$$a_0 = 0, \quad b_0 = 1, \quad f(a_0) = -\frac{1}{2}, \quad f(b_0) = \frac{1}{2}, \quad \frac{a_0+b_0}{2} = \frac{1}{2}, \quad f(\frac{a_0+b_0}{2}) = -\frac{3}{8},$$

$$a_1 = \frac{1}{2}, \quad b_1 = 1, \quad \frac{a_1+b_1}{2} = \frac{3}{4}, \quad f(\frac{a_1+b_1}{2}) = f(\frac{3}{4}) = -\frac{5}{64},$$

$$a_2 = \frac{3}{4}, \quad b_2 = 1, \quad \frac{a_2+b_2}{2} = \frac{7}{8}, \quad f(\frac{a_2+b_2}{2}) = f(\frac{7}{8}) = \frac{87}{512},$$

$$a_3 = \frac{3}{4}, \quad b_3 = \frac{7}{8}, \quad \frac{a_3+b_3}{2} = \frac{13}{16}, \quad f(\frac{a_3+b_3}{2}) = f(\frac{13}{16}) = \frac{149}{409}$$

$$a_4 = \frac{3}{4}, \quad b_4 = \frac{13}{16}, \ldots.$$

2) Apply Newton's method with $x_0 = 0.8$ and the secant method with $x_{-1} = 0.8$ and $x_0 = 1.2$ to the equation $x^3 - x^2 - x + 1 = 0$. Evaluate the first three approximations of the actual root.

Solution - We have to apply

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad \text{Newton},$$

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}, \qquad \text{Secant},$$

with $f(x) = x^3 - x^2 - x + 1$ and $f'(x) = 3x^2 - 2x - 1$. Notice that the equation $f(x) = 0$ has roots $x = 1$ (double, see next exercise) and $x = -1$ (simple). If we apply Newton's method we have

$$x_1 = 0.8 + \frac{0.072}{0.68} = 0.906,$$

$$x_2 = 0.906 + \frac{0.0168}{0.349} = 0.954,$$

$$x_3 = 0.954 + \frac{0.00413}{0.178} = 0.977,$$

whereas with the secant method we obtain

$$x_1 = 1.2 - \frac{1.2 - 0.8}{0.088 - 0.072} 0.088 = -1$$

and from here on the method produces always the same value, which is an actual root (but not the one we were supposedly looking for!).

3) In the previous example apply the modified Newton's method with $p = 2$ (i.e. $x_{k+1} = x_k - p\frac{f(x_k)}{f'(x_k)}$) and show that even after only three approximations the performance improved.

Solution - If we apply the modified Newton's method (in order to take into account the molteplicity of the root $x = 1$), we have

$$x_1 = 0.8 + 2\frac{0.072}{0.068} = 1.01176,$$

$$x_2 = 1.01176 - 2\frac{0.000278}{0.0474} = 0.988,$$

$$x_3 = 0.988 + 2\frac{0.000286}{0.0475} = 1.000042,$$

and the improvement with respect to the previous three approximations is evident.

4) The algebraic equation

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0$$

has a root at $\bar{x} = 1.368808107$. Put the previous equation in the equivalent form

$$x = \frac{20}{x^2 + 2x + 10},$$

and apply the iterative method $x_{n+1} = g(x_n)$ with the initial value $x_0 = 1$ to determine the first three approximations of $\bar{x}$.

Solution - Notice that

$$g'(x) = -\frac{20(2x+2)}{(x^2 + 2x + 10)^2}$$

and in the interval $[1, 2]$ is rather easy to see that $|g'(x)| \leq \frac{120}{169} < 1$. This ensures that we are dealing with a contractive mapping and the procedure converges to the actual root. Indeed we have

$$x_1 = \frac{20}{1 + 2 + 10} = \frac{20}{13} = 1.538,$$

$$x_2 = \frac{20}{(1.538)^2 + 2(1.538) + 10} = \frac{20}{2.365 + 3.076 + 10} = 1.295,$$

$$x_3 = \frac{20}{(1.295)^2 + 2(1.295) + 10} = \frac{20}{1.677 + 2.59 + 10} = 1.402,$$

$$x_4 = \frac{20}{(1.402)^2 + 2(1.402) + 10} = \frac{20}{1.966 + 2.804 + 10} = 1.354.$$

5) Apply the secant method to determine the first four approximations of the positive root of $\sin x - \frac{x}{2} = 0$ with $x_0 = \frac{\pi}{2}$ and $x_1 = \pi$.

Solution - We need to apply the formula

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$$

to obtain

$$x_2 = \pi - \frac{\pi - \frac{\pi}{2}}{f(\pi) - f(\frac{\pi}{2})} f(\pi) = \pi - \frac{\frac{\pi}{2}}{-\frac{\pi}{2} - (1 - \frac{\pi}{4})}(-\frac{\pi}{2}) = \pi - \frac{\frac{\pi^2}{4}}{1 + \frac{\pi}{4}},$$

$$x_3 = \pi - \frac{\frac{\pi^2}{4}}{1 + \frac{\pi}{4}} - \frac{\pi - \frac{\frac{\pi^2}{4}}{1+\frac{\pi}{4}} - \pi}{f(\pi - \frac{\frac{\pi^2}{4}}{1+\frac{\pi}{4}}) - f(\pi)} f(\pi - \frac{\frac{\pi^2}{4}}{1 + \frac{\pi}{4}})$$

6) Evaluate $\int_0^{\frac{\pi}{2}} \sin x \, dx$ using midpoint, trapezoidal and Simpson's rule with $n = 4$. Compare the performances of the different methods.

Solution - Let us start with the midpoint rule. We have $H = \dfrac{\pi}{8}$, $x_k = \frac{\pi}{16}(2k+1)$ with $k = 0, 1, 2, 3$. Hence

$$\begin{aligned} I &= \frac{\pi}{8}[f(x_0) + f(x_1) + f(x_2) + f(x_3)] = \\ &= \frac{\pi}{8}[f(\frac{\pi}{16}) + f(\frac{3\pi}{16}) + f(\frac{5\pi}{16}) + f(\frac{7\pi}{16})]. \end{aligned}$$

Let us now come to the trapezoid rule. Here we have $x_k = k\,\dfrac{\pi}{8}$ with $k = 0, 1, 2, 3, 4$ and we obtain

$$\begin{aligned} I &= \frac{\pi}{8}[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4)] = \\ &= \frac{\pi}{8}[\frac{1}{2}f(0) + f(\frac{\pi}{8}) + f(\frac{\pi}{4}) + f(\frac{3\pi}{8}) + \frac{1}{2}f(\frac{\pi}{2})]. \end{aligned}$$

We conclude with Simpson's rule. In this case $H = \dfrac{\pi}{8}$ and $x_k = k\,\dfrac{\pi}{16}$ with $k = 0, 1, \ldots, 8$. We have

$$I = \frac{\pi}{48}[f(x_0) + 2(f(x_2) + f(x_4) + f(x_6)) + 4(f(x_1) + f(x_3) + f(x_5) + f(x_7)) + f(x_8)] =$$

$$= \frac{\pi}{48}[f(0) + 2(f(\frac{\pi}{8}) + f(\frac{\pi}{4}) + f(\frac{3\pi}{8})) + 4(f(\frac{\pi}{16}) + f(\frac{3\pi}{16}) + f(\frac{5\pi}{16}) + f(\frac{7\pi}{16})) + f(\frac{\pi}{2})].$$

7) Calculate $\int_1^{1.3} \sqrt{x}\, dx$ with the trapezoidal rule, knowing that

$$\{x_0 = 1,\ x_1 = 1.05,\ x_2 = 1.1,\ x_3 = 1.15,\ x_4 = 1.2,\ x_5 = 1.25,\ x_6 = 1.3\}$$

and

$$\{f(x_0) = 1,\ f(x_1) = 1.02470,\ f(x_2) = 1.04881,\ f(x_3) = 1.07238,$$

$$f(x_4) = 1.09544,\ f(x_5) = 1.11803,\ f(x_6) = 1.14017\}.$$

Compare with the correct value.

Solution - First of all notice that

$$\int_1^{1.3} \sqrt{x}\, dx = \frac{2}{3}[x^{3/2}]_1^{1.3} = 0.32148.$$

Coming to the trapezoidal rule, we have $H = 0.05$ and $x_k = 1 + 0.05k$ with $k = 0, 1, \dots, 6$. Hence

$$I = 0.05[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + \frac{1}{2}f(x_6)] =$$

$$= 0.05[\frac{1}{2} + 1.02470 + 1.04881 + 1.07238 + 1.09544 + 1.11803 + \frac{1.14017}{2}] = 0.32147.$$

8) Evaluate the same integral of the previous exercise, using Simpson's rule.

Solution - In this case $H = 0.1$ and $x_k = 1 + 0.05k$ with $k = 0, 1, \dots, 6$. We obtain

$$I = \frac{0.1}{6}[f(x_0) + 2(f(x_2) + f(x_4)) + 4(f(x_1) + f(x_3) + f(x_5)) + f(x_6)] =$$

$$= \frac{0.1}{6}[1 + 2(1.04881 + 1.09544) + 4(1.02470 + 1.07238 + 1.11803) + 1.14017] = 0.32148.$$

9) Let

$$\mathbf{f} = \{f_n\} = \{0, 1, 2, 1\}.$$

Evaluate its DFT $\mathbf{F} = \{F_k\}$ using the definition and check that the inversion formula actually reconstructs $\mathbf{f}$ from $\mathbf{F}$.

Solution - In this case we have

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 2, \quad f_3 = 1.$$

We obtain

$$F_0 = f_0 + f_1 + f_2 + f_3 = 4,$$

$$F_1 = (f_0 - f_2) - i(f_1 - f_3) = -2,$$

$$F_2 = (f_0 + f_2) - (f_1 + f_3) = 0,$$

$$F_3 = (f_0 - f_2) + i(f_1 - f_3) = -2.$$

Coming now to the inversion formula, we have

$$f_n = \frac{1}{N}\sum_{k=0}^{N-1} F_k W_N^{-kn}$$

where $W_N = e^{-\frac{2\pi i}{N}}$, which in our instance becomes $W_4 = e^{-\frac{\pi}{2}i} = -i\sin\frac{\pi}{2} = -i$. If we now apply the inversion formula, we obtain

$$f_n = \frac{1}{4}\sum_{k=0}^{3} F_k(-i)^{-kn}$$

and we obtain

$$\begin{aligned} f_0 &= \frac{1}{4}[F_0 + F_1 + F_2 + F_3] = 0, \\ f_1 &= \frac{1}{4}[F_0 + F_1(-i)^{-1} + F_3(-i)^{-3}] = 1, \\ f_2 &= \frac{1}{4}[F_0 + F_1(-i)^{-2} + F_3(-i)^{-6}] = 2, \\ f_3 &= \frac{1}{4}[F_0 + F_1(-i)^{-3} + F_3(-i)^{-9}] = 1. \end{aligned}$$