Elliptic and parabolic second-order PDEs with growing coefficients

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Consider a second order (non-degenerate) parabolic operator L,

$$Lu(t,x) = \sum_{i,j=1}^{d} a^{ij}(t,x)u_{x^i x^j}(t,x)$$

$$+\sum_{i=1}^{d} b^i(t,x)u_{x^i}(t,x) - \lambda u(t,x),$$

with $\lambda > 0$, acting on functions defined on $[T,+\infty)\times \mathbb{R}^d$ if $T \in (-\infty,+\infty)$ and on $\mathbb{R}^{d+1}$ if $T = -\infty$,

$$\mathbb{R}^{d+1} = \{(t,x) : t \in \mathbb{R}, x = (x^1, ..., x^d) \in \mathbb{R}^d\}.$$

We study the equation

$$u_t(t,x) + Lu(t,x) = f(t,x), \qquad (1)$$

$$(t,x) \in (T,+\infty) \times \mathbb{R}^d.$$

The main result are global Schauder estimates for the bounded solution u like

$$\sup_{t\geq T} \|u(t,\cdot)\|_{2+\alpha} \leq N \sup_{t\geq T} \|f(t,\cdot)\|_{\alpha}, \qquad (2)$$

assuming "measurability of the coefficients in time" and α-Hölder continuity in the space variable,

We also do not assume the boundeness of b.

Schauder estimates of the form (2) for the heat equation were first proved in [Brandt 1969] (see also [Knerr 1980]).

Hypothesis 1 *(i) The matrix* $(a^{ij}(t,x))$ *is symmetric and there exist* $K \in (0,\infty)$ *such that*

$$K|\xi|^2 \geq \sum_{i,j=1}^{d} a^{ij}(t,x)\xi_i\xi_j \geq \frac{1}{K}|\xi|^2,$$

$$\text{for } (t,x) \in \mathbb{R}^{d+1},\ \xi \in \mathbb{R}^d;$$

(ii) a^{ij}, b^i *and* f *are measurable functions on* $\mathbb{R}^{d+1}$*; b is also locally bounded;*

(iii) For some $\alpha \in (0,1)$,

$$|a^{ij}(t,x)-a^{ij}(t,y)|+|b^i(t,x)-b^i(t,y)| \leq K|x-y|^\alpha,$$

for all $t \in \mathbb{R}$, $i,j = 1,...,d$, *and* $x,y \in \mathbb{R}^d$ *such that*

$$|x-y| \leq 1. \tag{3}$$

(iii) there exist positive constants F_0 *and* F_α *such that*

$$|f(t,x)| \leq F_0 \tag{4}$$

$$|f(t,x)-f(t,y)| \leq F_\alpha |x-y|^\alpha,$$

whenever $t \in \mathbb{R}$, $x,y \in \mathbb{R}^d$.

Remark In the paper we also consider the general case in which in the expression of L, $\lambda u(t,x)$ is replaced by

$$c(t,x)u(t,x),$$

where $c(t,x) \geq \lambda > 0$, $(t,x) \in \mathbb{R}^{d+1}$, but c can be unbounded from above, i.e.,

$$|c(t,x)-c(t,y)| \leq K|x-y|^\alpha, \quad t \in \mathbb{R}, \;\; |x-y| \leq 1;$$

but then we assume

$$|f(t,x)| \leq F_0 c(t,x), \quad t \in \mathbb{R}, \; x \in \mathbb{R}^d.$$

Notation

- $C^\alpha = C^\alpha(\mathbb{R}^d)$, $\alpha \in (0,1)$, is the Banach space of real functions g on $\mathbb{R}^d$ with finite norm

$$\|g\|_\alpha = \|g\|_0 + [g]_\alpha,$$

where

$$\|g\|_0 = \sup_{x \in \mathbb{R}^d} |g(x)|, \quad [g]_\alpha = \sup_{x \neq y} \frac{|g(x)-g(y)|}{|x-y|^\alpha},$$

- $C^{2+\alpha} = C^{2+\alpha}(\mathbb{R}^d)$ is the Banach space of real functions g on $\mathbb{R}^d$ with finite norm

$$\boxed{\|g\|_{2+\alpha} = \|g\|_0 + \|Dg\|_0 + \|D^2 g\|_0 + [D^2 g]_\alpha,}$$

$Dg = (g_{x^1}, ..., g_{x^d})$, $D^2 g = (g_{x^i x^j}; \, i, j = 1, ..., d)$,

For any $T \in (-\infty, +\infty)$, we define

$$\mathbb{R}^{d+1}_T = [T, \infty) \times \mathbb{R}^d,$$

and, if $T = -\infty$, we set $\mathbb{R}^{d+1}_T = \mathbb{R}^{d+1}$.

Definition of solution

Let $C^{2+\alpha}(T)$ be the set of bounded functions $u(t, x)$ defined on $\mathbb{R}^{d+1}_T$ such that

(i) u is continuous in $\mathbb{R}_T^{d+1}$;

(ii) for finite $t \geq T$, we have $u(t,\cdot) \in C^{2+\alpha}$ and $\|u(t,\cdot)\|_{2+\alpha}$ is bounded in t;

(iii) there exists a locally bounded function $g(t,x)$ on $\mathbb{R}_T^{d+1}$ such that, for $x \in \mathbb{R}^d$ and any finite $t > s \geq T$,

$$u(t,x) - u(s,x) = \int_s^t g(r,x)\,dr. \qquad (5)$$

For such a function u we write $u_t = g$.

A solution to our equation (1) is a mapping $u \in \mathcal{C}^{2+\alpha}(T)$ such that, for any $x \in \mathbb{R}^d$ and finite $t > s$ with $s \geq T$, we have

$$u(t,x) - u(s,x) = \int_s^t f(r,x)\,dr - \int_s^t Lu(r,x)\,dr. \qquad (6)$$

Theorem 2 *(a priori estimates) Let* $u \in \mathcal{C}^{2+\alpha}(T)$ *satisfying* (1). *Then:*

(i) There is a constant $N = N(\lambda, \alpha, d, K)$ *such that, for all finite* $t \geq T$*, we have*

$$\|u(t, \cdot)\|_{2+\alpha} \leq N(F_0 + F_\alpha). \tag{7}$$

(ii) (elliptic case) If f *and the coefficients of* L *are independent of* t*, then, for any solution* $u \in C^{2+\alpha}$,

$$\|u\|_{2+\alpha} \leq N(F_0 + F_\alpha).$$

Theorem 3 *(i) There exists a unique* $u \in \mathcal{C}^{2+\alpha}(T)$ *which satisfies* (1).

(ii) (elliptic case) If coefficients of L *and* f *are independent of* t *then there exists a unique* $u \in C^{2+\alpha}$ *satisfying* $Lu = f$.

The proof of previous theorem uses the following new maximum principle:

Theorem 4 *If* $u \in \mathcal{C}^{2+\alpha}(T)$ *is a solution to* (1). *Then, for any finite* $t \geq T$, $x \in \mathbb{R}^d$,

$$|u(t,x)| \leq \frac{F_0}{\lambda}.$$

Remark In the paper we also prove Schauder estimates for Cauchy problems associated with L. These are consequences of the previous theorems.

Remark In the special case of f and the coefficients of L independent of t, we get the estimate

$$\|u\|_{2+\alpha} \leq N\|f\|_{\alpha}.$$

This shows that the Schauder constant in the original (elliptic) Schauder estimates involving bounded coefficients (see [Gilbarg-Trudinger 1983]) are in fact independent of the L^{∞}-norm of the lower order coefficients.

TWO MOTIVATIONS AND REFERENCES

(Motivation I) Recent interest in Schauder estimates for elliptic and parabolic PDEs with unbounded coefficients. Such equations arise in the study of SDEs and in Mathematical Finance.

Related Papers

[Cannarsa-Vespri 1987], [Da Prato-Lunardi 1995], [Lunardi-Vespri 1995], [Cerrai 1996, 2002], [Lunardi 1998], [Priola 2003], [Bertoldi-L. Lorenzi 2004].

We mention [Da Prato-Lunardi 1995]. They consider constant $a_{ij}(t,x) = a_{ij}$ and $b_i(t,x) = \langle Ax, e_i \rangle$, $x \in \mathbb{R}^d$, for a fixed $d \times d$ matrix A, i.e.,

$$Lu(x) = \sum_{i,j=1}^{d} a^{ij} u_{x^i x^j}(x) + \langle Ax, Du(x) \rangle - \lambda u(x), \tag{8}$$

and prove Schauder estimates for L (the so-called

Ornstein-Uhlenbeck operator).

After [Da Prato - Lunardi 95] a question arises:

Schauder estimates are true for (8) **if we allow** a^{ij} **to be variable, bounded and Hölder continuous ??** (possible applications to SPDEs)

An attempt: To get a-priori estimates (assuming to have a solution $u \in C^{2+\alpha}(\mathbb{R}^d)$) in the classical case one uses a *localization procedure,* multiplying u by a suitable test function $\eta_{x_0}(x)$, $x_0 \in \mathbb{R}^d$, such that

$$Supp(\eta_{x_0}) \subset B(x_0, \delta)$$

where $\delta > 0$ is chosen small enough. If coefficients are bounded one gets

$$\|u\,\eta_{x_0}\|_{2+\alpha} \le C\|f\|_\alpha,$$

with C independent of x_0. *This does not work in our situation since we will have to estimate the "bad term"*

$$\boxed{\langle Ax, D\eta_{x_0}(x)\rangle u(x).}$$

The previous question has only been answered with the present paper. However we mention related results.

- [Lunardi 97] proves Schauder estimates assuming in addition that there exists

$$\lim_{x\to\infty} a^{ij}(x) = \tilde{a}^{ij}, \qquad i, j = 1, \ldots, d.$$

- [Lunardi - Vespri 95] assumes that $a^{ij} \in C_b^1(\mathbb{R}^d)$ and moreover

$$\langle Da^{ij}(x), Ax\rangle \ \in \ L^{\infty}(\mathbb{R}^d), \quad i, j = 1, \ldots, d.$$

- [Cerrai 96, 2002], [Lunardi 98] and [Bertoldi and Lorenzi 2005] deal with autonomous operators L of the form

$$Lu = \sum_{i,j=1}^{d} a^{ij}(x) u_{x^i x^j}(x) + \sum_{i=1}^{d} b^i(x) u_{x^i}(x),$$

in which coefficients a^{ij}, b^i can growth more than linearly. However coefficients must be smooth enough in order to apply the Bernstein method or stochastic methods.

(Motivation II) To complete the theory of second order parabolic operators with coefficients which are Hölder continuous in the space variable and discontinuous in time.

Indeed as far as we know after the pioneering works [Brandt 1969] and [Knerr 1980] the existence and uniqueness of solutions to (1) was not completely established even in the case of bounded coefficients.

We should mention Schauder estimates of [L. Lorenzi 2000] for Cauchy problems associated to L when coefficients are bounded, Hölder continuous in the space variable and discontinuous in time.

Some ideas from the proof of the priori estimates

We sketch the various steps which we need.

Step I Let

$$b^i = 0, \quad a^{ij}(t,x) = a^{ij}(t).$$

If $u \in \mathcal{C}^{2+\alpha}(T)$ with compact support then, for any $x \in \mathbb{R}^d$ and finite $t \geq T$, we have

$$[D^2 u(t,\cdot)]_\alpha \leq N \sup_{r>t} [(u_t + Lu)(r,\cdot)]_\alpha, \tag{9}$$

This estimate is based on an explicit representation formula for u available in this case.

Step II Let

$$b^i(t,x) = b^i(t), \quad a^{ij}(t,x) = a^{ij}(t),$$

Let $u \in \mathcal{C}^{2+\alpha}(T)$ such that, for any finite $S_2 > S_1 \geq T$, there exists $K = K_{S_1,S_2} > 0$ with $u(t,x) = 0$, for any $|x| > K$, $t \in [S_1, S_2]$.

Then, for any finite $t \geq T$ *(with summation convention)*,

$$[D^2 u(t,\cdot)]_\alpha$$

$$\leq N \sup_{s>t} [(u_t + a^{ij} u_{x^i x_j} + b^i u_{x^i} - \lambda u)(s,\cdot)]_\alpha .$$

To prove this, we may assume that $u \in \mathcal{C}^{2+\alpha}(T)$ with compact support (in the general case, we use $u^n(s,x) = \zeta(s/n)u(s,x)$, with $\zeta \in C_0^\infty$, $\zeta(0) = 1$)

Define

$$B(t) = \int_0^t b(s)\, ds, \qquad v(t,x) = u(t, x + B(t)).$$

Note that $v \in \mathcal{C}^{2+\alpha}(T)$ and has compact support, since b is locally bounded.

By inserting v in place of u in (9) we obtain the assertion.

Step III Let $u \in \mathcal{C}^{2+\alpha}(T)$ be a solution; we show that

$$\sup_{t \geq T} [D^2 u(t,\cdot)]_\alpha \leq N(F_0 + F_\alpha) + N \sup_{t \geq T} \|u(t,\cdot)\|_2. \tag{10}$$

To prove (10) fix an $\varepsilon \in (0, 1/2)$ and $\zeta \in C_0^\infty(\mathbb{R}^d)$ with support in $B(0, 2\epsilon)$ and such that $\zeta(x) = 1$ for $|x| \leq \varepsilon$.

Let $(t_0, x_0) \in \mathbb{R}^{d+1}_T$ and introduce $x(t) = x^{x_0,t_0}(t)$ as a solution (not necessarily unique) of

$$x(t) = x_0 + \int_{t_0}^t b(s, x(s))\, ds, \quad t \in \mathbb{R},$$

where $b(t,x)$ is the vector with coordinates $b^i(t,x)$, $i = 1, \ldots, d$. Set

$$a_0^{ij}(t) = a^{ij}(t, x(t)), \quad b_0(t) = b(t, x(t)),$$

$$L_0 = a_0^{ij}(t) D_{x^i x^j} + b_0(t) D_{x^i} - \lambda,$$

$$\eta(t,x) = \zeta(x - x(t)), \quad v(t,x) = u(t,x)\,\eta(t,x).$$

Observe that if $\eta(t,x) \neq 0$, then $|x - x(t)| \leq 2\varepsilon$; moreover, when $|x - x(t)| \leq 2\varepsilon$ we have

$$|a^{ij}(t,x) - a_0^{ij}(t)| \leq 2^\alpha K \varepsilon^\alpha,$$

$$|b(t,x) - b_0(t)| \leq 2^\alpha K \varepsilon^\alpha d,$$

It is crucial that

$$\boxed{\eta_t(t,x) + b_0^i(t)\eta_{x_i}(t,x) = 0}$$

(the "bad term" disappears)

Applying the previous step v, for any $x \in \mathbb{R}^d$ such that $|x - x_0| \le \varepsilon$, we have $\eta(t_0, x) = 1$ and

$$I := \frac{|D^2 u(t_0, x) - D^2 u(t_0, x_0)|}{|x - x_0|^\alpha} =$$

$$\frac{|D^2 v(t_0, x) - D^2 v(t_0, x_0)|}{|x - x_0|^\alpha}$$

$$\le N \sup_{s > t_0} [(v_t + L_0 v)(s, \cdot)]_\alpha.$$

Here

$$v_t + L_0 v =$$

$$= \eta f + \eta(L_0 - L)u + u\, a_0^{ij} \eta_{x^i x^j} + 2a_0^{ij} \eta_{x^i} u_{x^j}.$$

It is now standard to see that

$$I \le N(\varepsilon)(F_\alpha + F_0) + N\varepsilon^\alpha \sup_{s > t_0} [D^2 u(s, \cdot)]_\alpha$$

$$+ N(\varepsilon) \sup_{s > t_0} \|u(s, \cdot)\|_2$$

where $N = N(d, \alpha, K, \lambda)$ and $N(\varepsilon) = N(\varepsilon, d, \alpha, K, \lambda)$. Due to the arbitrariness of x_0 and x, we obtain

$$[u(t_0, \cdot)]_{2+\alpha} \leq N(\varepsilon)(F_0 + F_\alpha)$$

$$+N\varepsilon^\alpha \sup_{s>t_0}[D^2u(s, \cdot)]_\alpha + N(\varepsilon) \sup_{s>t_0} \|u(s, \cdot)\|_2.$$

Upon taking the sup of both sides with respect to $t_0 \geq T$ we conclude

$$\sup_{t\geq T}[D^2u(t, \cdot)]_\alpha \leq N(\varepsilon)(F_0 + F_\alpha)$$

$$+N\varepsilon^\alpha \sup_{t\geq T}[D^2u(t, \cdot)]_\alpha + N(\varepsilon) \sup_{t\geq T} \|u(t, \cdot)\|_2.$$

After choosing ε appropriately, we finally get the assertion.

Step IV. We have to prove a maximum principle: if $u \in \mathcal{C}^{2+\alpha}(T)$ is a solution then it holds:

$$|u(t,x)| \le \frac{F_0}{\lambda}, \quad (t,x) \in \mathbb{R}^{d+1}_T.$$

This maximum principle is also a non-standard result; indeed we are not assuming that $u(\cdot, x)$ is differentiable.

From Step III, using the maximum principle and the next interpolation inequality

$$\|v\|_2 \le N(\varepsilon)\|v\|_0 + \varepsilon[D^2 v]_\alpha, \quad \varepsilon > 0, \quad v \in C^{2+\alpha},$$

we get the desired a priori estimate. ∎

Finally, as far as the time regularity of solutions is concerned, we have

Theorem 5 *Let T_1, T_2, and R be finite numbers such that $T \le T_1 < T_2$ and $R > 0$. Then, for any solution $u \in \mathcal{C}^{2+\alpha}(T)$, we have:*

$$|t-s|^{-1}\,|u(s,x)-u(t,x)|$$

$$+|t-s|^{-\frac{1+\alpha}{2}}|Du(s,x)-Du(t,x)|$$

$$+\,|t-s|^{-\alpha/2}\,|D^2u(s,x)-D^2u(t,x)| \le N,$$

whenever $|x| \le R$ and $s,t \in [T_1,T_2]$, $t < s$, where N depends only on λ, α, d, K, F_0, F_α, R, and sup norms of $|b(t,0)|, c(t,0), |f(t,0)|$ over $[T_1,T_2]$.

We use

Lemma 6 *For any $u \in \mathcal{C}^{2+\alpha}(T)$ and finite $t, s \geq T$ and $x \in \mathbb{R}^d$ we have*

$$|D^2 u(t,x) - D^2 u(s,x)| \leq NI|t-s|^{\alpha/2}, \quad (11)$$

$$|Du(t,x) - Du(s,x)| \leq NI|t-s|^{(1+\alpha)/2},$$

where N depends only on d and

$$I = \sup_{r \in [s,t]} \big([u_t(r,\cdot)]_\alpha + [u(r,\cdot)]_{2+\alpha}\big).$$