Some qualitative properties of solutions to second–order elliptic equations and parabolic equations

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We study local properties of weak solutions to the equations

$$\boxed{\boxed{\mathcal{L}u = 0}} \quad (*)$$

and

$$\boxed{\boxed{\frac{\partial u}{\partial t} = \mathcal{L}u}} \quad (**)$$

where $\mathcal{L}$ is a second-order elliptic operator on a domain in $\mathbb{R}^n$.

Properties: local boundedness, continuity, Harnack inequalities, isolated singularities.

1 Elliptic Equations

Let $\Omega \subset \mathbb{R}^n$ be a domain, $n \geq 3$. The divergence type equation

$$\nabla \cdot a \cdot \nabla u := \sum_{i,j=1}^{n} \partial_{x_i} a_{ij} \partial_{x_j} u = 0 \tag{1}$$

with a uniformly elliptic matrix of measurable bounded coefficients, i.e. $\exists \nu > 0, M > 0$ s.t. for almost all $x \in \Omega$

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_j \xi_j \geq \nu |\xi|^2, \quad \xi \in \mathbb{R}^n \quad \text{and} \quad |a_{ij}(x)| \leq M.$$

Weak solution

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x)u(x)\varphi(x)dx = 0 \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

- E.DeGiorgi (1957) - local boundedness, Hölder continuity
- J.Nash (1958) - Hölder continuity for parabolic
- J.Moser (1960,1961) - local boundedness, Hölder continuity, Harnack inequality

Let u be a positive week solution to (1) and $B_{4R} \subset \Omega$. Then

$$\sup_{B_R} u \leq C \inf_{B_R} u.$$

Perturbations: Lower order terms can destroy properties

$-\Delta u + \frac{c}{|x|^2}u = 0$ - no Harnack, LB, isolated singularities are different.

- J.Serrin (1964)-Moser's technique - lower order terms in terms of L^p-conditions and a class of quasi-linear equations.

- Ladyzhenskaja, Uraltseva (1968) - DeGiorgi's technique

$$\text{For} \quad \nabla \cdot a \cdot \nabla u + Vu = 0$$

typical condition $V \in L^{n/2+\epsilon}$. For $V \in L^{n/2}$ some properties fail (e.g. no LB)

Kato class $K_n(\mathbb{R}^n)$

$$V \in K_n \iff \lim_{R \to 0} \sup_{x \in \mathbb{R}^n} \int_{B_R(x)} \frac{|V(y)|dy}{|x-y|^{n-2}} = 0$$

Typical singularity of K_n

$$\frac{1}{|x|^2(\log \frac{1}{|x|})^{1+\varepsilon}}, \quad \varepsilon > 0,$$

but not

$$\frac{1}{|x|^2 \log \frac{1}{|x|}}.$$

- Aizenman, Simon (1982) - probabilistic approach

$$\Delta u + Vu = 0,\ V \in K_n$$

LB, continuity (no HC), Harnack inequality

- Chiarenza, Fabes, Garofalo (1986) - real functions techniques and Green's functions

$$-\nabla \cdot a \cdot \nabla u + Vu = 0,\ V \in K_n$$

- Kurata (1994)

$$-\nabla \cdot a \cdot \nabla u + b \cdot \nabla u + Vu = 0,\ V,\ |b|^2 \in K_n$$

Quasi-linear equations

$$-\Delta_p u + Vu|u|^{p-2} = 0, \quad p > 1. \tag{2}$$

Extension of Kato class condition

$$V \in K_n \;\Leftrightarrow\; \lim_{R\to 0} \sup_{x\in\mathbb{R}^n} \int_0^R \left\{ \frac{1}{r^{n-2}} \int_{B_r(x)} |V(y)|dy \right\} \frac{dr}{r} = 0.$$

$$\int_{B_R(x)} \frac{|V(y)|dy}{|x-y|^{n-2}} = (n-2)\int_0^R \frac{1}{r^{n-2}} \int_{B_r(x)} |V(y)|dy \frac{dr}{r}$$
$$+ \quad \frac{1}{R^{n-2}} \int_{B_R(x)} |V(y)|dy$$

- M.Biroli (2001) - nonlinear Kato class K_p, local boundedness, Harnack, continuity

$$\lim_{R\to 0} \sup_{x\in\Omega} \int_0^R \left\{ \frac{1}{r^{n-p}} \int_{B_r(x)} |V(y)| dy \right\}^{\frac{1}{p-1}} \frac{dr}{r} = 0 \qquad (3)$$

Non-standard growth conditions (joint result with I.I.Skrypnik)

$$-\operatorname{div} \mathbf{A}(x, u, \nabla u) + a_0(x, u, \nabla u) = 0, \quad x \in \Omega \tag{4}$$

$\mathbf{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $a_0 : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\mathbf{A}(\cdot, \zeta, \xi), a_0(\cdot, \zeta, \xi)$ are measurable and $\mathbf{A}(x, \cdot, \cdot), a_0(x, \cdot, \cdot)$ are continuous for a.a. $x \in \Omega$.

$$\begin{array}{rcl} \mathbf{A}(x, u, \xi)\xi & \geq & c_1|\xi|^{p(x)}, \quad 1 < p_1 < p(x) < p_2 < n, \\ |\mathbf{A}(x, u, \xi)| & \leq & c_2|\xi|^{p(x)-1}, \\ |a_0(x, u, \xi)| & \leq & g(x)|u|^{p(x)-1} + f(x), \end{array} \tag{5}$$

with some positive constants c_1, c_2 and nonnegative functions $f,\ g$.

We say that $u \in W(\Omega)$ is a weak solution to (4) if

$$\int_\Omega \mathbf{A}(x,u,\nabla u)\nabla\varphi dx + \int_\Omega a_0(x,u,\nabla u)\varphi dx = 0 \tag{6}$$

for any $\varphi \in \overset{\circ}{W}(\Omega)$, where $W(\Omega)$ is the Banach space

$$W(\Omega) = \{f \in W^{1,1}(\Omega) \,:\, \int_\Omega |\nabla f(x)|^{p(x)}\, dx < \infty\}$$

with the norm

$$\|f\|_{W(\Omega)} = \inf\left\{\lambda > 0 \,:\, \int_\Omega \left|\tfrac{1}{\lambda}\nabla f(x)\right|^{p(x)} dx \le 1\right\} + \inf\left\{\lambda > 0 \,:\, \int_\Omega \left|\tfrac{1}{\lambda} f(x)\right|^{p(x)} dx \le 1\right\}.$$

$$|p(x) - p(y)| \leq \frac{A}{\ln |x - y|^{-1}}, \quad x, y \in \Omega, \ |x - y| < \frac{1}{2}.$$

The model equation (so-called, $p(x)$-Laplace equation)

$$\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = 0$$

Alkhutov (1997) - Hölder continuity, Harnack

Class $\mathcal{K}_p$:

$$\lim_{R\to 0}\sup_{x\in\Omega}\int_0^R\left\{\frac{1}{r^{n-p(x)}}\int_{B_r(x)}|g(z)|dz\right\}^{\frac{1}{p(x)-1}}\frac{dr}{r}=0$$

Theorem 1.1. *Let* $u\geq 0$ *be a sln to* (4) *in* Ω *with* $f,g\in\mathcal{K}_p$*.*

Then $\exists\, K_2,\ \theta>0$ *s.t. for any* $x_0\in\Omega$ *such that for* $B_{4R}(x_0)\subset\Omega$

$$\sup_{B_{\frac{R}{2}}(x_0)}u(x)\leq K_2[\inf_{B_{\frac{R}{2}}(x_0)}u(x)+\theta f(R)+R^{\frac{1}{2}}], \qquad (7)$$

$K_2,\ \theta$ *depend on* $\int_{B_{4R}(x_0)}u(x)^{(p^++1)(p^+-p^-)}dx$ *and*

n,p_1,p_2,c_1,c_2,A *and* $p^+=\max_{B_{4R}(x_0)}p(x)$*,* $p^-=\min_{B_{4R}(x_0)}p(x)$*,*

$$f(R)=\sup_{x_0\in\Omega}\sup_{x\in B_R(x_0)}\int_0^{2R}\left(\frac{1}{r^{n-p(x)}}\int_{B_r(x)}f(z)dz\right)^{\frac{1}{p(x)-1}}\frac{dr}{r}.$$

Denote

$$\omega(R) = osc_{B_R} u \equiv \sup_{B_R} u - \inf_{B_R} u.$$

The above theorem leads

$$\omega(R/2) \leq (1-\tau)\omega(R) + \gamma f(R), \quad \tau \in (0,1),\ \gamma > 0.$$

Hence – continuity if $f(R) \to 0$ ($R \to 0$),

but NOT Hölder continuity in general.

Example 1.2. *Let us consider the homogeneous equation*

$$-\Delta_p u + g u|u|^{p-2} = 0 \tag{8}$$

in the unit ball in $\mathbb{R}^n$, $1 < p < n$, *with potential*

$$g(x) \asymp (n-p)\alpha^{p-1}|x|^{-p}\left(\log\frac{e}{|x|}\right)^{1-p}, \quad \alpha > 0.$$

Then $u(x) = \left(\log\frac{e}{|x|}\right)^{-\alpha}$ *is a solution to* (8), *which is 1 at the boundary of the unit ball, nonnegative inside and zero at the origin. Harnack inequality is not valid for* (8), *the reason – stronger singularity of* g *than it is allowed by the conditions of Theorem 1.1.* $\tilde{g}(x) \sim c|x|^{-p}\left(\log\frac{1}{|x|}\right)^{1-p+\varepsilon}$ *is in* K_p *for* $\varepsilon > 0$ *but not for* $\varepsilon = 0$.

Isolated singularities Let $x_0 \in \Omega,\ R < 1$ s.t. $B_R(x_0) \subset \Omega$. Let u

$$\Delta u = 0 \quad \text{in } \Omega \setminus \{x_0\}$$

If $u(x) = o\left(|x - x_0|^{-n+2}\right),\ n > 2,$ then the singularity is removable.

- J.Serrin (1964) - positive solutions to general quasi-linear elliptic eqns, L^p-conditions on lower order terms

$$u(x) = o\left(|x - x_0|^{-\frac{n-p}{p-1}}\right),\ 1 < p < n, \tag{9}$$

- J.-L.Vazquez, C.Yarur (1976) $-\Delta u + Vu = 0, \quad V \in K_n$

- L& I.I.Skrypnik (2008)

$$-\Delta_p u + Vu|u|^{p-2} = 0,$$

for $V \in K_p$ if $p \geq 2$ and $V \in K_{2,p}$ if $1 < p < 2$. Class $K_{2,p}$:

$$\lim_{R\to 0} \sup_{x\in\Omega} \int_0^R \frac{1}{r^{n-p}} \int_{B_r(x)} |V(y)|dy \frac{dr}{r} = 0.$$

Example 1.3. $u(x) = |x|^{\frac{p-n}{p-1}} \left(\log \frac{1}{|x|}\right)^{-\alpha}$ *solves* $-\Delta_p u + Vu^{p-1} = 0$ *in* $B_1(0) \setminus \{0\}$ *with* $V \asymp c|x|^{-p} \left(\log \frac{1}{|x|}\right)^{-1}$

$\tilde{g}(x) = c|x|^{-p} \left(\log \frac{1}{|x|}\right)^{-\beta}$ *is from* K_p *for* $\beta > p-1$*, and from* $K_{2,p}$ *for* $\beta > 1$*.*

2 Parabolic Equations

$\Omega_T = \Omega \times (0, T), \Omega \in \mathbb{R}^n$

$$\partial_t u = \nabla \cdot a \cdot \nabla u \equiv \sum_{i,k=1}^{n} \partial_i a_{ik} \partial_k u$$

a_{ik} - measurable, uniformly elliptic

- J.Nash (1958) - Hölder continuity
- Laduzhenskaya, Solonnikov, Ural'tseva (1962) - Hölder continuity
- J.Moser (1964) - parabolic Harnack inequality

  $(x_0, t_0) \in \Omega_T,\ Q_\rho = B_\rho(x_0) \times (t_0 - \rho^2, t_0 + \rho^2]$.

For $Q_{2\rho} \subset \Omega_T \;\; \exists \gamma > 0$

$$u(x_0, t_0) \leq \gamma \inf_{B_\rho(x_0)} u(x, t_0 + \rho^2).$$

- K.-Th.Sturm (1994) - probabilistic approach, Harnack inequality
- Qi Zhang (1996)

$$\partial_t u = \nabla \cdot a \cdot \nabla u + V u, \quad V \in K_n$$

Potential $V \in L^1_{loc}((0, \infty) \times \mathbb{R}^N)$ is in the *non-autonomous Kato class* NK if

$$\lim_{\alpha \to 0} N_\alpha^-(V) = 0 \quad \text{and} \quad \lim_{\alpha \to 0} N_\alpha^+(V) = 0,$$

where for $\alpha > 0$

$$N_\alpha^{\pm}(V) = \sup_{x,s} \int_0^\alpha \int_{\mathbb{R}^N} k_t(x-y)|V(s \pm t, y)|\, dy\, dt$$

$k_t(x) = (4\pi t)^{-N/2} e^{-\frac{|x-y|^2}{4t}}$, and $V(\tau, \cdot) := 0$ for $\tau < 0$.

- L& Semenov (2000)

$$\partial_t u = \nabla \cdot a(t,x) \cdot \nabla u - b(t,x) \cdot \nabla u + \nabla \hat{b}(t,x) u + V(t,x) u$$

$$b = (b_j)_{j=1}^n,\ \hat{b} = (\hat{b}_j)_{j=1}^n$$

$$c_1 t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{c_2(t-s)}} \le r(t,x;s,y) \le c_3 t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{c_4(t-s)}}$$

($r(t,x;s,y)$ - fundamental solution)

$$k_\alpha(t,x) = (4\pi\alpha t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4\alpha t}}$$

$$w := b \cdot a^{-1} \cdot b, \quad \hat{w} := \hat{b} \cdot a^{-1} \cdot \hat{b}, \quad W = w + \hat{w}$$

$$n_\varphi^- = \sup_{z,t} \int_{t-h}^{t} \int_{\mathbb{R}^N} k_\varkappa(t-\tau, y-z)|W(\tau,\cdot)|dy \frac{d\tau}{\varphi(t-\tau)} \ ,$$

$$n_\varphi^+ = \sup_{z,s} \int_{s}^{s+h} \int_{\mathbb{R}^N} k_\varkappa(\tau-s, y-z)|W(\tau,\cdot)|dy \frac{d\tau}{\varphi(\tau-s)},$$

$$\varphi(0) = 0 \text{ and } \int_0^T \frac{\varphi(\tau)}{\tau} d\tau < \infty$$

Condition: $n_\varphi^\pm \to 0$ as $h \to 0$

Global bound for F.S. to

$$\Delta u + b\nabla u - u_t = 0 \text{ in } \mathbb{R}^n \times (0, \infty).$$

(L& Qi Zhang 2004) $\mathbf{b} \in L^2_{loc}(\mathbb{R}^n)$, q - F.S.

(i) there exists a constant $B > 0$ such that for any $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\mathbf{b}|^2 \phi^2 dx \leq B \int_{\mathbb{R}^n} |\nabla \phi|^2 dx; \tag{10}$$

(ii) there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^n} |\mathrm{div}\mathbf{b}| \phi^2 dx \leq (2 - \delta) \int_{\mathbb{R}^n} |\nabla \phi|^2. \tag{11}$$

(iii) the quantity

$$K((\mathrm{div}\mathbf{b})^-) \equiv \sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(\mathrm{div}\mathbf{b})^-(y)}{|x-y|^{n-2}}dy \qquad (12)$$

is sufficiently small.

Then $q(t,x,y) \le \dfrac{c_1}{t^{n/2}}\exp(-c_2|x-y|^2/t).$

Suppose

$$K_1(\mathbf{b}) \equiv \sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\mathbf{b}(y)|}{|x-y|^{n-1}}dy < \infty,$$

$$K(|\mathbf{b}|^2) \equiv \sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\mathbf{b}(y)|^2}{|x-y|^{n-2}}dy < \infty.$$

Then there exist positive constants c_1 and c_2 such that, for any $x, y \in \mathbb{R}^N$ and $t > 0$,

$$q(t, x, y) \geq \frac{c_1}{t^{N/2}} \exp(-c_2 |x - y|^2 / t).$$

Quasi-linear Equations

Parabolic p-Laplace equation

$$\partial_t u - \Delta_p u = 0, \quad \text{in } \Omega_T, \quad p > 2. \tag{13}$$

p-Laplace equation (13) admits explicit similarity solution ($p > 2$)

(Barenblatt solution)

$$\Gamma_p(x,t) = \frac{1}{t^{n/\lambda}} \left(1 - \gamma_p \left(\frac{|x|}{t^{1/\lambda}} \right)^{p/(p-1)} \right)_+^{(p-1)/(p-2)}, \quad t > 0,$$

where $\gamma_p = \lambda^{-\frac{1}{p-1}} \frac{p-2}{p}$, $\lambda = n(p-2) + p$.

- E.DiBenedetto (1983-84) – Hölder continuity, intrinsic Harnack inequality:

  Let u be a nonnegative solution to (13) Let $(x_0, t_0) \in \Omega_T$ s.t. $u(x_0, t_0) > 0$. Consider the parabolic cylinder

$$Q_\rho^\theta(x_0, t_0) = B_\rho(x_0) \times (t_0 - \theta\rho^p, t_0 + \theta\rho^p),$$

  where $\theta = \left(\frac{c}{u(x_0,t_0)}\right)^{p-2}$ and $c > 0$ is fixed.

$$\exists \gamma > 0 \qquad u(x_0, t_0) \leq \gamma \inf_{B_\rho(x_0)} u(x, t_0 + \theta\rho^p).$$

Intrinsic scaling is necessary – Barenblatt solution provides an example

General divergence type quasi-linear equations

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,\nabla u) = a_0(x,t,u,\nabla u), \quad (x,t) \in \Omega_T \tag{14}$$

$$\begin{aligned}
\mathbf{A}(x,t,u,\zeta)\zeta &\geq c_1|\zeta|^p, \quad \zeta \in \mathbb{R}^n, \quad 2 < p < n, \\
|\mathbf{A}(x,t,u,\zeta)| &\leq c_2|\zeta|^{p-1} + g_1(x)|u|^{p-1} + f_1(x), \\
|a_0(x,t,u,\zeta)| &\leq c_2|\zeta|^{p-1} + g_2(x)|u|^{p-1} + f_2(x).
\end{aligned}$$

Weak local solutions

$u \in C(0,T;L^2_{loc}(\Omega)) \cap L^p_{loc}(0,T;W^{1,p}_{loc}(\Omega))$ s.t. $\forall \Omega_1 \Subset \Omega$ and $t_1, t_2 \subset (0,T]$

$$\int_{\Omega_1} u\varphi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega_1} \{-u\varphi_t + \mathbf{A}(x,\tau,u,\nabla u)\cdot \nabla\varphi\} dx d\tau$$
$$= \int_{t_1}^{t_2} \int_{\Omega_1} a_0(x,\tau,u,\nabla u) dx d\tau$$

for all locally bdd $\varphi \in W^{1,2}_{loc}(0,T;L^2(\Omega_1)) \cap L^p_{loc}(0,T;\overset{\circ}{W}{}^{1,p}(\Omega_1))$

For $g_1, g_2 = 0$ and f_1, f_2 in L^p-classes, Y.Chen (1984),

E.DiBenedetto and A.Friedman (1985) – Hölder continuity.

Intrinsic Harnack inequality – long standing problem

E. Di Benedetto, U. Gianazza and V. Vespri (Acta Mathematica 2008)

for f_1, f_2, g_1, g_2 constants.

(L& I.I.Skrypnik)

Let g_1, f_1 be s.t.

$$\lim_{R\to 0} \sup_{x\in\Omega} \int_0^R \left\{ \frac{1}{r^{n-p}} \int_{B_r(x)} |f_1(z)|^{\frac{p}{p-1}} + |g_1(z)|^{\frac{p}{p-1}} dz \right\}^{\frac{1}{p}} \frac{dr}{r} = 0$$

Let $g_2,\ f_2 \in K_p$. Then weak solutions to (14) are locally bounded and continuous. Intrinsic Harnack inequality holds for positive solutions.

One cannot expect Hölder continuity under Kato class conditions!

Let u be a weak solution to (14) in Ω_T. Let $(y,s)\in\Omega_T$ be an arbitrary point. Consider the cylinder $Q^{\theta}_{4\rho}(y,s)\subset\Omega_T$,

$$Q^{\theta}_{\rho}(y,s)=B_\rho(y)\times(s-\theta\rho^p,s),\quad \theta>0.$$

$$\mu_+\geq \operatorname*{ess\,sup}_{Q^{\theta}_{4\rho}(y,s)} u(x,t),\quad \mu_-\leq \operatorname*{ess\,inf}_{Q^{\theta}_{4\rho}(y,s)} u(x,t),\quad \omega\geq\mu_+-\mu_-.$$

$$\mathcal{F}_1(R)=\sup_{x\in\Omega}\int_0^R\left(\frac{1}{r^{n-p}}\int_{B_r(x)}(g_1+f_1)^{\frac{p}{p-1}}dz\right)^{\frac{1}{p}}\frac{dr}{r}, \tag{15}$$

$$\mathcal{F}_2(R)=\sup_{x\in\Omega}\int_0^R\left(\frac{1}{r^{n-p}}\int_{B_r(x)}(g_2+f_2)dz\right)^{\frac{1}{p-1}}\frac{dr}{r}. \tag{16}$$

Theorem. Fix $\xi, a \in (0,1)$, $(\xi\omega)^{p-2} \geq \frac{1}{\theta}$. $\exists\, B \geq 1$ and $\nu \in (0,1)$ depending only on the data and θ, ξ, ω and a such that if

$$\left|\{(x,t) \in Q^{\theta}_{2\rho}(y,s) \,:\, u(x,t) \leq \mu_- + \xi\omega\}\right| \leq \nu |Q^{\theta}_{2\rho}(y,s)|,$$

then either $\xi\omega \leq B(\rho + \mathcal{F}_1(2\rho) + \mathcal{F}_2(2\rho))$, or

$$u(x,t) \geq \mu_- + a\xi\omega \;\text{ for almost all (a.a.) }\; (x,t) \in Q^{\theta}_{\rho}(y,s).$$

Likewise,if

$$\left|\{(x,t) \in Q^{\theta}_{2\rho}(y,s) \,:\, u(x,t) \geq \mu_+ - \xi\omega\}\right| \leq \nu |Q^{\theta}_{2\rho}(y,s)|,$$

then either $\xi\omega \leq B(\rho + \mathcal{F}_1(2\rho) + \mathcal{F}_2(2\rho))$, or

$$u(x,t) \leq \mu_+ - a\xi\omega \;\text{ for almost all (a.a.) }\; (x,t) \in Q^{\theta}_{\rho}(y,s).$$

Expansion of positivity

There exist numbers $B,\ b_1 < b_2$ and $\sigma \in (0,1)$ depending only on the data such that if

$$u(x,s) \geq \mu_- + N \quad \text{for } x \in B_\rho(y),$$

then either $N \leq B(\rho + \mathcal{F}_1(2\rho) + \mathcal{F}_2(2\rho))$, or

$$u(x,t) \geq \mu_- + \sigma N \quad \text{for a.a. } x \in B_{2\rho}(y),$$

for all $s + N^{2-p} b_1 \rho^p \leq t \leq s + N^{2-p} b_2 \rho^p$.

If on the other hand

$$u(x,s) \leq \mu_+ - N \quad \text{for } x \in B_\rho(y),$$

then either $N \leq B(\rho + \mathcal{F}_1(2\rho) + \mathcal{F}_2(2\rho))$, or

$$u(x,t) \leq \mu_+ - \sigma N \quad \text{for a.a. } x \in B_{2\rho}(y),$$

for all t s.t. $s + N^{2-p} b_1 \rho^p \leq t \leq s + N^{2-p} b_2 \rho$.

On a proof of local boundedness (Kilpeläinen and Malý technique)

Let $(x_0, t_0) \in \Omega_T$, $Q_R = B_R(x_0) \times (t_0 - R^2, t_0 + R^2)$ s.t. $Q_{2R} \subset \Omega_T$. Fix $(y, s) \in Q_{\frac{R}{2}}(x_0, t_0)$. One can find the sequences $(l_j)_j$, $(\delta_j)_j$, $\delta_j = l_{j+1} - l_j \geq 0$, s.t.

$\rho_j = R2^{-j}$, $B_j = B_{\rho_j}(y)$, $Q_j = B_j \times (s - \delta_j^{2-p}\rho_j^p, s + \delta_j^{2-p}\rho_j^p)$.

the inequality holds

$$\frac{(l_{j+1} - l_j)^{p-2}}{\rho_j^{n+p}} \iint_{Q_j \cap \{u > l_j\}} \frac{u}{l_{j+1}} \left(\frac{u - l_j}{l_{j+1} - l_j} \right)^{(1+\frac{1}{pn})(p-1)} dx d\tau$$

$$+ \sup_t \frac{1}{\rho_j^n} \int_{(B_j \times \{t\}) \cap \{u > l_j\}} \frac{u}{l_{j+1}} \left(\frac{u - l_j}{l_{j+1} - l_j} \right) dx \leq \varkappa$$

with $\varkappa$ dependent on the known data only, and

$$\begin{aligned}\delta_j &\leq \frac{1}{2}\delta_{j-1} \\ &+ \gamma\left(\rho_j^{p-n}\int_{B_j}\mathcal{F}_1(x)dx\right)^{\frac{1}{p}} + \gamma\left(\rho_j^{p-n}\int_{B_j}\mathcal{F}_2(x)dx\right)^{\frac{1}{p-1}}.\end{aligned}$$

where $\mathcal{F}_1 = (g_1 + f_1)^{\frac{p}{p-1}}$, $\mathcal{F}_2 = g_2 + f_2$.

Tools: energy estimates and Sobolev imbedding

Summing up w.r.t. j: $\quad l_J \leq \gamma \delta_0 +$

$$+\sum_{j=1}^{\infty}\left(\rho_j^{p-n}\int_{B_j}\mathcal{F}_1(x)dx\right)^{\frac{1}{p}}+\sum_{j=1}^{\infty}\left(\rho_j^{p-n}\int_{B_j}\mathcal{F}_2(x)dx\right)^{\frac{1}{p-1}}$$

with $\delta_0 \leq \gamma \sup_t \left(R^{-n}\int_\Omega u^{p+\frac{1}{pn}(p-1)}dx\right)^{\frac{p-1}{(1+\frac{1}{pn})(p-1)^2+1}}$

$$+\gamma \sup_t \left(R^{-n}\int_\Omega u^2 dx\right)^{\frac{p-1}{p}},$$

Thus $l_j \longrightarrow l$.

For a Lebesgue point (y, s) of $u(u - l)_+^{(1+\frac{1}{pn})(p-1)}$ one concludes : $u(y, s) \leq l$.

The sequences of positive numbers $(l_j)_{j\in\mathbb{N}}$ and $(\delta_j)_{j\in\mathbb{N}}$ are defined inductively as follows.

Set $l_0 = 1$ and assume that $l_1, l_2, \ldots, l_j$ and $\delta_0, \delta_1, \ldots, \delta_{j-1}$ have been already chosen. Let us show how to chose l_{j+1} and δ_j.

Define the sequence $(\alpha_j)_{j\in\mathbb{N}}$ by

$$\alpha_j = \rho_j + \frac{1}{\mathcal{F}_1(R)}\left(\frac{1}{\rho_j^{n-p}}\int_{B_j} F_1(x)dx\right)^{\frac{1}{p}} + \frac{1}{\mathcal{F}_2(R)}\left(\frac{1}{\rho_j^{n-p}}\int_{B_j} F_2(x)dx\right)^{\frac{1}{p-1}},$$

For $l > l_j + \alpha_j$ set

$$A_j(l) = \frac{(l-l_j)^{p-2}}{\rho_j^{n+p}} \iint_{\widetilde{L}_j} \frac{u}{l}\left(\frac{u-l_j}{l-l_j}\right)^{(1+\lambda)(p-1)} \xi_j^{k-p}\,dx\,d\tau + \sup_t \frac{1}{\rho_j^n}\int_{\widetilde{L}_j(t)} \frac{u}{l} G\left(\frac{u-l_j}{l-l_j}\right)\xi_j^k\,dx,$$

where $\widetilde{L}_j = \widetilde{Q}_j \cap \Omega_T \cap \{u(x,t) > l_j\}$,
$\widetilde{Q}_j = B_j \times (s-(l-l_j)^{2-p}\rho_j^p, s+(l-l_j)^{2-p}\rho_j^p)$.

Fix a positive number $\varkappa \in (0,1)$ depending on n, p, c_1, c_2, which will be specified later. If

$$A_j(l_j+\alpha_j) \le \varkappa,$$

we set $l_{j+1} = l_j + \alpha_j$.

Note that $A_j(l) \searrow 0$ as $l \longrightarrow \infty$. So if

$$A_j(l_j + \alpha_j) > \varkappa,$$

there exists $\bar{l} > l_j + \alpha_j$ such that $A_j(\bar{l}) = \varkappa$. In this case we set $l_{j+1} = \bar{l}$.

In both cases we set $\delta_j = l_{j+1} - l_j$. Note that our choices guarantee that $\widetilde{Q}_j \subset Q_j$

$$A_j(l_{j+1}) \leq \varkappa.$$

Open questions

- $p < 2$ Parabolic p-Laplace equation with lower order terms
- Parabolic $p(x)$-Laplace equation with lower order terms
- Degeneracy in structure conditions of the type

$$\mathbf{A}(x,t,u,\zeta)\zeta \geq w(x)|\zeta|^p, \quad \zeta \in \mathbb{R}^n,$$

where w is a weight function satisfying certain admissibility conditions