Advances in Differential Equations Volume 8, Number 11, November 2003, Pages 1341–1376

CONTINUITY OF WEAK SOLUTIONS OF A SINGULAR PARABOLIC EQUATION

Ugo Gianazza[1]
Dipartimento di Matematica "F. Casorati," Università degli Studi di Pavia
via Ferrata 1, 27100 Pavia

Vincenzo Vespri
Dipartimento di Matematica "U. Dini," Università degli Studi di Firenze
viale Morgagni 67/A, 50134 Firenze

(Submitted by: E. DiBenedetto)

Abstract. We prove the continuity of bounded, weak solutions of the singular parabolic equation $\beta(u)_t = Lu$, where Lu is a second-order, uniformly elliptic operator in divergence form with bounded and measurable coefficients and $\beta(\cdot)$ is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$ exhibiting an arbitrary but finite number of jumps.

1. Introduction

Let $\beta(\cdot)$ denote a maximal monotone graph having two jumps, respectively at $s = 0$ and at $s = 1$. More precisely, we assume that $\beta(\cdot)$ is given by

$$\beta(s) = \begin{cases} \beta_1(s) - \nu_1, & s < 0 \\ [-\nu_1, 0], & s = 0 \\ \beta_2(s), & 0 < s < 1 \\ [1, 1 + \nu_2], & s = 1 \\ \beta_3(s) + \nu_2, & s > 1, \end{cases}$$

where $\beta_i(\cdot)$ are monotone-increasing functions in their respective domains of definition, almost-everywhere differentiable, such that

$$0 < \alpha_0 \le \beta_i'(s) \le \alpha_1 \qquad i = 1, 2, 3,$$

for two positive constants α_0 and α_1, and

$$\beta_1(0) = \beta_2(0) = 0, \qquad \beta_2(1) = \beta_3(1) = 1.$$

Accepted for publication: June 2003.

AMS Subject Classifications: 35K65, 35B65.

[1] Research partially supported by M.I.U.R. through F.A.R. funds and by I.M.A.T.I.–C.N.R.

Consider now the singular parabolic equation

$$\frac{\partial}{\partial t}\beta(u) - \operatorname{div}\bar{\mathbf{a}}(x,t,u,\nabla_x u) + b(x,t,u,\nabla_x u) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega_T). \tag{1.1}$$

Here, Ω is a domain in $\mathbf{R}^N$, for $T > 0$ we denote by Ω_T the cylindrical domain $\Omega_T \equiv \Omega \times (0,T)$, $\bar{\mathbf{a}}$ is a map from $\mathbf{R}^{2N+2}$ into $\mathbf{R}^N$, and b maps $\mathbf{R}^{2N+2}$ into $\mathbf{R}$.

Before stating the assumptions on $\bar{\mathbf{a}}$ and b, let us first define the functional framework.

For $q, r \geq 1$ we denote by $L_{q,r}(\Omega_T)$ the Banach space of those measurable functions mapping $\Omega_T \to \mathbf{R}$, with norm defined by

$$\|u\|^r_{q,r,\Omega_T} = \int_0^T \|u\|^r_{q,\Omega}(t)\,dt, \quad \text{where} \quad \|u\|^q_{q,\Omega}(t) = \int_\Omega |u(x,t)|^q\,dx.$$

When $q = r = 2$, $L_{2,2}(\Omega_T)$ coincides with the Hilbert space $L_2(\Omega_T)$ whose inner product $(\cdot,\cdot)_{2,\Omega_T}$ generates the norm $\|\cdot\|_{2,\Omega_T} \equiv \|\cdot\|_{2,2,\Omega_T}$.

Let $W_2^{1,0}(\Omega_T)$ denote the Hilbert space with inner product

$$(u,v)_{W_2^{1,0}(\Omega_T)} = (u,v)_{2,\Omega_T} + \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_{2,\Omega_T},$$

while $W_2^{1,1}(\Omega_T)$ denotes the Hilbert space with inner product

$$(u,v)_{W_2^{1,1}(\Omega_T)} = (u,v)_{W_2^{1,0}(\Omega_T)} + \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right)_{2,\Omega_T}.$$

With $\overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ we denote the space of those elements in $W_2^{1,1}(\Omega_T)$ whose trace on $\partial\Omega \times (0,T]$ is zero. On the coefficients $\bar{\mathbf{a}} = (a_1, a_2, \ldots, a_N)$ and b we suppose the following:

$$\begin{aligned}
&a_i, b \text{ are measurable over } \Omega_T \times \mathbf{R}^{N+1}, \quad i = 1,2,\ldots,N;\\
&\sum_{i=1}^N a_i(x,t,v,\bar{p})p_i \geq C_0(|v|)|\bar{p}|^2 - \varphi_0(x,t);\\
&|a_i(x,t,v,\bar{p})| \leq \mu_0(|v|)|\bar{p}| + \varphi_1(x,t), \quad i = 1,2,\ldots,N;\\
&|b(x,t,v,\bar{p})| \leq \mu_1(|v|)|\bar{p}|^2 + \varphi_2(x,t),
\end{aligned} \tag{1.2}$$

where $C_0(\cdot) : \mathbf{R}^+ \to \mathbf{R}^+$ is continuous, decreasing, and strictly positive, $\mu_i(\cdot) : \mathbf{R}^+ \to \mathbf{R}^+$ are continuous and increasing $(i = 0,1)$, and φ_i $(i = 0,1,2)$ are nonnegative and satisfy $\|\varphi_0, \varphi_2\|_{\hat{q},\hat{r},\Omega_T}$, $\|\varphi_1\|_{2\hat{q},2\hat{r},\Omega_T} \leq \mu_2$. Here μ_2 is a given constant and $\hat{q}$ and $\hat{r}$ are positive numbers linked by

$$\frac{1}{\hat{r}} + \frac{N}{2\hat{q}} = 1 - \gamma_1, \quad \hat{q} \in [\frac{N}{2(1-\gamma_1)}, \infty], \quad \hat{r} \in [\frac{1}{1-\gamma_1}, \infty], \quad 0 < \gamma_1 < 1.$$

The partial differential equation in (1.1) is meant weakly; namely we have

Definition 1.1. By a weak solution of (1.1) in Ω_T we mean a function $u \in W_2^{1,1}(\Omega_T)$ defined by $u \equiv \beta^{-1}(w)$, where w is a function defined in Ω_T such that $w \subset \beta(u)$, the inclusion being understood in the sense of the graphs, and w and u satisfy

$$\begin{aligned} \int_\Omega w(x,\tau)\varphi(x,\tau)\,dx \Big|_{t_0}^{t} + \int_{t_0}^{t}\int_\Omega [-w(x,\tau)\frac{\partial}{\partial\tau}\varphi(x,\tau) + \\ + \bar{\mathbf{a}}(x,\tau,u,\nabla_x u)\cdot\nabla_x\varphi + b(x,\tau,u,\nabla_x u)\varphi]\,dx\,d\tau = 0 \end{aligned} \tag{1.3}$$

for all $\varphi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and for all intervals $[t_0, t] \subset (0, T]$.

We are not concerned here with the existence of solutions of (1.1) (under this point of view see the references given in [1]). Moreover, our results are local in nature and simply follow from (1.3), without the need to associate to (1.1) any particular boundary-value problem.

When we talk about the data, we mean β_i, ν_1, ν_2, N, and M (see below for the origin of M), and when we say that a particular quantity depends on the data, it means that it can be determined just in terms of β_i, ν_1, ν_2, N, and M.

Our goal is to prove local continuity for the weak solutions of (1.1). Therefore, the main result is the following:

Theorem 1.2. *Let u be a weak solution of* (1.1) *in the sense of Definition* 1.1. *Assume that u satisfies*

$$\|u\|_{\infty,\Omega_T} \le M \tag{1.4}$$

for some given $M > 0$. Then u is continuous in Ω_T. Moreover, for every compact subset $\mathcal{K} \subset \Omega_T$, there exists a continuous, nonnegative, increasing function

$$s \quad \mapsto \quad \omega_{data,\mathcal{K}}(s), \qquad \omega_{data,\mathcal{K}}(0) = 0$$

that can be determined a priori only in terms of the data and the distance from $\mathcal{K}$ to the parabolic boundary of Ω_T such that

$$|u(x_1,t_1) - u(x_2,t_2)| \le \omega_{data,\mathcal{K}}(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

for every pair of points $(x_i, t_i) \in \mathcal{K}$, $i = 1, 2$.

2. GENERAL REMARKS

It is well known that equations like (1.1) arise for example in the study of problems connected to phase transitions (for more details under this point of view, see [4], [6], and also the references therein).

The fact that the graph β considered here has only two jumps is not essential: actually any *finite* number of jumps would be acceptable, as will be clear in the sketch of the proof.

Moreover, even if we stated the result assuming a very general elliptic operator in divergence form, for the sake of simplicity all the calculations will be carried through for the Laplacian; this is because all the estimates needed in the proof do not depend on the precise form of the operator, but just on its strong ellipticity.

The problem we consider here has already been dealt with in [4], using a technique which gives the general result in the case of $N = 2$, but for $N \geq 3$ works only for the particular case of the Laplacian. So, in a sense, our work here can be seen as a proper generalization of that result. A lot of what we do here relies on the analogous lemmata proved in [4]. However, in the effort to keep this paper as self-contained as possible, we reproduce here most of the statements involved and also we develop all the relative calculations, whenever there is any (even slight) difference.

As we already explained in Section 1, once we consider the Laplacian instead of the more general elliptic operator, the weak solution u satisfies

$$\int_\Omega w(x,\tau)\varphi(x,\tau)\,dx\,\Big|_{t_0}^{t} + \int_{t_0}^{t}\int_\Omega [-w(x,\tau)\frac{\partial}{\partial\tau}\varphi(x,\tau) + \nabla_x u \nabla_x \varphi]\,dx\,d\tau = 0 \tag{2.1}$$

for all $\varphi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and for all intervals $[t_0, t] \subset (0, T]$ with $w \in \beta(u)$.

Let us now introduce the auxiliary function

$$v(x,t) = \beta_0(u(x,t)) = \begin{cases} \beta_1(u(x,t)) & \text{on } [u < 0] \\ 0 & \text{on } [u = 0] \\ \beta_2(u(x,t)) & \text{on } [0 < u < 1] \\ 1 & \text{on } [u = 1] \\ \beta_3(u(x,t)) & \text{on } [u > 1] \end{cases} \tag{2.2}$$

and set

$$w(x,t) = v(x,t) - \nu_1(x,t)\chi_{[v\leq 0]} + \nu_2(x,t)\chi_{[v\geq 1]}, \tag{2.3}$$

where

$$\nu_1(x,t) = \begin{cases} \nu_1 & (x,t) \in [v<0] \\ -w(x,t) & (x,t) \in [v=0] \end{cases} \tag{2.4}$$

and

$$\nu_2(x,t) = \begin{cases} 1+\nu_2 & (x,t) \in [v>0] \\ +w(x,t) & (x,t) \in [v=1]. \end{cases} \tag{2.5}$$

It is easy to verify that if $u \in W_2^{1,1}(\Omega_T)$, then also $v \in W_2^{1,1}(\Omega_T)$, and so it is enough to prove the continuity of v.

Substituting (2.3) into (2.1), we obtain

$$\begin{aligned} &\int_\Omega (v(x,t) - \nu_1(x,t)\chi_{[v\le 0]} + \nu_2(x,t)\chi_{[v\ge 1]})\varphi(x,\tau)\,dx\,\Big|_{t_0}^{t} \\ &+ \int_{t_0}^{t}\int_\Omega \Big[-(v(x,t) - \nu_1(x,t)\chi_{[v\le 0]} + \nu_2(x,t)\chi_{[v\ge 1]})\frac{\partial}{\partial\tau}\varphi(x,\tau) \\ &\qquad + \nabla_x v \nabla_x \varphi\Big]\,dx\,d\tau = 0 \end{aligned} \tag{2.6}$$

for all $\varphi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and for all intervals $[t_0,t] \subset (0,T]$. This can be seen as the weak formulation of

$$\frac{\partial}{\partial t}\beta(v) - \Delta v \ni 0 \qquad \text{in } \mathcal{D}'(\Omega_T),$$

where β is the maximal monotone graph

$$\beta(s) = \begin{cases} s-\nu_1 & s<0 \\ [-\nu_1, 0] & s=0 \\ s & 0<s<1 \\ [1, 1+\nu_2] & s=1 \\ s+\nu_2 & s>1. \end{cases} \tag{2.7}$$

In the following, we will assume β as in (2.7) for the sake of simplicity. Moreover, it is evident that the previous scaling for s, in particular the fact that the jumps occur at $s=0$ and at $s=1$, leads to no loss of generality.

Since the calculations in the proof are rather involved, let us try and sketch the main ideas. Let's consider the parabolic cylinder $Q(\rho,\theta\rho^2) \subset \Omega_T$ given by

$$Q(\rho,\theta\rho^2) = K_\rho \times (-\theta\rho^2, 0), \qquad \theta \in \mathbf{N},\ \theta \ge 1$$

(see also the next section), and let's partition it into smaller subcylinders of the kind

$$Q_i(\rho,\rho^2) = K_\rho \times (-(i+1)\rho^2, -i\rho^2), \quad i = 0,1,2,\dots,\theta-1, \quad i \in \mathbf{N}.$$

Let $\tilde{Q}$ be one of these subcylinders. If we set

$$\mu_- = \inf_{Q(2\rho^2, 2\theta\rho^2)} u, \quad \mu_+ = \sup_{Q(2\rho^2, 2\theta\rho^2)} u, \quad \omega \geq \mu_+ - \mu_-, \tag{2.8}$$

the key step of the proof will be to show that in a proper subset of $\tilde{Q}$ the solution u satisfies $u \geq \mu_- + \frac{\omega}{2^{s_0}}$, where s_0 is a proper integer such that $s_0 \geq 2$. Then we can apply the logarithmic lemma of [4] and conclude that $u \geq \mu_- + \frac{\omega}{2^{s_1}}$ for a proper s_1 in a suitable subcylinder, whose upper base is contained in the upper base of $Q(\rho, \theta\rho^2)$.

This gives as an immediate consequence that there exists a family of shrinking cylinders Q_n around $(0, 0)$ such that the essential oscillation ω_n of u in Q_n tends to zero as n tends to infinity in a way controlled just by the data (a similar idea is developed in [2] for the boundary continuity of the solution of a singular parabolic equation with β exhibiting just a single jump).

In the following we want to study the continuity of u at the point $(x_0, t_0) \in \Omega \times (0, T)$. However, it is always possible to translate the axes such that $(x_0, t_0) = (0, 0)$. Therefore, for the sake of simplicity, all the statements relative to the continuity result will be given for $(0, 0)$, assuming that the point belongs to the parabolic domain we are working in.

As for (2.8), thanks to the particular structure of the graph β we are considering, there is no loss of generality in assuming $\mu_+ > 1$ and $\mu_- < 0$; otherwise, we would be working with a graph like the one studied in [1] and the continuity would directly follow from that result.

Moreover, we can assume $\mu_- \simeq 0$ and $\mu_+ \simeq 1$, since a possible loss of continuity cannot occur in the regions $[u < 0]$ or $[u > 1]$, where by classical results (see [5]) the solution is known to be continuous.

3. PRELIMINARY ESTIMATES

In the following we present all the preliminary estimates we will need in the proof of Theorem 1.2. Many of the results given here have already been stated and proved in [4] under slightly different assumptions; therefore, we will mostly concentrate on the differences, referring for the rest to [4], when no difference occurs.

For $\rho > 0$ denote by K_ρ the cube of wedge 2ρ centered at the origin; i.e.,

$$K_\rho \equiv \{x \in \mathbf{R}^N : \max_{1 \leq i \leq N} |x_i| < \rho\}.$$

We let $[y + K_\rho]$ denote the cube centered at y and congruent to K_ρ; i.e.,

$$[y + K_\rho] \equiv \{x \in \mathbf{R}^N : \max_{1 \leq i \leq N} |x_i - y_i| < \rho\}.$$

For $\theta > 0$ denote by $Q(\rho, \theta\rho^2)$ the cylinder of cross section K_ρ, height $\theta\rho^2$, and vertex at the origin; i.e., $Q(\rho, \theta\rho^2) \equiv K_\rho \times (-\theta\rho^2, 0)$. For a point $(y, s) \in \mathbf{R}^{N+1}$, let $[(y, s) + Q(\rho, \theta\rho^2)]$ be the cylinder of vertex at (y, s) and congruent to $Q(\rho, \theta\rho^2)$; i.e.,

$$[(y, s) + Q(\rho, \theta\rho^2)] \equiv [y + K_\rho] \times (s - \theta\rho^2, s).$$

For $k \in \mathbf{R}$ the truncations $(u - k)_+$ and $(u - k)_-$ are defined by

$$(u-k)_+ \equiv \max\{u-k, 0\}; \qquad (u-k)_- \equiv \max\{k-u, 0\}.$$

We have the following:

Proposition 3.1. *There exists a constant* $\gamma = \gamma(data)$ *such that for every* $k \in [0, 1]$ *and for every cylinder*

$$[(y, s) + Q(\sigma\rho, \sigma\theta\rho^2)] \subset [(y, s) + Q(\rho, \theta\rho^2)], \qquad \sigma \in (0, 1),$$

we have

$$\begin{aligned}
&\sup_{s-\sigma\theta\rho^2 \le t \le s} \Big(\int_{[y+K_{\sigma\rho}]} |(u-k)_-|^2(x,t)\,dx + \nu_1 k |[y + K_{\sigma\rho}] \cap \{u(\cdot, t) \le 0\}| \Big) \\
&\qquad\qquad + \int\!\!\int_{(y,s)+Q(\sigma\rho,\sigma\theta\rho^2)} |\nabla_x (u-k)_-|^2\,dx\,d\tau \\
&\quad \le \gamma \left(\frac{1}{(1-\sigma)\theta\rho^2} + \frac{1}{(1-\sigma)^2\rho^2} \right) \int\!\!\int_{[(y,s)+Q(\rho,\theta\rho^2)]} |(u-k)_-|^2\,dx\,d\tau \\
&\qquad\qquad + \frac{2\gamma\nu_1}{(1-\sigma)\theta\rho^2} \int\!\!\int_{[(y,s)+Q(\rho,\theta\rho^2)]} (u-k)_-\,dx\,d\tau;
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\sup_{s-\sigma\theta\rho^2 \le t \le s} \Big(\int_{[y+K_{\sigma\rho}]} |(u-k)_+|^2(x,t)\,dx + (1+\nu_2)(1-k) |[y + K_{\sigma\rho}] \cap \{u(\cdot, t) \ge 1\}| \Big) \\
&\qquad\qquad + \int\!\!\int_{(y,s)+Q(\sigma\rho,\sigma\theta\rho^2)} |\nabla_x (u-k)_+|^2\,dx\,d\tau \\
&\quad \le \gamma \Big(\frac{1}{(1-\sigma)\theta\rho^2} + \frac{1}{(1-\sigma)^2\rho^2} \Big) \int\!\!\int_{[(y,s)+Q(\rho,\theta\rho^2)]} |(u-k)_+|^2\,dx\,d\tau \\
&\qquad\qquad + \frac{2\gamma(1+\nu_2)}{(1-\sigma)\theta\rho^2} \int\!\!\int_{[(y,s)+Q(\rho,\theta\rho^2)]} (u-k)_+\,dx\,d\tau;
\end{aligned} \tag{3.2}$$

moreover,

$$\begin{aligned}
&\sup_{s-\sigma\theta\rho^2 \le t \le s} \Big(\int_{[y+K_{\sigma\rho}]} |(u-k)_-|^2(x,t)\,dx + \nu_1 k |[y + K_{\sigma\rho}] \cap \{u(\cdot, t) \le 0\}| \Big) \\
&\qquad\qquad + \int\!\!\int_{(y,s)+Q(\sigma\rho,\sigma\theta\rho^2)} |\nabla_x (u-k)_-|^2\,dx\,d\tau
\end{aligned}$$

$$\leq \gamma\Big(\frac{1}{(1-\sigma)\theta\rho^2}+\frac{1}{(1-\sigma)^2\rho^2}\Big)\int\int_{[(y,s)+Q(\rho,\theta\rho^2)]}|(u-k)_-|^2\,dx\,d\tau$$
$$+\frac{2\gamma k}{(1-\sigma)\theta\rho^2}|\{u(x,t)\leq 0\}\cap[(y,s)+Q(\rho,\theta\rho^2)]|; \tag{3.3}$$

$$\sup_{s-\sigma\theta\rho^2\leq t\leq s}\Big(\int_{[y+K_{\sigma\rho}]}|(u-k)_+|^2(x,t)\,dx+(1+\nu_2)(1-k)|[y+K_{\sigma\rho}]\cap\{u(\cdot,t)\geq 1\}|\Big)$$
$$+\int\int_{(y,s)+Q(\sigma\rho,\sigma\theta\rho^2)}|\nabla_x(u-k)_+|^2\,dx\,d\tau$$
$$\leq \gamma\left(\frac{1}{(1-\sigma)\theta\rho^2}+\frac{1}{(1-\sigma)^2\rho^2}\right)\int\int_{[(y,s)+Q(\rho,\theta\rho^2)]}|(u-k)_+|^2\,dx\,d\tau$$
$$+\frac{2\gamma(1-k)}{(1-\sigma)\theta\rho^2}|\{u(x,t)\geq 1\}\cap[(y,s)+Q(\rho,\theta\rho^2)]|. \tag{3.4}$$

Proof. Taking into account the expression of β, as we have already seen, the weak formulation of our equation becomes

$$\int_\Omega (u(x,\tau)-\nu_1(x,\tau)\chi_{[u\leq 0]}+\nu_2(x,\tau)\chi_{[u\geq 1]})\varphi(x,\tau)\,dx\Big|_{t_0}^{t}$$
$$+\int_{t_0}^{t}\int_\Omega[-(u(x,\tau)-\nu_1(x,\tau)\chi_{[u\leq 0]}+\nu_2(x,\tau)\chi_{[u\geq 1]})\frac{\partial\varphi}{\partial\tau}+\nabla_x u\nabla_x\varphi]\,dx\,d\tau=0$$

for all $\varphi\in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and all intervals $[t_0,t]\subset(0,T]$. We recall that

$$\nu_1(x,t)=\begin{cases}\nu_1 & (x,t)\in[u<0]\\ -w_1(x,t) & (x,t)\in[u=0],\end{cases}$$

$$\nu_2(x,t)=\begin{cases}1+\nu_2 & (x,t)\in[u>1]\\ w_2(x,t) & (x,t)\in[u=1],\end{cases}$$

where w_1 and w_2 are respectively a selection out of $[-\nu_1,0]$ and $[1,1+\nu_2]$, and $0\leq w_1(x,t)\leq\nu_1$ and $1\leq w_2(x,t)\leq 1+\nu_2$. First of all, let us remark that we can rewrite the previous equation as

$$-\int_\Omega \nu_1(x,\tau)\chi_{[u\leq 0]}\varphi(x,\tau)\,dx\Big|_{t_0}^{t}+\int_\Omega\nu_2(x,\tau)\chi_{[u\geq 1]}\varphi(x,\tau)\,dx\Big|_{t_0}^{t}$$
$$+\int_{t_0}^{t}\int_\Omega[\frac{\partial u}{\partial\tau}\varphi+\nabla_x u\nabla_x\varphi]\,dx\,d\tau \tag{3.5}$$
$$+\int_{t_0}^{t}\int_\Omega(\nu_1(x,\tau)\chi_{[u\leq 0]}-\nu_2(x,\tau)\chi_{[u\geq 1]})\frac{\partial\varphi}{\partial\tau}\,dx\,d\tau=0.$$

Now, up to translation, we may assume that (y,s) coincides with the origin. Let us consider $x \mapsto \zeta_1(x)$, a nonnegative, piecewise-smooth cutoff function in K_ρ satisfying

$$\begin{cases} \zeta_1(x) \equiv 1 \text{ on } K_{\sigma\rho}, \quad \sigma \in (0,1) \\ \zeta_1(x) = 0 \text{ for } x \in \partial K_\rho \\ |\nabla_x \zeta_1(x)| \le \frac{1}{(1-\sigma)\rho}. \end{cases}$$

Moreover, we denote by $t \mapsto \zeta_2(t)$ the cutoff function defined by

$$\zeta_2(t) = \begin{cases} 0 \quad \text{for } t \in (-\infty, -\theta\rho^2) \\ \frac{t+\theta\rho^2}{(1-\sigma)\theta\rho^2} \qquad \text{for} \quad t \in (-\theta\rho^2, -\sigma\theta\rho^2) \\ 1 \quad \text{for } t \ge -\sigma\theta\rho^2. \end{cases}$$

If we now set $\varphi(x,t) = \pm(u-k)_\pm \zeta_1(x)^2 \zeta_2(t)^2$, with $k \in \mathbf{R}$, and substitute, we obtain

$$\begin{aligned} &- \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\le 0]}[\pm(u-k)_\pm \zeta_1(x)^2\zeta_2(\tau)^2]\,dx\,\Big|_{-\theta\rho^2}^{t} \\ &+ \int_{-\theta\rho^2}^{t}\int_{K_\rho} \nu_1(x,\tau)\chi_{[u\le 0]}\frac{\partial}{\partial\tau}[\pm(u-k)_\pm \zeta_1(x)^2\zeta_2(\tau)^2]\,dx\,d\tau \\ &+ \int_{K_\rho} \nu_2(x,\tau)\chi_{[u\ge 1]}[\pm(u-k)_\pm \zeta_1(x)^2\zeta_2(\tau)^2]\,dx\,\Big|_{-\theta\rho^2}^{t} \\ &- \int_{-\theta\rho^2}^{t}\int_{K_\rho} \nu_2(x,\tau)\chi_{[u\ge 1]}\frac{\partial}{\partial\tau}[\pm(u-k)_\pm \zeta_1(x)^2\zeta_2(\tau)^2]\,dx\,d\tau \\ &+\int_{-\theta\rho^2}^{t}\int_{K_\rho}[\pm\frac{\partial u}{\partial\tau}(u-k)_\pm\zeta_1(x)^2\zeta_2(\tau)^2 \pm \nabla_x u \nabla_x((u-k)_\pm\zeta_1(x)^2\zeta_2(\tau)^2)]dx\,d\tau = 0. \end{aligned}$$

We now put

$$\begin{aligned} \Phi_1^\pm(k,-\theta\rho^2,t,\zeta) &= \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\le 0]}[\pm(u-k)_\pm \zeta_1(x)^2\zeta_2(\tau)^2]\,dx\,\Big|_{-\theta\rho^2}^{t} \\ &- \int_{-\theta\rho^2}^{t}\int_{K_\rho} \nu_1(x,\tau)\chi_{[u\le 0]}\frac{\partial}{\partial\tau}[\pm(u-k)_\pm \zeta_1(x)^2\zeta_2(\tau)^2]\,dx\,d\tau \end{aligned}$$

and

$$\begin{aligned} \Phi_2^\pm(k,-\theta\rho^2,t,\zeta) &= -\int_{K_\rho} \nu_2(x,\tau)\chi_{[u\ge 1]}[\pm(u-k)_\pm \zeta_1(x)^2\zeta_2(\tau)^2]\,dx\,\Big|_{-\theta\rho^2}^{t} \\ &+ \int_{-\theta\rho^2}^{t}\int_{K_\rho} \nu_2(x,\tau)\chi_{[u\ge 1]}\frac{\partial}{\partial\tau}[\pm(u-k)_\pm \zeta_1(x)^2\zeta_2(\tau)^2]\,dx\,d\tau \end{aligned}$$

with $t \in (-\theta\rho^2, 0)$ and rewrite the previous relation as

$$\int_{-\theta\rho^2}^{t} \int_{K_\rho} [\pm \frac{\partial u}{\partial \tau}(u-k)_\pm \zeta_1(x)^2 \zeta_2(\tau)^2 \pm \nabla_x u \nabla_x ((u-k)_\pm \zeta_1(x)^2 \zeta_2(\tau)^2)]\, dx\, d\tau$$
$$= \Phi_1^\pm(k, -\theta\rho^2, t, \zeta) + \Phi_2^\pm(k, -\theta\rho^2, t, \zeta).$$

We now have

$$\int_{-\theta\rho^2}^{t} \int_{K_\rho} \pm \frac{\partial u}{\partial \tau}(u-k)_\pm \zeta_1^2 \zeta_2^2 \, dx\, d\tau = \frac{1}{2} \int_{-\theta\rho^2}^{t} \int_{K_\rho} (\frac{\partial}{\partial \tau} |(u-k)_\pm|^2) \zeta_1^2 \zeta_2^2 \, dx\, d\tau$$
$$= \frac{1}{2} \int_{K_\rho} |(u-k)_\pm|^2 \zeta_1^2 \zeta_2^2 \, dx \Big|_{-\theta\rho^2}^{t} - \int_{-\theta\rho^2}^{t} \int_{K_\rho} |(u-k)_\pm|^2 \zeta_1^2 \zeta_2 \frac{\partial \zeta_2}{\partial \tau}\, dx\, d\tau,$$

and the first term calculated at $t = -\theta\rho^2$ is zero. Furthermore,

$$\int_{-\theta\rho^2}^{t} \int_{K_\rho} \pm \nabla_x u \nabla_x ((u-k)_\pm \zeta_1^2 \zeta_2^2)\, dx\, d\tau$$
$$= \int_{-\theta\rho^2}^{t} \int_{K_\rho} |\nabla_x (u-k)_\pm|^2 \zeta_1^2 \zeta_2^2 \, dx\, d\tau \pm 2 \int_{-\theta\rho^2}^{t} \int_{K_\rho} \zeta_1 \zeta_2^2 (u-k)_\pm \nabla_x u \nabla_x \zeta_1 \, dx\, d\tau.$$

Therefore,

$$\frac{1}{2} \int_{K_\rho} |(u-k)_\pm|^2 \zeta_1^2 \zeta_2^2 \, dx \Big|_{-\theta\rho^2}^{t} + \int_{-\theta\rho^2}^{t} \int_{K_\rho} |\nabla_x (u-k)_\pm|^2 \zeta_1^2 \zeta_2^2 \, dx\, d\tau$$
$$\leq \int_{-\theta\rho^2}^{t} \int_{K_\rho} |(u-k)_\pm|^2 \zeta_1^2 \zeta_2 \frac{\partial \zeta_2}{\partial \tau}\, dx\, d\tau$$
$$+2 \int_{-\theta\rho^2}^{t} \int_{K_\rho} \zeta_1 \zeta_2^2 |(u-k)_\pm| |\nabla_x (u-k)_\pm| |\nabla_x \zeta_1|\, dx\, d\tau + \Phi_1^\pm + \Phi_2^\pm.$$

By Hölder's inequality,

$$\int_{-\theta\rho^2}^{t} \int_{K_\rho} \zeta_1 \zeta_2^2 |(u-k)_\pm| |\nabla_x (u-k)_\pm| |\nabla_x \zeta_1|\, dx\, d\tau$$
$$\leq \epsilon \int_{-\theta\rho^2}^{t} \int_{K_\rho} |\nabla_x (u-k)_\pm|^2 \zeta_1^2 \zeta_2^2 \, dx\, d\tau +$$
$$+ C(\epsilon) \int_{-\theta\rho^2}^{t} \int_{K_\rho} \zeta_2^2 |(u-k)_\pm|^2 |\nabla_x \zeta_1|^2 \, dx\, d\tau.$$

Therefore, relying on the properties of the test functions, we can conclude that

$$\sup_{-\sigma\theta\rho^2 \leq t \leq 0} \int_{K_{\sigma\rho}} |(u-k)_\pm|^2 (x,t)\, dx + \int\int_{Q(\sigma\rho, \sigma\theta\rho^2)} |\nabla_x (u-k)_\pm|^2 \, dx\, d\tau$$

$$\leq \gamma \left(\frac{1}{(1-\sigma)\theta\rho^2} + \frac{1}{(1-\sigma)^2\rho^2} \right) \int\int_{Q(\rho,\theta\rho^2)} |(u-k)_\pm|^2 \, dx \, d\tau + \gamma\Phi_1^\pm + \gamma\Phi_2^\pm, \tag{3.6}$$

where γ is a proper constant depending only upon the data. Let us now concentrate on $\Phi_1^\pm$ and $\Phi_2^\pm$ and show how we can give a finer estimate of these two quantities when $k \in [0,1]$.

Let us first take

$$\Phi_1^\pm(k, -\theta\rho^2, t, \zeta) = \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\leq 0]}[\pm(u-k)_\pm \zeta_1^2\zeta_2^2] \, dx \Big|_{-\theta\rho^2}^{t}$$
$$- \int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\leq 0]} \frac{\partial}{\partial\tau}[\pm(u-k)_\pm \zeta_1(x)^2\zeta_2(\tau)^2] \, dx \, d\tau.$$

If $k \geq 0$ it is immediate to conclude that $\Phi_1^+ = 0$, and the same holds for Φ_1^- when $k = 0$. Let us now take $k > 0$ and concentrate on Φ_1^-. We have

$$\begin{aligned}
\Phi_1^- &= - \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\leq 0]}[(u-k)_- \zeta_1^2\zeta_2^2] \, dx \Big|_{-\theta\rho^2}^{t} \\
&\quad + \int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\leq 0]} \frac{\partial}{\partial\tau}[(u-k)_- \zeta_1^2\zeta_2^2] \, dx \, d\tau \\
&= - \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\leq 0]}(k-u)\zeta_1^2\zeta_2^2 \, dx \Big|_{-\theta\rho^2}^{t} \\
&\quad + \int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\leq 0]}(k-u)\zeta_1^2 \frac{\partial\zeta_2^2}{\partial\tau} \, dx \, d\tau \\
&\quad + \int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\leq 0]}\zeta_1^2\zeta_2^2 \frac{\partial(k-u)}{\partial\tau} \, dx \, d\tau \\
&= - \int_{K_\rho} \nu_1(x,\tau)u_- \zeta_1^2\zeta_2^2 \, dx \Big|_{-\theta\rho^2}^{t} - k \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\leq 0]}\zeta_1^2\zeta_2^2 \, dx \Big|_{-\theta\rho^2}^{t} \\
&\quad + \int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\leq 0]}\zeta_1^2(k-u) \frac{\partial\zeta_2^2}{\partial\tau} \, dx \, d\tau \\
&\quad + \int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_1(x,\tau)\zeta_1^2\zeta_2^2 \frac{\partial u_-}{\partial\tau} \, dx \, d\tau \\
&\leq -k \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\leq 0]}\zeta_1^2\zeta_2^2 \, dx \Big|_{-\theta\rho^2}^{t} \qquad (3.7) \\
&\quad + 2 \int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_1(x,\tau)\zeta_1^2(k-u)\zeta_2 \frac{\partial\zeta_2}{\partial\tau} \, dx \, d\tau
\end{aligned}$$

$$- 2 \int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_1 \zeta_1^2 u_- \zeta_2 \frac{\partial \zeta_2}{\partial \tau} \, dx \, d\tau,$$

and since the last quantity is negative we conclude

$$\begin{aligned} \Phi_1^-(k, -\theta\rho^2, t, \zeta) &\le 2\nu_1 \int_{-\theta\rho^2}^{t} \int_{K_\rho} (u-k)_- \frac{\partial \zeta_2}{\partial \tau} \, dx \, d\tau \\ &- k \int_{K_\rho} \nu_1(x,\tau) \chi_{[u\le 0]} \zeta_1^2 \zeta_2^2 \, dx \, \Big|_{-\theta\rho^2}^{t} \\ &\le \frac{2\nu_1}{(1-\sigma)\theta\rho^2} \int_{-\theta\rho^2}^{0} \int_{K_\rho} (u-k)_- \, dx \, d\tau - k \int_{K_\rho} \nu_1(x,\tau) \chi_{[u\le 0]} \zeta_1^2 \zeta_2^2 \, dx \, \Big|_{-\theta\rho^2}^{t} \end{aligned} \tag{3.8}$$

for any $k \ge 0$. Let's now take

$$\begin{aligned} \Phi_2^{\pm}(k, -\theta\rho^2, t, \zeta) = &- \int_{K_\rho} \nu_2(x,\tau) \chi_{[u\ge 1]} [\pm (u-k)_\pm \zeta_1^2 \zeta_2^2] \, dx \, \Big|_{-\theta\rho^2}^{t} \\ &+ \int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_2(x,\tau) \chi_{[u\ge 1]} \frac{\partial}{\partial \tau} [\pm (u-k)_\pm \zeta_1^2 \zeta_2^2] \, dx \, d\tau. \end{aligned}$$

If $k \le 1$ it is immediate to conclude that $\Phi_2^- = 0$, and the same holds for Φ_2^+ when $k = 1$. In the case of $k < 1$, estimates analogous to the ones given above hold. So we can conclude that for any $k \le 1$ we have

$$\begin{aligned} \Phi_2^+(k, -\theta\rho^2, t, \zeta) \le &\frac{2(1+\nu_2)}{(1-\sigma)\theta\rho^2} \int_{-\theta\rho^2}^{0} \int_{K_\rho} (u-k)_+ \, dx \, d\tau \\ &-(1-k) \int_{K_\rho} \nu_2(x,\tau) \chi_{[u\ge 1]} \zeta_1^2 \zeta_2^2 \, dx \, \Big|_{-\theta\rho^2}^{t}. \end{aligned} \tag{3.9}$$

If we now substitute (3.9) and (3.8) into (3.6), shift on the left-hand side the corresponding negative term, and take the supremum over $t \in (-\theta\rho^2, 0)$ for this same term, we come up with (3.1) and (3.2).

To prove (3.3) it is enough to estimate in a finer way the term

$$\int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_1(x,\tau) \chi_{[u\le 0]} \zeta_1^2 (k-u) \frac{\partial \zeta_2^2}{\partial \tau} \, dx \, d\tau$$

in (3.7). Namely,

$$\begin{aligned} &\int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_1(x,\tau) \chi_{[u\le 0]} \zeta_1^2 (k-u) \frac{\partial \zeta_2^2}{\partial \tau} \, dx \, d\tau \\ &\le 2k \int_{-\theta\rho^2}^{0} \int_{K_\rho} \nu_1 \chi_{[u\le 0]} \zeta_1 \zeta_2 \frac{\partial \zeta_2}{\partial \tau} \, dx \, d\tau + 2 \int_{-\theta\rho^2}^{0} \int_{K_\rho} \nu_1 \zeta_1 \zeta_2 u_- \frac{\partial \zeta_2}{\partial \tau} \, dx \, d\tau, \end{aligned}$$

and the last term cancels the equivalent one with the negative sign. The same procedure is to be followed for (3.4). □

Remark 3.2. The main difference with the usual parabolic energy inequality lies in the presence of the L^1 term on the right-hand side of (3.1)–(3.2) and (3.3)–(3.4). If k is close to zero or to one, the term $(u-k)_\pm$ is larger than $|(u-k)_\pm|^2$, and therefore (3.1)–(3.2) and (3.3)–(3.4) give poorer estimates than the corresponding ones considered in [5], which in turn yield the Hölder continuity of the solution. Therefore, the main difficulty will be in dealing with the regions where $u \simeq 0$ or $u \simeq 1$.

Remark 3.3. If $k < 0$, it is easy to see that we reduce to

$$\begin{aligned}
&\sup_{s-\sigma\theta\rho^2 \le t \le s} \Big(\int_{[y+K_{\sigma\rho}]} |(u-k)_-|^2(x,t)\,dx + \nu_1 k |[y+K_{\sigma\rho}] \cap \{u(\cdot,t) \le 0\}| \Big) \\
&+ \int\int_{(y,s)+Q(\sigma\rho,\sigma\theta\rho^2)} |\nabla_x (u-k)_-|^2\,dx\,d\tau \qquad (3.10)\\
&\le \gamma \Big(\frac{1}{(1-\sigma)\theta\rho^2} + \frac{1}{(1-\sigma)^2\rho^2} \Big) \int\int_{[(y,s)+Q(\rho,\theta\rho^2)]} |(u-k)_-|^2\,dx\,d\tau,
\end{aligned}$$

and an analogous estimate holds for $(u-k)_+$ if $k > 1$. This is no surprise considering what we said above about the continuity of u in the sets $[u < 0]$ and $[u > 1]$.

Let us now set

$$H_k^\pm = \sup_{[(y,s)+Q(\rho,\theta\rho^2)]} (u-k)_\pm,$$

take $c \in (0, H_k^\pm)$, and define

$$\Psi(H_k^\pm, (u-k)_\pm, c) = \ln^+ \Big(\frac{H_k^\pm}{H_k^\pm - (u-k)_\pm + c} \Big).$$

We have

Proposition 3.4. *There exists a constant* $\gamma = \gamma(data)$ *such that for every* $k \in [0,1]$ *and for every cylinder*

$$[(y,s)+Q(\sigma\rho,\theta\rho^2)] \subset [(y,s)+Q(\rho,\theta\rho^2)], \qquad \sigma \in (0,1),$$

we have

$$\begin{aligned}
&\sup_{s-\theta\rho^2 \le t \le s} \Big(\int_{[y+K_{\sigma\rho}]} |(u-k)_-|^2(x,t)\,dx + \nu_1 k |[y+K_{\sigma\rho}] \cap \{u(\cdot,t) \le 0\}| \Big) \\
&+ \int\int_{(y,s)+Q(\sigma\rho,\theta\rho^2)} |\nabla_x (u-k)_-|^2\,dx\,d\tau \qquad (3.11)
\end{aligned}$$

$$\leq \gamma\Big[\frac{1}{(1-\sigma)^2\rho^2}\int\int_{[(y,s)+Q(\rho,\theta\rho^2)]}|(u-k)_-|^2\,dx\,d\tau$$
$$+\int_{[y+K_\rho]}|(u-k)_-|^2(x,s-\theta\rho^2)\,dx+\nu_1\int_{[y+K_\rho]}(u-k)_-(x,s-\theta\rho^2)\,dx\Big];$$
$$\sup_{s-\theta\rho^2\leq t\leq s}\Big(\int_{[y+K_{\sigma\rho}]}|(u-k)_+|^2(x,t)\,dx$$
$$+(1+\nu_2)(1-k)|[y+K_{\sigma\rho}]\cap\{u(\cdot,t)\geq 1\}|\Big)$$
$$+\int\int_{(y,s)+Q(\sigma\rho,\theta\rho^2)}|\nabla_x(u-k)_+|^2\,dx\,d\tau$$
$$\leq \gamma\Big[\frac{1}{(1-\sigma)^2\rho^2}\int\int_{[(y,s)+Q(\rho,\theta\rho^2)]}|(u-k)_+|^2\,dx\,d\tau \tag{3.12}$$
$$+\int_{[y+K_\rho]}|(u-k)_+|^2(x,s-\theta\rho^2)\,dx+(1+\nu_2)\int_{[y+K_\rho]}(u-k)_+(x,s-\theta\rho^2)\,dx\Big];$$

moreover,

$$\sup_{s-\theta\rho^2\leq t\leq s}\int_{K_{\sigma\rho}}\Psi^2(x,t)\,dx\leq\gamma\Big[\int_{K_\rho}\Psi^2(x,s-\theta\rho^2)\,dx \tag{3.13}$$
$$+\int_{K_\rho}\Psi(x,s-\theta\rho^2)\,dx+\frac{1}{(1-\sigma)^2\rho^2}\int\int_{[(y,s)+Q(\rho,\theta\rho^2)]}\Psi(x,\tau)\,dx\,d\tau\Big].$$

Proof. Once again, as in the previous proof, we may assume that (y,s) coincides with the origin. Let us start with the proof of (3.11) and (3.12). Take $\varphi=\pm(u-k)_\pm\zeta_1(x)^2$ with ζ_1 as in the proof of Proposition 3.1. Substituting in equation (3.5), we have

$$-\int_{K_\rho}\nu_1(x,\tau)\chi_{[u\leq 0]}[\pm(u-k)_\pm\zeta_1(x)^2]\,dx\Big|_{-\theta\rho^2}^{t}$$
$$+\int_{-\theta\rho^2}^{t}\int_{K_\rho}\nu_1(x,\tau)\chi_{[u\leq 0]}\frac{\partial}{\partial\tau}[\pm(u-k)_\pm\zeta_1(x)^2]\,dx\,d\tau$$
$$+\int_{K_\rho}\nu_2(x,\tau)\chi_{[u\geq 1]}[\pm(u-k)_\pm\zeta_1(x)^2]\,dx\Big|_{-\theta\rho^2}^{t}$$
$$-\int_{-\theta\rho^2}^{t}\int_{K_\rho}\nu_2(x,\tau)\chi_{[u\geq 1]}\frac{\partial}{\partial\tau}[\pm(u-k)_\pm\zeta_1(x)^2]\,dx\,d\tau$$
$$+\int_{-\theta\rho^2}^{t}\int_{K_\rho}[\pm\frac{\partial u}{\partial\tau}(u-k)_\pm\zeta_1(x)^2\pm\nabla_x u\nabla_x((u-k)_\pm\zeta_1(x)^2)]\,dx\,d\tau=0.$$

As before, we label the first two terms as $-\Phi_1^\pm$ and the third and fourth as $-\Phi_2^\pm$. Moreover,

$$\begin{aligned}&\int_{-\theta\rho^2}^t \int_{K_\rho} \pm\frac{\partial u}{\partial \tau}(u-k)_\pm \zeta_1(x)^2\,dx\,d\tau = \frac{1}{2}\int_{-\theta\rho^2}^t \int_{K_\rho} \frac{\partial}{\partial\tau}(|(u-k)_\pm|^2\zeta_1^2)\,dx\,d\tau\\ &= \frac{1}{2}\int_{K_\rho}(|(u-k)_\pm|^2\zeta_1^2)(x,t)\,dx - \frac{1}{2}\int_{K_\rho}(|(u-k)_\pm|^2\zeta_1^2)(x,-\theta\rho^2)\,dx\\ &\geq \frac{1}{2}\Big[\Big(\sup_{-\theta\rho^2\leq t\leq 0}\int_{K_{\sigma\rho}}|(u-k)_\pm|^2(x,t)\,dx\Big) - \int_{K_\rho}|(u-k)_\pm|^2(x,-\theta\rho^2)\,dx\Big]\end{aligned}$$

and

$$\begin{aligned}&\int_{-\theta\rho^2}^t \int_{K_\rho} \pm\nabla_x u\nabla_x((u-k)_\pm\zeta_1^2)\,dx\,d\tau = \int_{-\theta\rho^2}^t \int_{K_\rho} |\nabla_x(u-k)_\pm|^2\zeta_1^2\,dx\,d\tau\\ &\pm 2\int_{-\theta\rho^2}^t \int_{K_\rho} \nabla_x(u-k)_\pm(u-k)_\pm\zeta_1\nabla_x\zeta_1\,dx\,d\tau\\ &\geq \int_{-\theta\rho^2}^t \int_{K_{\sigma\rho}} |\nabla_x(u-k)_\pm|^2\,dx\,d\tau\\ &- 2\int_{-\theta\rho^2}^t \int_{K_\rho} |\nabla_x(u-k)_\pm|(u-k)_\pm\zeta_1|\nabla_x\zeta_1|\,dx\,d\tau.\end{aligned}$$

Hence, working as before, we come up with

$$\begin{aligned}&\sup_{-\theta\rho^2\leq t\leq 0}\int_{K_{\sigma\rho}}|(u-k)_\pm|^2(x,t)\,dx + \int_{-\theta\rho^2}^0\int_{K_{\sigma\rho}}|\nabla_x(u-k)_\pm|^2\,dx\,d\tau\\ &\leq \frac{\gamma}{(1-\sigma)^2\rho^2}\int_{-\theta\rho^2}^0\int_{K_\rho}|(u-k)_\pm|^2\,dxd\tau \qquad (3.14)\\ &+\gamma\int_{K_\rho}|(u-k)_\pm|^2(x,-\theta\rho^2)\,dx + \gamma\Phi_1^\pm + \gamma\Phi_2^\pm,\end{aligned}$$

where for the sake of simplicity we omitted the arguments in $\Phi_1^\pm$ and $\Phi_2^\pm$. Once again we have to estimate the last two terms. It is not difficult to see that, as before, $\Phi_1^+ = 0$ for $k \geq 0$ and $\Phi_1^- = 0$ for $k = 0$. On the other hand, if $k > 0$ we have

$$\Phi_1^- \leq \int_{K_\rho}\nu_1(u-k)_-(x,\tau)\,dx\Big|_{\tau=-\theta\rho^2} - k\int_{K_\rho}\nu_1(x,\tau)\chi_{[u\leq 0]}\zeta_1^2\,dx\Big|_{\tau=t}. \qquad (3.15)$$

Analogously, $\Phi_2^- = 0$ for $k \le 1$ and $\Phi_2^+ = 0$ for $k = 1$, and if $k < 1$ we have

$$\Phi_2^+ \le \int_{K_\rho} (1+\nu_2)(u-k)_+(x,\tau)\,dx\Big|_{\tau=-\theta\rho^2} - (1-k)\int_{K_\rho} \nu_2(x,\tau)\chi_{[u\ge 1]}\zeta_1^2\,dx\Big|_{\tau=t}. \tag{3.16}$$

Once more, if we substitute (3.15) and (3.16) into (3.14), shift on the left-hand side the corresponding negative term, and take the supremum over $t \in (-\theta\rho^2, 0)$ for this same term, we come up with (3.11) and (3.12).

Let us now come to the proof of (3.13). Take $\varphi = \pm\Psi\Psi'\zeta_1(x)^2$ with ζ_1 as in the proof of Proposition 3.1. Substituting in equation (3.5), we obtain

$$\begin{aligned}
&-\int_{K_\rho} \nu_1(x,\tau)\chi_{[u\le 0]}[\pm\Psi\Psi'\zeta_1(x)^2]\,dx\,\Big|_{-\theta\rho^2}^{t}\\
&+\int_{-\theta\rho^2}^{t}\int_{K_\rho} \nu_1(x,\tau)\chi_{[u\le 0]}\frac{\partial}{\partial\tau}[\pm\Psi\Psi'\zeta_1(x)^2]\,dx\,d\tau\\
&+\int_{K_\rho} \nu_2(x,\tau)\chi_{[u\ge 1]}[\pm\Psi\Psi'\zeta_1(x)^2]\,dx\,\Big|_{-\theta\rho^2}^{t}\\
&-\int_{-\theta\rho^2}^{t}\int_{K_\rho} \nu_2(x,\tau)\chi_{[u\ge 1]}\frac{\partial}{\partial\tau}[\pm\Psi\Psi'\zeta_1(x)^2]\,dx\,d\tau\\
&+\int_{-\theta\rho^2}^{t}\int_{K_\rho} [\pm\frac{\partial u}{\partial\tau}\Psi\Psi'\zeta_1(x)^2 \pm \nabla_x u\nabla_x(\Psi\Psi'\zeta_1(x)^2)]\,dx\,d\tau = 0.
\end{aligned}$$

We use the same notation as before for $\Phi_1^\pm$ and $\Phi_2^\pm$. It is easy to see that for Ψ to be positive, we need to take $(u-k)_\pm > c > 0$. Hence,

$$\begin{aligned}
&\int_{-\theta\rho^2}^{t}\int_{K_\rho} \pm\frac{\partial u}{\partial\tau}\Psi\Psi'\zeta_1^2\,dx\,d\tau = \int_{-\theta\rho^2}^{t}\int_{K_\rho} \frac{\partial(u-k)_\pm}{\partial\tau}\Psi\Psi'\zeta_1^2\,dx\,d\tau\\
&= \frac{1}{2}\int_{-\theta\rho^2}^{t}\int_{K_\rho} \frac{\partial(\Psi^2\zeta_1^2)}{\partial\tau}\,dx\,d\tau\\
&= \frac{1}{2}\int_{K_\rho}(\Psi^2\zeta_1^2)(x,t)\,dx - \frac{1}{2}\int_{K_\rho}(\Psi^2\zeta_1^2)(x,-\theta\rho^2)\,dx.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_{-\theta\rho^2}^{t}\int_{K_\rho} \pm\nabla_x u\nabla_x(\Psi\Psi'\zeta_1^2)\,dx\,d\tau\\
&= \int_{-\theta\rho^2}^{t}\int_{K_\rho} \nabla_x(u-k)_\pm\nabla_x(\Psi\Psi'\zeta_1^2)\,dx\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\theta\rho^2}^{t}\int_{K_\rho} (\Psi\Psi')'\zeta_1^2|\nabla_x(u-k)_\pm|^2\,dx\,d\tau\\
&+ 2\int_{-\theta\rho^2}^{t}\int_{K_\rho} \nabla_x(u-k)_\pm\Psi\Psi'\zeta_1\nabla_x\zeta_1\,dx\,d\tau\\
&= \int_{-\theta\rho^2}^{t}\int_{K_\rho} (1+\Psi)\Psi'^2\zeta_1^2|\nabla_x(u-k)_\pm|^2\,dx\,d\tau\\
&+ 2\int_{-\theta\rho^2}^{t}\int_{K_\rho} \nabla_x(u-k)_\pm\Psi\Psi'\zeta_1\nabla_x\zeta_1\,dx\,d\tau.
\end{aligned}
$$

Therefore, since $\Psi \geq 0$ and $0 \leq \Psi' \leq \frac{1}{c}$, we obtain

$$
\begin{aligned}
&\sup_{-\theta\rho^2\leq t\leq 0}\int_{K_{\sigma\rho}} \Psi^2(x,t)\,dx + \int_{-\theta\rho^2}^{0}\int_{K_\rho}(1+\Psi)\Psi'^2\zeta_1^2|\nabla_x(u-k)_\pm|^2\,dx\,d\tau\\
&\leq \frac{1}{2}\int_{K_\rho}\Psi^2(x,-\theta\rho^2)\,dx + 2\int_{-\theta\rho^2}^{0}\int_{K_\rho}|\nabla_x(u-k)_\pm|\Psi\Psi'\zeta_1|\nabla_x\zeta_1|\,dx\,d\tau\\
&\qquad + \Phi_1^\pm + \Phi_2^\pm\\
&\leq \frac{1}{2}\int_{K_\rho}\Psi^2(x,-\theta\rho^2)\,dx + \int_{-\theta\rho^2}^{0}\int_{K_\rho}|\nabla_x(u-k)_\pm|^2\Psi\Psi'^2\zeta_1^2\,dx\,d\tau\\
&+ \int_{-\theta\rho^2}^{0}\int_{K_\rho}\Psi|\nabla_x\zeta_1|^2\,dx\,d\tau + \Phi_1^\pm + \Phi_2^\pm\\
&\leq \frac{1}{2}\int_{K_\rho}\Psi^2(x,-\theta\rho^2)\,dx + \int_{-\theta\rho^2}^{0}\int_{K_\rho}|\nabla_x(u-k)_\pm|^2(1+\Psi)\Psi'^2\zeta_1^2\,dx\,d\tau\\
&+ \frac{1}{(1-\sigma)^2\rho^2}\int_{-\theta\rho^2}^{0}\int_{K_\rho}\Psi(x,\tau)\,dx\,d\tau + \Phi_1^\pm + \Phi_2^\pm.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sup_{-\theta\rho^2\leq t\leq 0}\int_{K_{\sigma\rho}} \Psi^2(x,t)\,dx\\
&\leq \gamma\int_{K_\rho}\Psi^2(x,-\theta\rho^2)\,dx + \frac{\gamma}{(1-\sigma)^2\rho^2}\int_{-\theta\rho^2}^{0}\int_{K_\rho}\Psi(x,\tau)\,dx\,d\tau + \Phi_1^\pm + \Phi_2^\pm.
\end{aligned}
$$

Let us now consider $\Phi_1^\pm$. We have

$$
\Phi_1^\pm = \int_{K_\rho} \pm\nu_1(x,\tau)\chi_{[u\leq 0]}\Psi\Psi'\zeta_1^2\,dx\,\Big|_{-\theta\rho^2}^{t}
$$

$$- \int_{-\theta\rho^2}^{t} \int_{K_\rho} \nu_1(x,\tau)\chi_{[u\le 0]} \frac{\partial}{\partial\tau} \left[\pm\Psi\Psi'\zeta_1^2\right] \, dx\, d\tau.$$

If $k \ge 0$, it is immediate to see that $\Phi_1^+ = 0$ since we must have $(u-k)_+ > c \ge 0$. If we now come to Φ_1^- and consider $k \le 0$, we must have $(u-k)_- > c$; that is, $u < k - c < 0$! Hence

$$\begin{aligned}\Phi_1^- &= -\int_{K_\rho} \nu_1\zeta_1^2\Psi\Psi'\, dx \Big|_{-\theta\rho^2}^{t} + \int_{-\theta\rho^2}^{t}\int_{K_\rho} \nu_1\zeta_1^2 \frac{\partial(\Psi\Psi')}{\partial\tau}\, dx\, d\tau \\ &= -\int_{K_\rho} \nu_1\zeta_1^2\Psi\Psi'\, dx \Big|_{-\theta\rho^2}^{t} + \int_{K_\rho} \nu_1\zeta_1^2\Psi\Psi'\, dx \Big|_{-\theta\rho^2}^{t} = 0.\end{aligned}$$

Analogously, $\Phi_2^- = 0$ for $k \le 1$ and $\Phi_2^+ = 0$ for $k \ge 1$, so we only have to estimate Φ_1^- for $k > 0$ and Φ_2^+ for $k < 1$. Now we can apply the same techniques used in the previous proofs and conclude that

$$\begin{aligned}\sup_{-\theta\rho^2\le t\le 0} \int_{K_{\sigma\rho}} \Psi^2(x,t)\, dx &\le \gamma\Big[\int_{K_\rho} \Psi^2(x,-\theta\rho^2)\, dx \\ &+ \int_{K_\rho} \Psi(x,-\theta\rho^2)\, dx + \frac{1}{(1-\sigma)^2\rho^2} \int\int_{Q(\rho,\theta\rho^2)} \Psi(x,\tau)\, dx\, d\tau\Big].\end{aligned}$$

□

Fix now $\theta > 0$, and consider a cylinder $[(y,s) + Q(2\rho, 2\theta\rho^2)] \subset \Omega_T$. Set

$$\mu_+ = \sup_{[(y,s)+Q(2\rho,2\theta\rho^2)]} u, \qquad \mu_- = \inf_{[(y,s)+Q(2\rho,2\theta\rho^2)]} u,$$

and denote by ω a positive number satisfying

$$\omega \ge \operatorname{osc}_{[(y,s)+Q(2\rho,2\theta\rho^2)]} u \equiv \mu_+ - \mu_-.$$

Let $\xi^\pm \in (0,1)$ and set

$$A^+_{\xi^+,\rho} \equiv \{(x,t) \in [(y,s) + Q(\rho,\theta\rho^2)] : \; u(x,t) > \mu_+ - \xi^+\omega\}, \tag{3.17}$$

$$A^-_{\xi^-,\rho} \equiv \{(x,t) \in [(y,s) + Q(\rho,\theta\rho^2)] : \; u(x,t) < \mu_- + \xi^-\omega\}. \tag{3.18}$$

Then we have the following:

Proposition 3.5. *Suppose that* $0 < \mu_+ - \xi^+\omega < \mu_+ - \frac{2}{3}\xi^+\omega < 1$. *Then there exists a number* ν^+ *depending upon the data and the numbers* ξ^+, θ, *and* ω *such that*

$$|A^+_{\xi^+,\rho}| < \nu^+ |Q(\rho,\theta\rho^2)| \tag{3.19}$$

implies

$$u(x,t) < \mu_+ - \frac{2}{3}\xi^+\omega \qquad \forall\, (x,t) \in [(y,s) + Q(\frac{\rho}{2}, \frac{\theta\rho^2}{2})]. \tag{3.20}$$

Remark 3.6. The proof shows that the number ν^+ is determined by

$$\nu^+ = C\frac{1}{\theta}\Big(\frac{\theta\xi^+\omega}{1+\theta\xi^+\omega}\Big)^{\frac{N+2}{2}}$$

with $C = C(\text{data})$.

Proposition 3.7. *Suppose that* $0 < \mu_- + \frac{2}{3}\xi^-\omega < \mu_- + \xi^-\omega < 1$*. Then there exists a number* ν^- *depending upon the data and the numbers* ξ^-, θ, *and* ω *such that*

$$|A^-_{\xi^-,\rho}| < \nu^-|Q(\rho,\theta\rho^2)| \tag{3.21}$$

implies

$$u(x,t) > \mu_- + \frac{2}{3}\xi^-\omega \qquad \forall\,(x,t) \in [(y,s) + Q(\frac{\rho}{2},\frac{\theta\rho^2}{2})]. \tag{3.22}$$

Remark 3.8. The proof shows that the number ν^- is determined by

$$\nu^- = C\frac{1}{\theta}\Big(\frac{\theta\xi^-\omega}{1+\theta\xi^-\omega}\Big)^{\frac{N+2}{2}}$$

with $C = C(\text{data})$.

Proof. Here we limit ourselves to the proof of (3.20), since (3.22) follows exactly in the same way. Moreover, as we have already done before, we can assume (y,s) to coincide with the origin. As in [4], let us consider the sequence of radii

$$\rho_n = \frac{\rho}{2} + \frac{\rho}{2^{n+1}}, \qquad \tilde\rho_n = \frac{\rho_n + \rho_{n+1}}{2} = \frac{\rho}{2} + \frac{3}{2}\frac{\rho}{2^{n+2}}$$

and the sequence of numbers

$$\xi_n = \frac{2}{3}\xi^+ + \frac{1}{3}\frac{\xi^+}{2^n}, \qquad k_n = \mu_+ - \xi_n\omega.$$

Due to the hypotheses above, $0 \le k_n \le 1$, for any $n \in \mathbf{N}$. Let us consider (3.2): since we will need to do it in the following, we can already take $\theta >> 1$; moreover, $\frac{1}{1-\sigma} < \frac{1}{(1-\sigma)^2}$ as $\sigma \in (0,1)$; finally, γ can take into account also ν_1 and ν_2, which in a sense are data of the problem. Therefore, we can rewrite (3.2) in the following way:

$$\sup_{-\sigma\theta\rho^2\le t\le 0}\int_{K_{\sigma\rho}} |(u-k_n)_+|^2(x,t)\,dx + \int\!\!\int_{Q(\sigma\rho,\sigma\theta\rho^2)} |\nabla_x(u-k_n)_+|^2\,dx\,d\tau$$
$$\le \frac{\gamma}{(1-\sigma)^2\rho^2}\Big[\quad \int\!\!\int_{Q(\rho,\theta\rho^2)} |(u-k_n)_+|^2\,dx\,d\tau$$

$$+\frac{1}{\theta}\int\int_{Q(\rho,\theta\rho^2)}(u-k_n)_+\,dx\,d\tau\Big]. \tag{3.23}$$

Let's now write (3.23) over the pair of cylinders

$$\tilde{Q}_n = K_{\tilde{\rho}_n}\times(-\theta\tilde{\rho}_n^2,0) \qquad Q_n = K_{\rho_n}\times(-\theta\rho_n^2,0)$$

with $1-\sigma = 2^{-(n+3)}$. Taking into account the definition of $A^+_{\xi^+,\rho}$, we have

$$\begin{aligned}
&\frac{\gamma}{(1-\sigma)^2\rho^2}\Big[\int\int_{Q(\rho,\theta\rho^2)}|(u-k_n)_+|^2\,dx\,d\tau + \frac{1}{\theta}\int\int_{Q(\rho,\theta\rho^2)}(u-k_n)_+\,dx\,d\tau\Big]\\
&\quad\le \gamma\frac{2^{2(n+3)}}{\frac{\rho^2}{4}(1+\frac{1}{2^n})^2}\Big[\sup_{Q_n}(u-k_n)_+^2\cdot|A^+_{\xi_n,\rho_n}| + \frac{1}{\theta}\sup_{Q_n}(u-k_n)_+\cdot|A^+_{\xi_n,\rho_n}|\Big]\\
&\quad\le \gamma\frac{4^{n+4}}{\rho^2\frac{2^n+1}{2^n}}[(\mu_+-(\mu_+-\xi_n\omega))^2+\frac{1}{\theta}(\mu_+-(\mu_+-\xi_n\omega))]|A^+_{\xi_n,\rho_n}|\\
&\quad\le \frac{4^{n+4}}{\rho^2}\xi_n^2\omega^2[1+\frac{1}{\theta\xi^+\omega}]|A^+_{\xi_n,\rho_n}| = \tilde{\gamma}\frac{4^n}{\rho^2}\xi_n^2\omega^2[1+\frac{1}{\theta\xi^+\omega}]|A^+_{\xi_n,\rho_n}|,
\end{aligned}$$

where $\tilde{\gamma}$ takes into account the purely numerical factor 4^4. From here on the proof proceeds exactly as in [4], to which we refer. □

Proposition 3.9. *Let the cylinder $[(y,s)+Q(\rho,\theta\rho^2)]$ be fixed, and let $\xi^\pm \in (0,1)$. If $0<\mu_+-\xi^+\omega<\mu_+-\frac{2}{3}\xi^+\omega<1$ and*

$$u(x,s-\theta\rho^2)\le\mu_+-\xi^+\omega \qquad \forall x\in[y+K_\rho],$$

then there exists a number $\nu^+\in(0,1)$ depending only on N and θ such that

$$|A^+_{\xi^+,\rho}|<\nu^+|Q(\rho,\theta\rho^2)| \tag{3.24}$$

implies

$$u(x,t)<\mu_+-\frac{2}{3}\xi^+\omega \qquad \forall\,(x,t)\in[(y,s)+Q(\frac{\rho}{2},\theta\rho^2)]. \tag{3.25}$$

On the other hand, if $0<\mu_-+\frac{2}{3}\xi^-\omega<\mu_-+\xi^-\omega<1$ and

$$u(x,s-\theta\rho^2)\ge\mu_-+\xi^-\omega \qquad \forall x\in[y+K_\rho],$$

then there exists a number $\nu^-\in(0,1)$ depending only on N and θ such that

$$|A^-_{\xi^-,\rho}|<\nu^-|Q(\rho,\theta\rho^2)| \tag{3.26}$$

implies

$$u(x,t)>\mu_-+\frac{2}{3}\xi^-\omega \qquad \forall\,(x,t)\in[(y,s)+Q(\frac{\rho}{2},\theta\rho^2)]. \tag{3.27}$$

Proof. As usual, let $(y,s) \equiv (0,0)$. Take now $n = 0,1,2,\dots$ and assume as in Proposition 3.5

$$\rho_n = \frac{\rho}{2} + \frac{\rho}{2^{n+1}}, \qquad \tilde{\rho}_n = \frac{\rho_n + \rho_{n+1}}{2} = \frac{\rho}{2} + \frac{3}{2}\frac{\rho}{2^{n+2}},$$

and

$$\xi_n = \frac{2}{3}\xi^+ + \frac{1}{3}\frac{\xi^+}{2^n}, \qquad k_n = \mu_+ - \xi_n\omega.$$

As before, due to the hypotheses above, $0 \le k_n \le 1$, for any $n \in \mathbf{N}$. Since

$$u(x, -\theta\rho^2) \le \mu_+ - \xi^+\omega, \qquad \forall x \in K_\rho,$$

the last integrals in (3.12) vanish. From here on the proof proceeds as in [4].

Analogous estimates hold if we work with $k_n = \mu_- + \xi_n\omega$ once we guarantee that $0 \le k_n \le 1$ (the same choices as above will do). □

The same remarks as in Remark 3.2 of [4] hold here.

Let $\xi^\pm \in (0,1)$ and set

$$A^+_{\xi^+,\rho}(t) \equiv \{x \in [y + K_\rho] : \ u(x,t) > \mu_+ - \xi^+\omega\}, \tag{3.28}$$

$$A^-_{\xi^-,\rho}(t) \equiv \{x \in [y + K_\rho] : \ u(x,t) < \mu_- + \xi^-\omega\}. \tag{3.29}$$

We have the following:

Proposition 3.10. *Let the cylinder $[(y,s) + Q(\rho, \theta\rho^2)]$ be fixed, and take $\xi_0^\pm \in (0,1)$ such that*

$$0 < \mu_+ - \xi_0^+\omega < \mu_+ - \frac{\xi_0^+}{4}\omega < 1$$

and

$$0 < \mu_- + \frac{\xi_0^-}{4}\omega < \mu_- + \xi_0^-\omega < 1.$$

If

$$u(x, s - \theta\rho^2) \le \mu_+ - \xi_0^+\omega \qquad \forall x \in [y + K_\rho], \tag{3.30}$$

then for every $\nu^+ \in (0,1)$, there exists a number $\xi^+ \in (0, \frac{1}{4}\xi_0^+)$ depending only on the data, ξ_0^+, and θ such that

$$|A^+_{\xi^+,\frac{1}{2}\rho}(t)| \le \nu^+ |K_{\frac{1}{2}\rho}|, \qquad \forall t \in (s - \theta\rho^2, s). \tag{3.31}$$

On the other hand, if

$$u(x, s - \theta\rho^2) \ge \mu_- + \xi_0^-\omega, \qquad \forall x \in [y + K_\rho], \tag{3.32}$$

then for every $\nu^- \in (0,1)$, *there exists a number* $\xi^- \in (0, \frac{1}{4}\xi_0^-)$ *depending only on the data,* ξ_0^-, *and* θ *such that*

$$|A^-_{\xi^-,\frac{1}{2}\rho}(t)| \le \nu^-|K_{\frac{1}{2}\rho}|, \qquad \forall t \in (s-\theta\rho^2, s). \tag{3.33}$$

Proof. As usual, let $(y,s) \equiv (0,0)$. We will prove (3.31), since (3.33) is completely analogous. Take Proposition 3.4 and put $k = \mu_+ - \xi_0^+\omega$, $\sigma = \frac{1}{2}$, $c = \xi^+\omega$, with $\xi^+ \in (0, \frac{1}{4}\xi_0^+)$ to be chosen. Since $k \in (0,1)$, (3.13) applies. Moreover, due to (3.30), the integrals on the right-hand side at time $t = -\theta\rho^2$ vanish. Therefore, we are left with

$$\sup_{-\theta\rho^2 \le t \le 0} \int_{K_{\frac{\rho}{2}}} \Psi^2(x,t)\,dx \le \frac{4\gamma}{\rho^2}\int_{-\theta\rho^2}^0 \int_{K_\rho} \Psi(x,\tau)\,dx\,d\tau$$

with

$$\Psi = \ln^+\Big(\frac{H_k^+}{H_k^+ - (u-k)_+ + \xi^+\omega}\Big).$$

Now put $4\gamma = \gamma^*$. It is easy to see that $\Psi \le \ln\frac{H_k^+}{\xi^+\omega}$, and since $H_k^+ = \sup_{Q(\rho,\theta\rho^2)}(u-k)_+$, we must have $H_k^+ \le \mu_+ - \mu_+ + \xi_0^+\omega = \xi_0^+\omega$. Hence,

$$\frac{\gamma^*}{\rho^2}\int_{-\theta\rho^2}^0 \int_{K_\rho} \Psi(x,\tau)\,dx\,d\tau \le \gamma^*\theta|K_\rho| \ln\frac{\xi_0^+}{\xi^+} \le 2\gamma^*\theta\ln(\frac{\xi_0^+}{2\xi^+}),$$

since $\xi^+ \in (0, \frac{1}{4}\xi_0^+)$. If we consider $A^+_{\xi^+,\frac{\rho}{2}}(t) = \{x \in K_{\frac{\rho}{2}} :\ u(x,t) > \mu_+ - \xi^+\omega\}$ as integration set for the integral on the left-hand side instead of the larger $K_{\frac{\rho}{2}}$, we have $H_k^+ - (u-k)_+ + \xi^+\omega \le 2\xi^+\omega$; hence, $\Psi \ge \ln(\frac{\xi_0^+}{2\xi^+})$. Then we obtain

$$\Big(\ln\frac{\xi_0^+}{2\xi^+}\Big)^2 |A^+_{\xi^+,\frac{\rho}{2}}(t)| \le \tilde{\gamma}\theta\ln(\frac{\xi_0^+}{2\xi^+})|K_{\frac{\rho}{2}}| \qquad \forall t \in (-\theta\rho^2, 0)$$

with $\tilde{\gamma} = 2\gamma^* = 8\gamma$; that is,

$$|A^+_{\xi^+,\frac{\rho}{2}}(t)| \le \frac{\tilde{\gamma}\theta}{\ln(\frac{\xi_0^+}{2\xi^+})}|K_{\frac{\rho}{2}}| \qquad \forall t \in (-\theta\rho^2, 0).$$

If we choose ξ^+ such that $\nu^+ \equiv \frac{\tilde{\gamma}\theta}{\ln(\xi_0^+/2\xi^+)}$ we are done.

It is just to remark that for (3.33) we have to choose

$$\nu^- \equiv \frac{\tilde{\gamma}\theta}{\ln(\frac{\xi_0^-}{2\xi^-})}.$$

$\square$

Let $[(y,s)+Q(\rho,\rho^2)]\subset\Omega_T$ be fixed, and consider the sets $A^{\pm}_{\xi^{\pm},\rho}(t)$ with $t\in(s-\rho^2,s)$ introduced in the previous lemma with the choice $\theta=1$ and the sets

$$A^{\pm}_{\xi^{\pm},\rho}=\int_{s-\rho^2}^{s}|A^{\pm}_{\xi^{\pm},\rho}(t)|\,dt.$$

We have the following:

Proposition 3.11. *Suppose that*

$$\int_{s-\rho^2}^{s}|A^{\pm}_{\xi^{\pm},\rho}(t)|\,dt\geq\nu^{\pm}|Q(\rho,\rho^2)|, \tag{3.34}$$

where $\nu^{\pm}$ are the numbers claimed in Proposition 3.5 *with $\theta=1$.*

Then for any $\lambda>1$ such that $\mu_+-\lambda\xi^+\omega\in(0,1)$ and for any $\eta\in(0,1)$ there exists a point $(y^*_+,s^*_+)\in[(y,s)+Q(\rho,\rho^2)]$, a number $\delta_+\in(0,1)$, and a cylinder

$$[(y^*_+,s^*_+)+Q(\delta_+\rho,\delta_+^2\rho^2)]\subset[(y,s)+Q(\rho,\rho^2)]$$

such that

$$\begin{aligned}|\{(x,t)\in[(y^*_+,s^*_+)+Q(\delta_+\rho,\delta_+^2\rho^2)]:u(x,t)>\mu_+-\lambda\xi^+\omega\}|&\\ >(1-\eta)|[(y^*_+,s^*_+)+Q(\delta_+\rho,\delta_+^2\rho^2)]|&\end{aligned} \tag{3.35}$$

with $\delta_+=\delta_+(data,\lambda,\eta,\xi^+,\omega)$.

On the other hand, for any $\lambda>1$ such that $\mu_-+\lambda\xi^-\omega\in(0,1)$ and for any $\eta\in(0,1)$ there exists a point $(y^*_-,s^*_-)\in[(y,s)+Q(\rho,\rho^2)]$, a number $\delta_-\in(0,1)$, and a cylinder

$$[(y^*_-,s^*_-)+Q(\delta_-\rho,\delta_-^2\rho^2)]\subset[(y,s)+Q(\rho,\rho^2)]$$

such that

$$\begin{aligned}|\{(x,t)\in[(y^*_-,s^*_-)+Q(\delta_-\rho,\delta_-^2\rho^2)]:u(x,t)<\mu_-+\lambda\xi^-\omega\}|&\\ >(1-\eta)|[(y^*_-,s^*_-)+Q(\delta_-\rho,\delta_-^2\rho^2)]|&\end{aligned} \tag{3.36}$$

with $\delta_-=\delta_-(data,\lambda,\eta,\xi^-,\omega)$.

Proof. Once we check that $\mu_+-\lambda\xi^+\omega$ or $\mu_-+\lambda\xi^-\omega$ is in $(0,1)$, the proof is completely analogous to the one given in [4] for $(5.2)_{\pm}$.

Just a minor change is needed in the proof of Lemma 7.1 of [4]: in fact, here we have to apply inequality (3.11) to the function $(u-k)_-$ with $k=\mu_+-\bar\lambda\xi^+\omega$ and $\bar\lambda=\frac{\lambda+1}{2}$ over the pair of cubes $[(\bar y,s^*)+Q(r,\epsilon^2r^2)]$, $[(\bar y,s^*_+)+Q(2r,\epsilon^2r^2)]$ with $\epsilon\in(0,1)$ to be chosen. If we keep just the first term on the left-hand side, we come up with

$$\int_{\bar y+K_r}|(u-(\mu_+-\bar\lambda\xi^+\omega))_-|^2(x,t)\,dx$$

$$\begin{aligned}&\le \gamma\Big[\frac{4}{r^2}\int_{-\epsilon^2 r^2}^{0}\int_{K_{2r}} |(u-(\mu_+ - \bar\lambda\xi^+\omega))_-|^2(x,\tau)\,dx\,d\tau\\ &+\int_{K_{2r}} |(u-(\mu_+ - \bar\lambda\xi^+\omega))_-|^2(x,s^*_+ - \epsilon^2 r^2)\,dx\\ &+\nu_1\int_{K_{2r}} (u-(\mu_+ - \bar\lambda\xi^+\omega))_-(x,s^*_+ - \epsilon^2 r^2)\,dx\Big].\end{aligned}$$

Now

$$\begin{aligned}&\int_{K_{2r}} |(u-(\mu_+ - \bar\lambda\xi^+\omega))_-|^2(x,s^*_+ - \epsilon^2 r^2)\,dx\\ &\le 2M\int_{K_{2r}} (u-(\mu_+ - \bar\lambda\xi^+\omega))_-(x,s^*_+ - \epsilon^2 r^2)\,dx\end{aligned}$$

with M as in (1.4). Hence,

$$\begin{aligned}&\int_{K_{2r}} |(u-(\mu_+ - \bar\lambda\xi^+\omega))_-|^2(x,s^*_+ - \epsilon^2 r^2)\,dx\\ &+\nu_1\int_{K_{2r}} (u-(\mu_+ - \bar\lambda\xi^+\omega))_-(x,s^*_+ - \epsilon^2 r^2)\,dx\\ &\le \gamma(data)\int_{K_{2r}} (u-(\mu_+ - \bar\lambda\xi^+\omega))_-(x,s^*_+ - \epsilon^2 r^2)\,dx,\end{aligned}$$

and the rest follows as in [4]. □

Remark 3.12. Propositions 8.1 and 9.1 of [4] hold in our context without any significant change. Therefore we directly refer to them, without reproducing them here explicitly.

4. PROOF OF THEOREM 1.2

To prove the continuity of u at a point $(y,s)\in\Omega_T$ we assume, up to a translation, that such a point coincides with the origin and will work within the cylinder $Q(\rho,\theta\rho^2)$ with $\theta=2^n$ and $n\in\mathbf{N}$ to be fixed later, anyway such that $Q(\rho,\theta\rho^2)\subseteq\Omega_T$.

The numbers μ_+, μ_-, and ω are defined as in Section 3.

We can now regard $Q(\rho,\theta\rho^2)$ as partitioned, up to a set of measure zero, into disjoint layers of the type

$$[(0,t_i)+Q(\rho,\rho^2)]\qquad t_i=-i\rho^2\qquad i=0,1,2,\dots,(\theta-1). \tag{4.1}$$

Let $\xi^\pm=\frac{1}{12}$, $\lambda=\frac{3}{2}$, and $\delta\in(0,1)$ as determined in Proposition 8.1 of [4]. Reducing δ if necessary, we can assume δ^{-1} to be an integer.

There is no loss of generality in assuming $0 \leq \mu_+ - \frac{1}{12}\omega < \mu_+ - \frac{1}{18}\omega \leq 1$ and $0 \leq \mu_- + \frac{1}{18}\omega < \mu_- + \frac{1}{12}\omega \leq 1$ since $\mu_+ \approx 1$, $\mu_- \approx 0$, and $\omega \geq \mu_+ - \mu_-$; this allows us to apply the lemmata proved in the previous sections, which all require $0 \leq k \leq 1$.

Now fix any box of type (4.1) and assume for the moment that its vertex coincides with the origin, so that such a layer coincides with $Q(\rho, \rho^2)$. Moreover, fix $m \in \mathbf{N}$ to be later specified. We partition this layer (that is, $Q(\rho, \rho^2)$), up to a set of measure zero, into $m^N m^2$ pairwise-disjoint cylinders.

If we denote their vertices by (x_l, t_h), each of these cylinders takes the form $[(x_l, t_h) + Q(\frac{1}{m}\rho, \frac{1}{m^2}\rho^2)]$, where

$$t_h = (1-h)\frac{1}{m^2}\rho^2 \qquad h = 1, 2, \ldots, m^2 \qquad \forall\, l = 1, 2, \ldots, m^N. \tag{4.2}$$

Therefore,

$$Q(\rho, \rho^2) = \bigcup_{h=1}^{m^2} \bigcup_{l=1}^{m^N} [(x_l, t_h) + Q(\frac{1}{m}\rho, \frac{1}{m^2}\rho^2)].$$

Within each $[(x_l, t_h) + Q(\frac{1}{m}\rho, \frac{1}{m^2}\rho^2)]$ consider coaxial cylinders of the type $[(x_l, \tau) + Q(r, r^2)]$ with

$$\tau \in [t_h - (\frac{1}{m^2}\rho^2 - r^2), t_h] \tag{4.3}$$

$$r \in [\frac{\delta}{m}\rho, \frac{1}{m}\rho]. \tag{4.4}$$

Then we have:

Proposition 4.1. *There exists a positive integer m, which can be determined a priori only in terms of ω and the data, such that for some cylinder $[(x_l, t_h) + Q(\frac{1}{m}\rho, \frac{1}{m^2}\rho^2)]$ making up the partition of $Q(\rho, \rho^2)$ and for some cylinder $[(x_l, \tau) + Q(r, r^2)] \subset [(x_l, t_h) + Q(\frac{1}{m}\rho, \frac{1}{m^2}\rho^2)]$ either*

$$|\{(x,t) \in [(x_l, \tau) + Q(r, r^2)] : \; u(x,t) > \mu_+ - \frac{1}{12}\omega\}| < \nu |Q(r, r^2)| \tag{4.5}$$

or

$$|\{(x,t) \in [(x_l, \tau) + Q(r, r^2)] : \; u(x,t) < \mu_- + \frac{1}{12}\omega\}| < \nu |Q(r, r^2)| \tag{4.6}$$

with

$$\nu = C(data)\left(\frac{\frac{\omega}{12}}{1 + \frac{\omega}{12}}\right)^{\frac{N+2}{N}}.$$

Proof. See Proposition 22.1 of [4]. □

Remark 4.2. $\omega \mapsto \delta(\omega)$ is an increasing function of ω with $\lim_{\omega\searrow 0}\delta(\omega) = 0$, whereas $\omega \mapsto m(\omega)$ is a decreasing function of ω with $\lim_{\omega\searrow 0} m(\omega) = +\infty$.

We can now give a refinement of the previous proposition.

Lemma 4.3. *For each box of the type* $[(0, t_i) + Q(\rho, \rho^2)]$ *making up the partition of* $Q(\rho, \theta\rho^2)$ *there exist two disjoint subcylinders of the type*

$$[(x_{l1}, t_{h1}) + Q(\frac{\delta}{2m}\rho, (\frac{\delta}{2m})^2\rho^2)], \qquad [(x_{l2}, t_{h2}) + Q(\frac{\delta}{2m}\rho, (\frac{\delta}{2m})^2\rho^2)],$$

with

$$t_{h1} = (1 - h_1)(\frac{\delta}{2m})^2\rho^2, \qquad h_1 = 1, 2, \ldots, (\frac{4m(\omega)}{\delta(\omega)})^{N+2}$$

for each $l = 1, 2, \ldots, q(\omega)$ *with* $q(\omega) = \frac{4m(\omega)}{\delta(\omega)}$, *and analogously for* t_{h2}, *such that both*

$$u(x,t) < \mu_+ - \frac{1}{27}\omega \qquad \forall\,(x,t) \in [(x_{l1}, t_{h1}) + Q(\delta_0\rho, {\delta_0}^2\rho^2)] \tag{4.7}$$

and

$$u(x,t) > \mu_- + \frac{1}{27}\omega \qquad \forall\,(x,t) \in [(x_{l2}, t_{h2}) + Q(\delta_0\rho, {\delta_0}^2\rho^2)] \tag{4.8}$$

hold with $\delta_0 = \delta_0(\omega) = \frac{\delta(\omega)}{4m(\omega)}$.

Proof. Paragraph 23 of [4] states that either

$$u(x,t) < \mu_+ - \frac{1}{18}\omega \qquad \forall\,(x,t) \in [(x_{l1}, t_{h1}) + Q(\delta_0\rho, {\delta_0}^2\rho^2)] \tag{4.9}$$

or

$$u(x,t) > \mu_- + \frac{1}{18}\omega \qquad \forall\,(x,t) \in [(x_{l2}, t_{h2}) + Q(\delta_0\rho, {\delta_0}^2\rho^2)]. \tag{4.10}$$

Suppose now that in a layer $[(0, t_i) + Q(\rho, \rho^2)]$ only (4.9) holds and that (4.10) is never satisfied. Then

$$|\{(x,t) \in [(x_l, \tau) + Q(r, r^2)] : \ u(x,t) < \mu_- + \frac{1}{18}\omega\}| < \nu|Q(r, r^2)|,$$

and so we can apply Proposition $(3.1)^-$ of [4] and conclude. An analogous method can be followed to prove the other implication. □

Remark 4.4. It is evident from Remark 4.2 that $\omega \mapsto \delta_0(\omega)$ is an increasing function of ω and $\lim_{\omega\searrow 0}\delta_0(\omega) = 0$.

Let us now consider one of the cylinders of (4.1), and let us denote it by $\tilde{Q}$, namely, $\tilde{Q} \equiv [(0, t_i) + Q(\rho, \rho^2)]$ for a proper value of t_i. Moreover, let us suppose that $Q^* \equiv [(x_l, \tau) + Q(\delta_0\rho, \delta_0^2\rho^2)] \subset [(0, t_i - \rho^2) + Q(\rho, \rho^2)]$ is a box which satisfies (4.8).

It is evident that

$$t_i - 2\rho^2 + \delta_0^2\rho^2 \le \tau \le t_i - \rho^2. \tag{4.11}$$

Now from Q^* we want to build a long, thin cylinder with vertex at the top of $[(0, t_i) + Q(\rho, \rho^2)]$. We set $4r = \delta_0\rho$ and consider the new cylinder given by $[(x_l, t_i) + Q(4r, 4\bar{\theta}r^2)]$ with

$$\rho^2 + \delta_0^2\rho^2 \le 4\bar{\theta}r^2 \le 2\rho^2$$

according to (4.11). Hence,

$$\frac{4(1+\delta_0^2)}{\delta_0^2} \le \bar{\theta} \le \frac{8}{\delta_0^2}. \tag{4.12}$$

Thanks to (4.8) we have

$$u(x, t_i - 4\bar{\theta}r^2) > \mu_- + \frac{1}{27}\omega \qquad \forall\, x \in [x_l + K_{4r}], \tag{4.13}$$

but this is exactly (3.32) with $\rho = 4r$ and $\xi_0^- = \frac{1}{27}$. Therefore, we can apply Proposition 3.10 and conclude that for every $\nu^- \in (0,1)$ there exists $\xi^- \in (0, \frac{1}{4}\xi_0^-) \equiv (0, \frac{1}{108})$ such that

$$|A_{\xi^-,2r}(t)| \le \nu^-|[x_l + K_{2r}]| \qquad \forall t \in (t_i - 4\bar{\theta}r^2, t_i),$$

where

$$A_{\xi^-,2r}(t) = \{x \in [x_l + K_{2r}] : \ u(x,t) < \mu_- + \xi^-\omega\}$$

and ξ^- depends only on $\bar{\theta}$, ν^- and the data through the relation

$$\nu^- = \frac{\gamma_1(\text{data})4\bar{\theta}}{\ln(\frac{1}{216\xi^-})}.$$

If we now integrate over $t \in (t_i - 4\bar{\theta}r^2, t_i)$, we obtain

$$|\{(x,t) \in [(x_l, t_i) + Q(2r, 4\bar{\theta}r^2)] : \ u(x,t) < \mu_- + \xi^-\omega\}| < \nu^-|Q(2r, 4\bar{\theta}r^2)|. \tag{4.14}$$

Once we fix $\nu^- = \frac{\gamma_2(\text{data})}{4\theta}$ such that $0 < \mu_- + \frac{2}{3}\xi^-\omega < \mu_- + \xi^-\omega < 1$, we have exactly satisfied the hypotheses of Proposition 3.9, and we can conclude that

$$u(x,t) > \mu_- + \frac{2}{3}\xi^-\omega \qquad \forall (x,t) \in [(x_l, t_i) + Q(r, 4\bar{\theta}r^2)].$$

Since $\xi^- < \frac{1}{4}\xi_0^-$, we have $\xi^- < \frac{1}{108}$. On the other hand, since $\mu_- + \frac{2}{3}\xi^-\omega > 0$, we must have $\xi^- \geq \epsilon_0$ for a proper value $\epsilon_0 > 0$, and that can be obtained with a suitable choice of ν^-.

Hence we have proved:

Lemma 4.5. *There exists a number* $\xi^- \in (\epsilon_0, \frac{1}{108})$, *with* $\epsilon_0 = \epsilon_0(\omega, data)$, *which can be determined a priori only in terms of* ω *and the data, such that*

$$u(x,t) > \mu_- + \xi^-\omega \qquad \forall (x,t) \in [(x_l, t_i) + Q(r, 4\bar\theta r^2)] \tag{4.15}$$

with $\bar\theta$ *as in* (4.12).

Remark 4.6. If $x_l = 0$ and $t_i = 0$, the previous lemma could directly lead to the proof of our continuity theorem as already remarked in [4] at the end of the proof of Proposition 24.1; we can always assume $t_i = 0$ without loss of generality. What is not true in general is $x_l = 0$. In [4] the proof of continuity is achieved by showing that there is a proper "spread of positivity" after a sufficiently long time. Here we follow a different approach.

Remark 4.7. Let us remark that the long, thin cylinder obtained in the procedure above has been stretched along the t-axis, but has reduced its radius by a factor of four.

Remark 4.8. Since in each layer $[(0, t_i) + Q(\rho, \rho^2)]$ there is a subcylinder which satisfies (4.8), it follows that the result of Lemma 4.5 can be repeated for any layer, with the possible exclusion of the bottom one.

Lemma 4.9. *In the cylinder* $Q(\rho, (\theta - 1)\rho^2)$ *we have*

$$|\{x \in K_\rho : \ u(\cdot, t) > \mu_- + \frac{\omega}{2^n}\}| \geq \gamma_0 |K_\rho| \tag{4.16}$$

$\forall\, t \in [-(\theta-1)\rho^2, 0]$ *and* $\forall\, n \in \mathbf{N}$ *such that* $0 \leq \mu_- + \frac{\omega}{2^n} \leq \mu_- + \xi\omega$, *where* ξ *is the constant given by Lemma* 4.5 *and previously denoted by* ξ^- *and* γ_0 *is a proper constant.*

Proof. According to (4.15) and to Remark 4.8, we have $u(x,t) > \mu_- + \xi\omega$ for a proper value of ξ and $\forall (x,t) \in B$, where B is the union of all $[(x_l, t_i) + Q(r, 4\bar\theta r^2)]$ given by (4.15). It is evident that

$$|B| \geq |Q(r, (\theta - 1)r^2)|.$$

Hence, for any $\bar t \in [-(\theta - 1)\rho^2, 0]$ we have

$$u(x, \bar t) > \mu_- + \xi\omega \qquad \forall x \in K_r.$$

Therefore, if we take $0 \leq \mu_- + \frac{\omega}{2^n} \leq \mu_- + \xi\omega$ we have

$$|\{u(x,\cdot) > \mu_- + \frac{\omega}{2^n}\}| \geq |\{u(x,\cdot) > \mu_- + \xi\omega\}| = r^N.$$

If we set

$$\gamma_0 < \frac{r^N}{\rho^N} = (\frac{\delta(\omega)}{16m(\omega)})^N$$

we are done. □

From now on, for the sake of simplicity we directly set θ instead of $\theta - 1$.

Lemma 4.10. *For any $\theta_0 \in (0,1)$ there exists an $n_1 = n_1(data, \omega, \theta_0)$ and a $\theta = 2^{n_1}$ such that*

$$|\{(x,t) \in Q(\rho, \theta\rho^2) : \ u(x,t) < \mu_- + \frac{\omega}{2^{n_1}}\}| \le \theta_0 |Q(\rho, \theta\rho^2)|. \tag{4.17}$$

Remark 4.11. Up to now θ has remained undefined. Lemma 4.10 definitely sets its value.

Proof. Let us set

$$A_{k,\rho}(t) = \{u(\cdot,t) < k\} \cap K_\rho \qquad t \in [-\theta\rho^2, 0],$$

$$A_{k,\rho} = \int_{-\theta\rho^2}^0 A_{k,\rho}(\tau)\, d\tau.$$

A well-known result due to De Giorgi (see for example [3]) states that, given $l > k$, we have

$$(l-k) A_{k,\rho}(t) \le C \frac{\rho^{N+1}}{|K_\rho \setminus A_{l,\rho}(t)|} \int_{A_{l,\rho}(t) \setminus A_{k,\rho}(t)} |\nabla_x u|(\cdot,t)\, dx. \tag{4.18}$$

Let us now apply (4.18) to $u(\cdot,t)$ with $l = \mu_- + \frac{\omega}{2^{n-1}}$, $k = \mu_- + \frac{\omega}{2^n}$, and $n \in \mathbf{N}$ such that

$$0 \le \mu_- + \frac{\omega}{2^n} \le \mu_- + \frac{\omega}{2^{n-1}} \le 1.$$

Since $l - k = \frac{\omega}{2^n}$, we obtain

$$\frac{\omega}{2^n} |A_{k,\rho}(t)| \le C \frac{\rho^{N+1}}{|K_\rho \setminus A_{l,\rho}(t)|} \int_{A_{l,\rho}(t) \setminus A_{k,\rho}(t)} |\nabla_x u(\cdot,t)|\, dx.$$

Thanks to (4.16) we have that

$$|K_\rho \setminus A_{l,\rho}(t)| \ge \gamma_0 K_\rho = \gamma_0 \rho^N$$

for any $t \in [-\theta\rho^2, 0]$ and for any $n \in \mathbf{N}$ such that $0 \le \mu_- + \frac{\omega}{2^n} \le \mu_- + \xi\omega$. Therefore,

$$\frac{\omega}{2^n} |A_{k,\rho}(t)| \le \frac{C\rho}{\gamma_0} \int_{A_{l,\rho}(t) \setminus A_{k,\rho}(t)} |\nabla_x u(\cdot,t)|\, dx.$$

If we now integrate over $[-\theta\rho^2, 0]$, square both sides and apply Hölder's inequality on the right-hand side, we obtain

$$\frac{\omega^2}{2^{2n}}|A_{\mu_-+\frac{\omega}{2^n},\rho}|^2 \le \Big(\frac{C\rho}{\gamma_0}\Big)^2\Big[\int_{-\theta\rho^2}^0\int_{K_\rho}|\nabla_x(u-(\mu_-+\frac{\omega}{2^n}))_-|^2\,dx\,d\tau\Big]\times$$
$$\times\,|A_{\mu_-+\frac{\omega}{2^{n-1}},\rho}\setminus A_{\mu_-+\frac{\omega}{2^n},\rho}|.$$

Let us now set

$$A_n = A_{\mu_-+\frac{\omega}{2^n},\rho}, \qquad A_{n-1} = A_{\mu_-+\frac{\omega}{2^{n-1}},\rho}.$$

We can then rewrite

$$|A_n|^2 \le \Big(\frac{C\rho}{\gamma_0}\Big)^2\frac{2^{2n}}{\omega^2}\Big[\int_{-\theta\rho^2}^0\int_{K_\rho}|\nabla_x(u-(\mu_-+\frac{\omega}{2^n}))_-|^2\,dx\,d\tau\Big]|A_{n-1}\setminus A_n|.$$

Now we have to estimate the integral on the right-hand side. If we apply (3.3) over the cylinders $Q(\rho,\theta\rho^2)$ and $Q(2\rho,2\theta\rho^2)$ we obtain

$$\begin{aligned}
&\int_{-\theta\rho^2}^0\int_{K_\rho}|\nabla_x(u-(\mu_-+\frac{\omega}{2^n}))_-|^2\,dx\,d\tau\\
&\le \frac{\gamma}{\rho^2}\Big[\int_{-2\theta\rho^2}^0\int_{K_{2\rho}}|(u-(\mu_-+\frac{\omega}{2^n}))_-|^2\,dx\,d\tau\\
&+\frac{\nu_1}{\theta}\int_{-2\theta\rho^2}^0\int_{K_{2\rho}}(u-(\mu_-+\frac{\omega}{2^n}))_-\,dx\,d\tau\Big]\\
&\le \frac{\gamma}{\rho^2}\Big[(\sup_{Q(2\rho,2\theta\rho^2)}|(u-(\mu_-+\frac{\omega}{2^n}))_-|^2)|Q(2\rho,2\theta\rho^2)|\\
&+\frac{1}{\theta}(\sup_{Q(2\rho,2\theta\rho^2)}(u-(\mu_-+\frac{\omega}{2^n}))_-)|Q(2\rho,2\theta\rho^2)|\Big]\\
&\le \frac{\gamma}{\rho^2}\Big[(\frac{\omega}{2^n})^2+\frac{1}{\theta}(\frac{\omega}{2^n})\Big]|Q(\rho,\theta\rho^2)|
\end{aligned}$$

where γ in different lines stands for different constants. Putting everything together, we find

$$|A_n|^2 \le \frac{C^2\gamma}{\gamma_0^2}[1+\frac{2^n}{\theta}]|Q(\rho,\theta\rho^2)||A_n\setminus A_{n-1}| \tag{4.19}$$

since $\omega > 1$.

Inequality (4.19) holds $\forall\theta$ and $\forall n > n_0$ as long as $\mu_- + \frac{\omega}{2^n} > 0$. Let us recall that n_0 is defined by the condition

$$\mu_- + \frac{\omega}{2^{n_0}} \le \mu_- + \xi\omega.$$

Let $n_1 > n_0$ to be later selected, and assume $\mu_- + \frac{\omega}{2^{n_1}} > 0$. Summing up (4.19) for $n = n_0, n_0+1, \dots, n_1 - 1$, we obtain

$$|A_{n_1}|^2 \le \frac{C^2\gamma}{\gamma_0^2}\frac{1}{n_1 - n_0}[1 + \frac{2^{n_1}}{\theta}]|Q(\rho, \theta\rho^2)||A_{n_0}| \tag{4.20}$$

since all sets $A_n \setminus A_{n-1}$ are disjoint and

$$\bigcup_{n=n_0}^{n_1-1} (A_n \setminus A_{n-1}) \subseteq A_{n_0}.$$

Therefore, we can further estimate

$$|A_{n_1}|^2 \le \frac{C_0}{n_1 - n_0}[1 + \frac{2^{n_1}}{\theta}]|Q(\rho, \theta\rho^2)|^2, \tag{4.21}$$

where C_0 takes into account all the constants given in (4.20). We can now fix θ, which has been left undecided up to now. In order to have

$$[1 + \frac{2^{n_1}}{\theta}] \simeq 1$$

it is enough to take $\theta = 2^{n_1}$. Then we have

$$|A_{n_1}|^2 \le \frac{C_1}{n_1 - n_0}|Q(\rho, \theta\rho^2)|^2$$

and we have proved the lemma, once we take n_1 such that

$$\frac{C_1}{n_1 - n_0} \le \theta_0^2, \qquad \mu_- + \frac{\omega}{2^{n_1}} > 0; \tag{4.22}$$

that is,

$$n_1 \ge n_0 + \frac{C_1}{\theta_0^2}, \qquad n_1 < \frac{\omega}{|\mu_-|\log 2}.$$

□

From now on we will assume $\theta = 2^{n_1}$. Moreover,

Corollary 4.12. *There exists* $\tilde{Q} = [(0, t_i) + Q(\rho, \rho^2)]$ *with* i *a proper index, such that*

$$|\{(x,t) \in \tilde{Q} :\ u(x,t) < \mu_- + \frac{\omega}{2^{n_1}}\}| \le \theta_0 |Q(\rho, \rho^2)| \tag{4.23}$$

with the same n_1 *and* θ_0 *as in Lemma* 4.10.

Proof. Suppose that for all $[(0, t_i) + Q(\rho, \rho^2)]$ making up the partition of $Q(\rho, \theta\rho^2)$, (4.23) is false. This means that for all $i = 0, 1, \dots$

$$|\{(x,t) \in [(0, t_i) + Q(\rho, \rho^2)] :\ u(x,t) < \mu_- + \frac{\omega}{2^{n_1}}\}| > \theta_0 |Q(\rho, \rho^2)|.$$

Summing up over i, we obtain

$$|\{(x,t) \in Q(\rho,\theta\rho^2): \ u(x,t) < \mu_- + \frac{\omega}{2^{n_1}}\}| > \theta_0 |Q(\rho,\theta\rho^2)|,$$

which is the exact opposite of (4.17) just proved. □

Let us then denote by $\tilde{Q}$ the particular cylinder which satisfies (4.23). We do not know its exact location between the layers of $Q(\rho,\theta\rho^2)$, we just know that it exists.

Let us now consider $\tilde{Q}$ and set

$$\rho_j = \frac{\rho}{2} + \frac{\rho}{2^{j+1}}. \tag{4.24}$$

We define the sequence of cylinders $Q_j = [(0,t_i) + Q(\rho_j,\rho_j^2)]$ (recall that i is fixed and determines $\tilde{Q}$).

Now, given $C_1 > 0$ and $k \in (0,1)$ proper real values, we have two possible alternatives, which we will study in the following.

I alternative

$$|\{u(x,t) \le 0\} \cap Q_j| \le C_1 |\{0 < u(x,t) < \frac{k}{2}\} \cap Q_j| \qquad \forall j \ge 1. \tag{4.25}$$

This case will be studied in Lemmata 4.14 and 4.15.

II alternative

$$\exists j \ge 1 \ \text{ such that } \ |\{u(x,t) \le 0\} \cap Q_j| > C_1 |\{0 < u(x,t) < \frac{k}{2}\} \cap Q_j|. \tag{4.26}$$

This case will be studied in Lemmata 4.16 and 4.18.

The combination of these two alternatives will lead to the final proof of our theorem.

Remark 4.13. If (4.25) holds, then the integral term dominates the measure term on the right-hand side of (3.3), whereas the opposite happens if (4.26) is satisfied.

Heuristically, (4.25) means that the singularity (actually $u \simeq 0$) plays a negligible role, and this is the favorable situation; (4.26) says that the singularity occupies a large set, but we will show that due to the parabolicity of the equation, the solution cannot grow too fast.

Lemma 4.14. *There exists n_* such that if*

$$|\{u(x,t) \le 0\} \cap Q_j| \le \frac{\omega}{2^{n_*}} |\{0 < u(x,t) < \mu_- + \frac{\omega}{2^{n_*+1}}\} \cap Q_j| \qquad \forall j \ge 1 \tag{4.27}$$

and

$$0 < \mu_- + \frac{\omega}{2^{n_1}} < \mu_- + \frac{\omega}{2^{n_*+2}} < \mu_- + \frac{\omega}{2^{n_*}} < 1, \tag{4.28}$$

then

$$u(x,t) \geq \mu_- + \frac{\omega}{2^{n_*+2}} \qquad \forall (x,t) \in [(0,t_i) + Q(\frac{\rho}{2}, \frac{\rho^2}{4})]. \tag{4.29}$$

Proof. (4.23) states that the region where $u(x,t) < \mu_- + \frac{\omega}{2^{n_1}}$ is very small. In order to verify (4.28), we just possibly need to take n_1 larger: it is immediate that if (4.23) holds with a certain n_1, it holds for any larger one. For the sake of simplicity, we assume $\tilde{Q} = Q(\rho, \rho^2)$; that is, we take $t_i = 0$. Let us consider (3.3) with $\theta = 1$. Then

$$\begin{aligned}
&\sup_{-\sigma\rho^2 \leq t \leq 0} \int_{K_{\sigma\rho}} |(u-k)_-|^2(x,t)\,dx + \int\int_{Q(\sigma\rho,\sigma\rho^2)} |\nabla_x (u-k)_-|^2\,dx\,d\tau \\
&\leq \gamma\Big[\frac{1}{(1-\sigma)^2\rho^2} \int\int_{Q(\rho,\rho^2)} |(u-k)_-|^2\,dx\,d\tau \\
&+ \frac{k}{(1-\sigma)^2\rho^2} |\{u(x,t) \leq 0\} \cap Q(\rho,\rho^2)|\Big],
\end{aligned}$$

where k stands for $\mu_- + \frac{\omega}{2^n}$ with $n \leq n_*$, where n_* is to be later specified. We have

$$\begin{aligned}
&\frac{k}{(1-\sigma)^2\rho^2} |\{u(x,t) \leq 0\} \cap Q(\rho,\rho^2)| \\
&\leq \frac{1}{(1-\sigma)^2\rho^2} (\frac{\omega}{2^n})^2 |\{u(x,t) < \mu_- + \frac{\omega}{2^{n+1}}\} \cap Q(\rho,\rho^2)|.
\end{aligned}$$

On the other hand,

$$(\frac{\omega}{2^{n+1}})^2 |\{u(x,t) < \mu_- + \frac{\omega}{2^{n+1}}\} \cap Q(\rho,\rho^2)| \leq \int\int_{Q(\rho,\rho^2)} |(u - (\mu_- + \frac{\omega}{2^n}))_-|^2\,dx\,d\tau.$$

Therefore,

$$\begin{aligned}
&\frac{k}{(1-\sigma)^2\rho^2} |\{u(x,t) \leq 0\} \cap Q(\rho,\rho^2)| \\
&\leq \frac{\gamma}{(1-\sigma)^2\rho^2} \int\int_{Q(\rho,\rho^2)} |(u - (\mu_- + \frac{\omega}{2^n}))_-|^2\,dx\,d\tau,
\end{aligned}$$

and so

$$\begin{aligned}
&\sup_{-\sigma\rho^2 \leq t \leq 0} \int_{K_{\sigma\rho}} |(u - (\mu_- + \frac{\omega}{2^n}))_-|^2(x,t)\,dx \\
&+ \int\int_{Q(\sigma\rho,\sigma\rho^2)} |\nabla_x (u - (\mu_- + \frac{\omega}{2^n}))_-|^2\,dx\,d\tau
\end{aligned}$$

$$\leq \frac{\gamma}{(1-\sigma)^2\rho^2} \int\int_{Q(\rho,\rho^2)} |(u - (\mu_- + \frac{\omega}{2^n}))_-|^2 \, dx \, d\tau \tag{4.30}$$

for $n_0 \leq n \leq n_*$ and not just for ρ but for all ρ_j given by (4.24). Now, by classical theory (see [5, pages 110–128]), it follows from (4.30) that there actually exists $n_* = n_*(\text{data}, \omega)$ such that

$$u(x,t) \geq \mu_- + \frac{\omega}{2^{n_*+2}} \qquad \forall (x,t) \in Q(\frac{\rho}{2}, \frac{\rho^2}{4}).$$

□

If we compare (4.23) with (4.28)–(4.29), the heuristical meaning is that the region where $u(x,t) < \mu_- + \frac{\omega}{2^{n_1}}$ ends up being completely outside the cylinder $[(0,t_i) + Q(\frac{\rho}{2}, \frac{\rho^2}{4})]$.

Let us now recall that

$$u(x,t) \geq \mu_- + \frac{\omega}{2^{n_*+2}} \qquad \forall (x,t) \in \frac{\tilde{Q}}{2},$$

where $\frac{\tilde{Q}}{2}$ stands for $[(0,t_i) + Q(\frac{\rho}{2}, \frac{\rho^2}{4})]$. Now set $4r = \frac{\rho}{2}$ and consider the cylinder $\bar{Q} \equiv Q(4r, \eta_1 r^2)$, where η_1 is given by imposing that the height of the cylinder must be $\frac{\rho^2}{4} + i\rho^2$, namely, $\eta_1 r^2 = \eta_1 (\frac{\rho}{8})^2 = \rho^2(\frac{1}{4} + i)$, $i \in \mathbf{N}$ denoting the position of $\tilde{Q}$ within $Q(\rho, \theta\rho^2)$. Therefore, $\eta_1 = 16(1+4i)$. Let us now set $\xi_0 = \frac{1}{2^{n_*+2}}$. Then we have

Lemma 4.15. *(First alternative concluded) There exists $\xi_1 \in (0, \xi_0)$, which can be determined a priori only in terms of the data and ω, such that*

$$u(x,t) > \mu_- + \xi_1 \omega \qquad \forall (x,t) \in Q(r, \frac{\eta_1}{4} r^2) \tag{4.31}$$

with

$$0 < \mu_- + \xi_1\omega < \mu_- + \xi_0\omega < 1.$$

Proof. It is the same as for Proposition 24.1 of [4]. □

Let us now consider the other possibility. From here on n_* will be the one given by Lemma 4.14. We have

Lemma 4.16. *If there exists Q_j such that*

$$|\{u(x,t) \leq 0\} \cap Q_j| > \frac{\omega}{2^{n_*}} |\{0 < u(x,t) < \mu_- + \frac{\omega}{2^{n_*+1}}\} \cap Q_j|, \tag{4.32}$$

then

$$u(x,t) > \mu_- + \frac{\omega}{2^{n_*+2}} \qquad \forall (x,t) \in \frac{Q_j}{2}. \tag{4.33}$$

Proof. Without loss of generality, we can assume $Q_j = \tilde{Q}$. In fact, if $Q_j \subset \tilde{Q}$, since $\frac{\rho}{2} < \rho_j < \rho$, we can always take into account that

$$|Q_j| \geq \frac{1}{2^{N+2}}|\tilde{Q}|$$

and therefore rescale all the constants by a factor $2^{-(N+2)}$. So we can start with

$$\begin{aligned}\frac{\omega}{2^{n_*}}|\{u(x,t) < \mu_- + \frac{\omega}{2^{n_*+1}}\} \cap \tilde{Q}| &< |\{u(x,t) \leq 0\} \cap \tilde{Q}| \\ &\leq |\{u(x,t) \leq \mu_- + \frac{\omega}{2^{n_1}}\} \cap \tilde{Q}| \leq \theta_0|\tilde{Q}|\end{aligned}$$

since $\tilde{Q}$ is chosen just to satisfy the last inequality. Then

$$|\{u(x,t) < \mu_- + \frac{\omega}{2^{n_*+1}}\} \cap \tilde{Q}| \leq \frac{2^{n_*}\theta_0}{\omega}|\tilde{Q}|,$$

where $\theta_0^2 \geq \frac{C}{n_1 - n_0}$ as stated in Lemma 4.10. Therefore, if we take n_1 so large that

$$\frac{2^{n_*}\theta_0}{\omega} < \nu^- \tag{4.34}$$

with ν^- as in Remark 3.8, then we can apply Proposition 3.5 and conclude that $u(x,t) > \mu_- + \frac{\omega}{2^{n_*+2}}$, $\forall(x,t) \in \frac{\tilde{Q}}{2}$. □

Remark 4.17. It is clear that n_1 must be very large in order to satisfy (4.34), and this shows that the starting cylinder $Q(\rho, \theta\rho^2)$ must be very long, since we set $\theta = 2^{n_1}$. This need to take n_1 large is strictly connected to what we were saying at the beginning of the proof of Lemma 4.14. Moreover, a situation similar to the one seen for the first alternative holds also in this case: again u is small outside $[(0,t_i) + Q(\frac{\rho}{2}, \frac{\rho^2}{4})]$.

Since

$$u(x,t) > \mu_- + \frac{\omega}{2^{n_*+2}} \qquad \forall(x,t) \in \frac{Q_j}{2}$$

and $Q_j \supset \frac{\tilde{Q}}{2}$, we directly obtain that

$$u(x,t) > \mu_- + \frac{\omega}{2^{n_*+2}} \qquad \forall(x,t) \in \frac{\tilde{Q}}{2}. \tag{4.35}$$

We can then reason as before and conclude that

Lemma 4.18. (Second alternative concluded) *There exists* $\xi_2 \in (0, \xi_0)$, *which can be determined a priori only in terms of the data and* ω, *such that*

$$u(x,t) > \mu_- + \xi_2\omega \qquad \forall(x,t) \in Q(\frac{r}{2}, \frac{\eta_2}{4}\frac{r^2}{4}) \tag{4.36}$$

with

$$0 < \mu_- + \xi_2\omega < \mu_- + \xi_0\omega < 1$$

and η_2 *defined analogously as* η_1*.*

Remark 4.19. As in [4], it can be easily shown that $\xi_1 = \xi_1(\omega)$, $\xi_2 = \xi_2(\omega)$ are such that $\lim_{\omega\searrow 0}\xi_i(\omega) = 0$, $i = 1, 2$.

Now we have all the elements to conclude. If we put

$$\bar{r} = \frac{r}{2}, \qquad \bar{\xi}(\omega) = \max\{\xi_1(\omega), \xi_2(\omega)\}, \qquad \bar{\theta} = \min\{1, \frac{\eta_1}{64}, \frac{\eta_2}{256}\},$$

from (4.31) and (4.36) we obtain that

$$\mathrm{osc}_{Q(\bar{r},\bar{\theta}\bar{r}^2)}u \le (1 - \bar{\xi}(\omega))\omega, \tag{4.37}$$

where ω is any number satisfying $\mathrm{osc}_{Q(\rho,\theta\rho^2)}u \le \omega$ with θ defined as before and with ρ such that $Q(2\rho, 2\theta\rho^2) \subseteq \Omega_T$.

Then we can consider a family of shrinking cylinders around the vertex $(x, t) = (0, 0)$ and reasoning exactly as in paragraph 25 of [4] we conclude.

Remark 4.20. All the proof is built working on $(u-k)_-$, but obvious and analogous implications as the ones proved above hold for $(u-k)_+$; therefore, the same result could be proved just working the other way around.

REFERENCES

[1] E. DiBenedetto, *Continuity of weak solutions to certain singular parabolic equations,* Ann. Mat. Pura Appl., 130 (1982), 131–177.

[2] E. DiBenedetto, *A boundary modulus of continuity for a class of singular parabolic equations,* J. Diff. Eq., 63 (1986), 418–447.

[3] E. DiBenedetto, "Degenerate Parabolic Equations," Springer Verlag, Series Universitext, New York, 1993.

[4] E. DiBenedetto and V. Vespri, *On the singular equation* $\beta(u)_t = \Delta u$*,* Archive Rat. Mech. Anal., 132 (1995), 3, 247–309.

[5] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type," A.M.S. Transl. Math. Mono., 23, Providence, RI, 1968.

[6] A. Visintin, A. Fasano, E. Magenes, and C. Verdi, *About mathematical models of phase transitions,* Bollettino U.M.I., (8), 1-B (1998), 1–47, 49–69, 71–81, 83–108.