# Unequal Contributions from Symmetric Agents in a Local Interaction Model

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Abstract

The main findings of the theory on the private provision of public goods under the assumptions of symmetric agents and normality are that (1) there exists a unique Nash equilibrium in which everybody contributes the same; and (2) this pattern is stable. We show that these findings no longer hold in a context characterized by local interaction. In this context, it is always possible to find preferences satisfying the assumption of normality such that the symmetric Nash equilibrium is unstable, and there exist asymmetric Nash equilibria which are locally stable.

1. Introduction: Public Goods, Nash Equilibria, and Local Interaction

Consider a group of four identical houses belonging to different owners, who are identical as to both preferences and incomes. The owners seek to protect their houses against burglaries by individually subscribing to a certain number

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of visits per night by security firms. The security guards are paid according to the number of visits that they make, and the price of each visit is assumed to be fixed. The houses are grouped so closely to each other that guard visits represent a pure public good for the four owners, each of whom then enjoys the level of protection corresponding to the total number of visits paid for all of them.

Bergstrom, Blume, and Varian (1986, henceforth BBV) show that this situation, if viewed as a noncooperative game among self-interested utility maximizers, has a unique equilibrium in which everybody incurs the same cost and enjoys the same level of protection.[1] Moreover, by using a "best reply adjustment process," Cornes (1980) shows that the unique Nash equilibrium is also stable.[2] We will refer to this situation as symmetric Nash equilibrium.

In the real world, voluntary contribution regimes usually generate unequal contributions by people who are equal in their observable circumstances such as income. Obviously, this may depend on the fact that different people have different preferences for either the public good or private consumption. Alternatively, it can be assumed (departing from the original BBV setting) that either people have different degrees of altruism which influence their contributions to the public good[3] or that individuals' actions are constrained by different moral obligations to contribute.[4] Is introducing differences in unobservable preferences and moral constraints the only way to account for heterogeneous contributions? Is there any other hypothesis of the BBV model that can be modified in order to explain why individuals with identical incomes behave differently?

An interesting question is how strongly the results of BBV depend on the implicit spatial structure of their model.[5] Suppose that the four houses in the example are located on a circle in such a way that each visit by a guard only protects the purchaser of the service, the neighbor to her right, and the neighbor to her left. As before, the most straightforward equilibrium that can be conjectured is that in which everybody incurs the same security cost

[1]Including Samuelson's seminal works (Samuelson 1954, 1955), see also Warr (1983), Andreoni (1988), and Cornes and Sandler (1984).

[2]Cornes (1980) proves local stability under a simple continuous adjustment process for a more general externalities model. See also Sandmo (1980).

[3]An example is the theory of warm-glow giving. See Andreoni (1989, 1990) and Diamond (2006).

[4]See Laffont (1975), Bilodeau and Gravel (2004).

[5]One of the first attempts to take account of the spatial location of public goods is Tiebout's model of local expenditures (Tiebout 1956). See Bloch and Zenginobuz (2006) for a recent development of Tiebout's model under the assumption of spillovers between communities, and Scotchmer (2002) for a survey of the literature related to Tiebout's theory. It is evident that Tiebout's model refers to a problem entirely different from the one considered by BBV. Indeed, instead of introducing local interaction to explain how much individuals contribute, Tiebout uses it to analyze where individuals decide to move.

and enjoys the same level of protection. However, there can now also exist asymmetric equilibria in which symmetric owners (i.e. they are identical in terms of income and preferences but belong to different neighbors of the same size) pay different costs and achieve different levels of utility. In particular, there can exist an equilibrium in which owners 1 and 3 pay for a positive number of guard visits, while owners 2 and 4 pay nothing. Intuitively, since 2 and 4 are not contributing at all, owners 1 and 3 might decide to pay a high amount for security in order to enjoy an acceptable level of protection. For if they respond to their neighbors' decisions by reducing their security costs, they end up with high risks of burglary, which is not optimal. On the other hand, owners 2 and 4 can find it optimal to keep the level of security implied by their neighbors' choices and to pay nothing for security services.[6]

In this paper, we present a version of the BBV model based on Ellison's model of local interaction (Ellison 1993). There are N symmetric individuals with identical preferences and income distributed around a circle. Each individual belongs to a neighborhood defined as the first k individuals on her right, the first k individuals on her left, and herself. The total contribution collected within her neighborhood represents the level of public good enjoyed by an individual.

We show that the assumption of 'locally enjoyed' public goods within overlapping neighborhoods radically affects both the properties of uniqueness and stability of the symmetric Nash equilibrium. In particular, for any size of neighborhood which does not include the population as a whole, it is always possible to find preferences satisfying the assumption of normality such that (a) the symmetric Nash equilibrium is unstable, (b) there exists at least one asymmetric Nash equilibrium which is locally stable, and (c) these asymmetric equilibria are characterized by clusters of positive contributors which coexist with clusters of noncontributors. Coexistence is a "natural" result in the sense that, rather than being caused by particular assumptions on either the behavior of agents or their personal characteristics (like preferences and income), it is a direct consequence of the spatial structure of the model.

Our model presents important similarities with respect to Eshel, Samuelson, and Shaked (1998, henceforth ESS) who adapt Ellison's model of local interaction to a public good games with two strategies in an evolutionary setting. In their model, each individual can either contribute a given amount which supplies one unit of utility to each member of her neighborhood and which costs a loss of utility equal to $c > 0$ or not contribute at all. In each period there is a positive probability that an agent's strategy is revised to imitate that of the most successful neighbor(s). Given these assumptions, ESS show

[6]Further examples of "locally enjoyed" public goods are local environmental protection, acoustic pollution, or systems of wireless Internet connections.

that positive contributors can survive if they are grouped together, so that the benefits of contributions are enjoyed primarily by other positive contributors, who then earn relatively high payoffs and are imitated. Finally, positive contributors manage to survive in the presence of mutations that constantly introduce free riders into the population.[7]

Although the result of coexistence of positive contributors and free riders represents an important similarity between ESS and our model, the two models differ significantly in both their assumptions and implications. With respect to assumptions, while in ESS people imitate others' discrete actions and can commit errors, we stick to a conventional setting in which individuals choose their continuous contribution as rational utility maximizers. As regards the results, while in ESS clusters of positive contributors need to be composed of at least three consecutive individuals to survive mutations (see proposition 2, p. 162), in our model, as shown in the houses example with local interaction, they do not have any lower bound in length. On the other hand, while in ESS clusters of both positive contributors and free riders do not have any upper bound in length,[8] in our model any equilibrium only composed of clusters can neither contain a string of free riders of length longer than k nor comprise a cluster of more than $k + 1$ consecutive positive contributors.

The work most closely related to ours is Bloch and Zenginobuz (2007). In this paper, agents are partitioned into communities and communities differ from each other in size. Each community chooses the level of public good which maximizes the sum of utilities of agents in that jurisdiction taking as given the decisions of other communities. Finally, each community enjoys positive spillovers from the public goods provided in different communities. Their main result is that uniqueness of the equilibrium vector of contributions depends on the nature of spillovers. In particular, if spillovers from the public good produced in one community is the same for all the other jurisdictions, there exists a unique equilibrium vector of contributions at which smaller jurisdictions contribute less than larger jurisdictions. On the other hand, when spillovers on other communities are not the same, equilibrium is unique if spillovers are low, while multiple equilibria exist for high spillovers.

Our paper significantly differs from Bloch and Zenginobuz in three aspects. First, while in their model the existence of multiple equilibria depends on the nature of spillovers across jurisdictions, in our model it depends directly on the intensity of preferences for the public good. Second, in our model we study both the property of (non)uniqueness of equilibria and their

[7]The existence of an evolutionary equilibrium characterized by the coexistence of positive contributors and free riders is also shown by Bergstrom and Stark (1993). See Bergstrom (2002) for a survey of models of local interaction based on the circular structure.
[8]For instance, both the situations in which everybody free rides and everybody contributes are absorbing sets in ESS.

stability properties. Third, in our paper we study in detail a class of asymmetric equilibria at which those who contribute enjoy a lower level of public good than those who free-ride.

The rest of the paper is organized as follows. Section 2 discusses the model's assumptions and gives some definitions which will be used to describe the equilibria of the model. Section 3 analyzes the main properties of the equilibria. Finally, Section 4 presents the conclusions and offers suggestions for further research. All the proofs are set out in the Appendix.

2. The Model: Assumptions and Definitions

Suppose that N individuals are distributed in clockwise order around a circle. Let us indicate with $P = \{1, 2, \ldots, N\}$ the set containing all the individuals in the population. Individuals are identical as to preferences and income. Each individual belongs to a neighborhood defined as the first k individuals on her right, the first k individuals on her left, and herself. That is, the neighborhood of the generic individual j can be rewritten as follows:

$$P_j = \{(j-k), \ldots, (j-1), j, (j+1), \ldots, (j+k)\}_{\mathrm{mod}\,N}. \tag{1}$$

Note that the previous definition implies an overlapping of neighborhoods. To see why, assume $N > 2k+1$ and consider two generic individuals j and $(j+1)_{\mathrm{mod}\,N}$. By definition, it follows that although j and $(j+1)_{\mathrm{mod}\,N}$ are neighbors, $P_j \neq P_{(j+1)_{\mathrm{mod}\,N}}$, because $P_j \backslash P_{(j+1)_{\mathrm{mod}\,N}} = \{(j-k)\}_{\mathrm{mod}\,N}$ and $P_{(j+1)_{\mathrm{mod}\,N}} \backslash P_j = \{(j+k+1)\}_{\mathrm{mod}\,N}$.

The preferences of individual j are described by a utility function, $U_j = U(c_j, Q_{P_j})$, which is strictly increasing in both the level of private consumption, c_j, and the level of public good collected in j's neighborhood, Q_{P_j}, strictly quasi-concave and twice differentiable.

Individual j must decide how to divide her income I (which is assumed to be strictly positive in order to avoid trivial results) between contribution q_j and consumption by taking the contributions of her neighbors as given and subjecting her decision to both a budget constraint and a nonnegativity constraint which rules out the possibility of an individual contributing a negative amount of resources. Formally, if we assume that both the price for a unit of private consumption and the price for a unit of public good are equal to one, the problem faced by individual j can be written as follows:

$$\begin{aligned} \max_{c_j, q_j} \quad & U_j = U(c_j, Q_{P_j}) \\ s.t. \quad & I = c_j + q_j, \\ & q_j \geqslant 0, \end{aligned} \tag{2}$$

where the level of public good enjoyed by individual j can be written as an additive function of the contributions of all the individuals in P_j:

$$Q_{P_j} = \sum_{s \in P_j} q_s. \tag{3}$$

Following BBV, let us define as $f(W_j)$ the demand function of contributor j when she is endowed with total wealth $W_j = I + Q_{P_{-j}}$ where P_{-j} is the set containing all members of P_j but j and $Q_{P_{-j}}$ is the total amount contributed by all members of P_j except j. This function simply represents the value of public good that individual j would choose for different values of W_j if she could ignore the nonnegativity constraint.[9] As in BBV, we assume $f(W_j)$ to be differentiable, with $0 < f'(W_j) = \frac{\partial f(W_j)}{\partial W_j} < 1$, which means that both public good and private consumption are normal goods.

Taking into account the nonnegativity constraint, we obtain the following equation which represents individual j's best response to $Q_{P_{-j}}$:

$$q_j^{BR} = \max[f(I + Q_{P_{-j}}) - Q_{P_{-j}}; 0], \quad \forall j \in P. \tag{4}$$

Note that, given the assumption of normality, individual j's contribution depends negatively on $Q_{P_{-j}}$. Let us state the concept of equilibrium used in the model.

DEFINITION 1: *For given N and k, an equilibrium $\Lambda(k, N)$, is a vector of N contributions, $(q_1^e, \ldots, q_N^e)$, such that*

$$q_j^e = \max\left[f\left(I + \sum_{s \in P_{-j}} q_s^e \right) - \left(\sum_{s \in P_{-j}} q_s^e \right); 0 \right], \forall j \in P.$$

In particular, if $2k + 1 \geq N$, then $P_j = P, \forall j$, and the spatial structure of the model coincides with the one assumed by BBV. Therefore, the symmetric Nash equilibrium is the unique equilibrium pattern of contributions and it is globally stable.

Let us introduce a second definition which will be used to describe a particular category of asymmetric Nash equilibria.

DEFINITION 2: *For a given k, a BBV community of size z denoted in what follows by BBV_z is a set B composed of z consecutive positive contributors such that (a) it is surrounded by h consecutive contributors who contribute nothing on each side, (b) every individual j in B enjoies the same level of public good Q_{P_j}, with $Q_{P_j} = \sum_{s \in B} q_s$.*

In other words, for a given k, a BBV_z is an "isolated" subgroup of z individuals with identical preferences and income who enjoy the same level of public good. The following lemma states the two main conditions on the shape of

[9]In other words, $f(W_j)$ represents the Engel curve of individual j.

a *BBV community* of size z which must hold in order to assure 'isolation' of its members.

LEMMA 1: *For a given k, consider a generic BBV community of size z. Then, (a) the BBV community is surrounded by (exactly) k consecutive null contributors on each side and (b)* $z \leq k + 1$.

The previous lemma implies that, if B is a BBV_z and $j \in B$, then, for a given k (a) everyone else in B is j's neighbor and (b) if d is a neighbor of j but $d \notin B$, then she is a null contributor. By the BBV model, everybody in a *BBV community* contributes the same strictly positive amount. For instance, Figure 1, with individuals on the horizontal axis and contributions on the vertical axis, shows a BBV_2 when $k = 2$.

The following lemma shows that the contribution of a generic individual in a *BBV community* inversely depends on the size of the community.

LEMMA 2: *Let j and y to be members of a* BBV_{z_1} *and a* BBV_{z_2}*, respectively, with* $1 \leq z_1 < z_2 \leq k + 1$. *Then,* $q_j > q_y$.

Turning to the dynamical structure of the model used to study the stability properties of the equilibrium patterns, let us suppose that individuals continuously adjust their contributions over time according to the following 'best reply' adjustment process:

$$\dot{q}_{j,t} = \frac{\partial q_{j,t}}{\partial t} = \mu\left[q_{j,t}^{BR} - q_{j,t}\right], \quad \mu > 0, \tag{5}$$

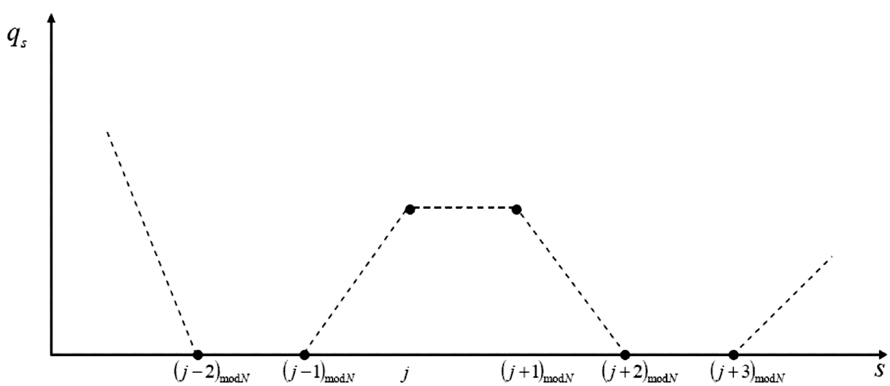

Figure 1: A BBV community of size 2 when $k = 2$

where μ represents the speed of adjustment, which is assumed to be the same across individuals.[10]

3. Properties of the Equilibria

Suppose that individual j's neighborhood is defined according to (1) and suppose that $N \geq 2k + 1$. In this case, given the adjustment process stated in the assumptions, we can specify the following dynamical system consisting of nonlinear, first order differential equations:[11]

$$\begin{cases} \dot{q}_1 = \mu \max\left[f\left(I + \sum_{s \in P_{-1}} q_s \right) - \sum_{s \in P_{-1}} q_s ; 0 \right] - \mu q_1, \\ \vdots \\ \dot{q}_j = \mu \max\left[f\left(I + \sum_{s \in P_{-j}} q_s \right) - \sum_{s \in P_{-j}} q_s ; 0 \right] - \mu q_j, \\ \vdots \\ \dot{q}_N = \mu \max\left[f\left(I + \sum_{s \in P_{-N}} q_s \right) - \sum_{s \in P_{-N}} q_s ; 0 \right] - \mu q_N. \end{cases} \tag{6}$$

At equilibrium, it must be that $\dot{q}_1 = \dot{q}_2 = \cdots = \dot{q}_N = 0$. By imposing this condition, system (6) becomes:

$$\begin{cases} q_1 = \max\left[f\left(I + \sum_{s \in P_{-1}} q_s \right) - \sum_{s \in P_{-1}} q_s ; 0 \right], \\ \vdots \\ q_j = \max\left[f\left(I + \sum_{s \in P_{-j}} q_s \right) - \sum_{s \in P_{-j}} q_s ; 0 \right], \\ \vdots \\ q_N = \max\left[f\left(I + \sum_{s \in P_{-N}} q_s \right) - \sum_{s \in P_{-N}} q_s ; 0 \right]. \end{cases} \tag{7}$$

Note that the generic equation j of system (7) represents the best response function of individual j evaluated at a specific point of time. This implies that any Nash equilibrium pattern of contributions is an equilibrium of the

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[10] Economists explain this adjustment mechanism in terms of a learning process. For further information on learning processes in game theory, see Fudeberg and Levine (1999).

[11] In order to simplify the notation, the subscript for time is henceforth omitted.

dynamical system (6). As regards the existence of an equilibrium pattern of contributions, theorem 2 of BBV (ibid. p. 33) can be extended to this context.

PROPOSITION 1: *For any* $k \geq 1$ *and for any* $N \geq 2k + 1$, *a Nash equilibrium exists.*

This proposition states nothing about the number and the stability properties of the equilibria. The question is whether, under the spatial structure assumed in our model, the symmetric Nash equilibrium continues to be characterized by the same properties of stability and uniqueness as envisaged by traditional theories on the private provision of public goods. In order to highlight how the circular structure of our model affects these properties, let us assume that $N \geq 2k + 2$, which implies that individuals in the population are not members of the same neighborhood. First of all, we show that under the assumption of normality the model admits (only) one symmetric Nash equilibrium in which everybody contributes the same amount.

PROPOSITION 2: *For any* $k \geq 1$ *and for any* $N \geq 2k + 2$, *there exists a unique symmetric Nash equilibrium in which* $q_j = \bar{q}(N) > 0, \forall j \in P$.

Concerning the dynamic properties of the symmetric Nash equilibrium, the following proposition shows that it is always possible to find preferences such that, under normality, the symmetric Nash equilibrium is unstable.

PROPOSITION 3: *For any* $k \geq 1$ *and for any* $N \geq 2k + 2$ *it is always possible to find a value* α_{ns}, *with* $0 < \alpha_{ns} < 1$, *such that if* $0 < f'(2k\bar{q}(N)) \leq \alpha_{ns}$, *the symmetric Nash equilibrium is not stable.*

To understand the intuition behind the last result, consider the following example. Suppose that N is even and $k = 1$. In this case, it is possible to show that if $f'(2k\bar{q}(N)) \leq 1/2$, then the symmetric Nash equilibrium is unstable. Suppose that, starting from the symmetric Nash equilibrium, each person in an even numbered position reduces her contribution by ϵ. In this case, if $f'(2k\bar{q}(N)) \leq 1/2$, each person in an odd numbered position reacts to the behavior of her immediate neighbors by increasing her contribution by more than ϵ. However, this induces a further negative reaction by those that initially perturbed the system. Therefore, rather than being dampened, the initial perturbation is amplified and drives the system to diverge from the symmetric Nash equilibrium.

Now, we show that it is always possible to find conditions on $f'(W)$ such that they do not violate the assumption of normality and imply the existence of some other asymmetric equilibria composed of clusters of positive contributors and clusters of null contributors only.

PROPOSITION 4: *For any* $k \geq 1$ *and for any* $N \geq 2k + 2$ *it is always possible to find a value* α_{nu}, *with* $0 < \alpha_{nu} < 1$, *such that if* $0 < f'(W) \leq \alpha_{nu}$, $\forall W > 0$, *in addition to the symmetric equilibrium, it is possible to find asymmetric Nash equilibria composed of BBV communities only.*

The intuition behind the last result is as follows. Consider two consecutive neighbors, a and b, who are identical in terms of preferences and incomes. Given the spatial structure of the model, the neighborhoods of a and b overlap but do not coincide. This implies that if a's neighborhood comprises more null contributors than b's, it can be optimal for a to contribute more than b in order to enjoy a sufficiently high level of public good. On the other hand, if b's neighborhood comprises more positive contributors than a's, the level of public good provided by her neighbors can be sufficiently high to crowd-out her contribution completely. Therefore, one can conjecture the existence of an equilibrium at which a behaves as a positive contributor while b contributes nothing.

Note that, for a given (and sufficiently large) N, it is possible to find several combinations of *BBV communities* which exhaust N. For instance, if $k = 2$ and $N = 20$, both a combination composed of five *BBV communities* of size 2 and a combination composed of five *BBV communities* of size 1 and one *BBV community* of size 3 exhaust N. Given this consideration, one can ask whether, for any possible combination of *BBV communities* which exhausts N, it is always possible to find conditions on $f'(W)$ such that they do not violate the assumption of normality and imply the combination to be an asymmetric Nash equilibrium. The following proposition states that there exists only one restriction on the shape of these particular equilibria.

PROPOSITION 5: *Let* C *be a generic combination of BBV communities only such that it exhausts* N *and does not present two consecutive BBV communities of size* $k + 1$. *There exists a value* α_C, *with* $0 < \alpha_C < 1$, *such that if* $0 < f'(W) \leq \alpha_C$, $\forall W > 0$, C *is an equilibrium*, $\Lambda_C(k, N)$.

These propositions state that it is always possible to find conditions on $f'(W)$ which satisfy the assumption of normality and such that, in addition to the unstable symmetric Nash equilibrium, there are asymmetric equilibria composed of clusters of null contributors and clusters of positive contributors who contribute the same strictly positive amount of resources. Although this conclusion is based on different assumptions, it is similar to the main result that there exist equilibria composed of clusters of *altruists* and clusters of *egoists* derived by Eshel, Samuelson, and Shaked (1998). Moreover, as in ESS model there can be clusters of *altruists* which are able to self-perpetuate over time, the next proposition states that any asymmetric equilibrium composed of *BBV communities* only is always stable according to the best reply dynamic process stated in (5).

PROPOSITION 6: *Let C be a generic combination of BBV communities only such that it exhausts N and does not present two consecutive BBV communities of size* $k + 1$. *Suppose that when* $0 < f'(W) \leq \alpha_C, \forall W > 0$, *then C is an equilibrium,* $\Lambda_C(k, N)$. *If* $f'(W) < \alpha_C, \forall W > 0$, *then* $\Lambda_C(k, N)$ *is locally stable.*

Let us explain the intuition behind this result. Consider a generic asymmetric equilibrium composed of *BBV communities* only. By definition, a *BBV community* is an 'isolated' subgroup of contributors who enjoy the strictly positive contribution of each other. Therefore, it is reasonable to conjecture that the same property of stability of the equilibrium in the BBV model continues to hold within each of the *BBV communities*. What about the behavior of the null contributors in the asymmetric equilibrium? The conditions on $f'(W)$ stated in proposition 6 guarantee that, for any (sufficiently) small perturbation around the asymmetric equilibrium, the null contributors donot systematically diverge from contributing nothing.

Finally, Corrollary 1 follows from the previous propositions.

COROLLARY 1: *For any* $k \geq 1$ *and for any* $N \geq 2k + 2$ *it is always possible to find a value* α^*, *with* $0 < \alpha^* < 1$, *such that if* $0 < f'(W) \leq \alpha^*, \forall W > 0$, *the symmetric Nash equilibrium is unstable and there exists an asymmetric Nash equilibrium which is locally stable.*

These results are somewhat counterintuitive because they state that, under particular conditions on $f'(W)$, a population of identical individuals is more likely to exhibit patterns characterized by the coexistence of clusters of positive contributors and clusters of null contributors rather than a fairer and more reasonable situation in which everybody contributes the same amount.

They also rise important normative questions about the distribution of welfare. Indeed, it is not difficult to show that the welfare of a generic null contributor in an asymmetric Nash equilibrium is always higher than that of a positive contributor in the symmetric Nash equilibrium. This is due by two facts. First, the null contributor enjoyes a level of public good which is strictly higher than that of a contributor in the symmetric equilibrium. Second, since she contributes nothing, she can use her entire income to increase her private consumption.[12] This consideration becomes even more relevant when the number of null contributors at an asymmetric equilibrium gets larger and larger. For a given N, consider the combination of *BBV communities* only such that if $N - (k+1)\lfloor \frac{N}{k+1} \rfloor = 0$ (where $\lfloor \frac{N}{k+1} \rfloor$ is obtained by rounding down $\frac{N}{k+1}$

[12]In particular, given the definition of *BBV community*, it is possible to determine the upper and the lower bound of the utility of a generic null contributor j located between two *BBV communities*. Indeed, j reaches the leastlevel of utility when he enjoies the contribution of k contributors in a *BBV community* of size $k + 1$ and 1 contributor in a *BBV community* of size k, while he reaches the highest level of utility when he enjoies the contribution of k individuals who belong to a *BBV community* of size k and 1 individual who belongs to a *BBV community* of size 1.

to the unity), the combination is composed of $\frac{N}{k+1}$ *BBV communities* of size 1 while if $N-(k+1)\lfloor\frac{N}{k+1}\rfloor>0$, the combination is composed of $\lfloor\frac{N}{k+1}\rfloor-1$ *BBV communities* of size 1 and one *BBV community* ofsize $1+N-(k+1)\lfloor\frac{N}{k+1}\rfloor$. Proposition 5 implies that there exist conditions on $f'(W)$ which do not violate the assumption of normality and such that this pattern of contributions is an equilibrium. Suppose these conditions are satisfied and let us indicate this equilibrium with $\underline{\Delta}(k, N)$. It follows that, among the possible asymmetric equilibria composed of *BBV communities* only, (a) $\underline{\Delta}(k, N)$ presents $k\lfloor\frac{N}{k+1}\rfloor$ null contributors and $N-k\lfloor\frac{N}{k+1}\rfloor$ positive contributors and, (b) it presents highest number of null contributors. Moreover, by comparing $\underline{\Delta}(k, N)$ with the symmetric Nash equilibrium, it follows that $k\lfloor\frac{N}{k+1}\rfloor$ contributors in $\underline{\Delta}(k, N)$ reach a level of utility which is strictly higher than that of a generic contributor in the symmetric equilibrium.[13]

4. Conclusions

That "individuals with identical preferences and income contribute the same amount of resources" is the most natural and reasonable finding by the theory on the private provision of public goods. Does it continue to hold when we modify the spatial structure of the model? By introducing Ellison's symmetric "local interaction structure" (Ellison 1993) into the model developed by Bergstrom, Blume and Varian (1986) with agents who have identical preferences and are endowed with the same income, we have shown that it is always possible to find preferences satisfying the assumption of normality such that (a) the symmetric Nash equilibrium is unstable and, (b) there exists at least one asymmetric Nash equilibrium which is locally stable.

These results have important policy implications. Indeed they suggest that when public goods are only such for overlapping sets of nearby agents—as often happens, for instance, with neighboring countries in face of environmental issues—and agents act noncooperatively, there is a natural and systematic tendency to reach "unfair" asymmetric equilibria in which free riders enjoy higher levels of public good than heavy contributors.

We envisage at least two directions in which our work could be extended. First, an interesting issue is the extent to which our results change under heterogeneity in either incomes or preferences. Second, this paper has examined only a particular class of asymmetric Nash equilibria, namely those composed of clusters of positive contributors, all providing the same amount of resources, and clusters of total free riders. Generally, however, there exist

[13]For instance, if $N=19$ and $k=3$, the combination of *BBV communities* with the highest number of null contributors is the one composed by three *BBV communities* of size 1 and one *BBV community* of size 4 which presents 12 null contributors and 7 positive contributors. This means that twelve individuals in the asymmetric equilibrium reach a level of utility which is strictly higher than that of a generic contributor in the symmetric equilibrium.

other asymmetric Nash equilibria which exhibit less intuitive patterns of contributions. Subsequent research should study the shape of these equilibria and examine their properties.

Appendix: Proofs of the Propositions

Proof of Lemma 1: Let $B = \{j, (j+1), \ldots, (j+z-1)\}_{\text{mod}\,N}$ be a BBV community of size z and let $\{(j-h), \ldots, (j-1), (j+z), \ldots, (j+z+h)\}_{\text{mod}\,N}$ be the null contributors surrounding it. First, we show that $h = k$. If $h > k$, then, $\forall s \in \{(j-h), (j+z+h)\}_{\text{mod}\,N}$, $q_s = 0$ if and only if $f(I + z\bar{q}(z)) \leq z\bar{q}(z)$, where $\bar{q}(z)$ is the contribution of a generic individual in B. By the intermediate value theorem, the previous condition can be rewritten as $f(I + (z-1)\bar{q}(z)) + \bar{q}(z) f'(W) \leq z\bar{q}(z)$, for some $W \in (0, \bar{q}(z)]$. Notice that, by Equation (4), the first term on the left hand side of the previous condition is equal to $z\bar{q}(z)$. Therefore, $q_s = 0$ if $f'(W) \leq 0$, for some W in $(0, \bar{q}(z)]$, which violates the assumption of normality. If $h < k$, then, by the previous considerations, there are members of B who enjoy the strictly positive contributions of some contributors outside B, which contradicts the definition of BBV community. Second, we show that $z \leq k+1$. Suppose that B is a BBV community of size z, with $z > k+1$. By definition, every individual in B contributes the same amount $\bar{q}(z)$ and by the first part of this proof, B is surrounded by k null contributors on each side. Consider the first two individuals in B, $\{j, (j+1)\}_{\text{mod}\,N}$. Since we are assuming $z > k+1$, $Q_{P_{-j}} = (k-1)\bar{q}(z) < k\bar{q}(z) = Q_{P_{-(j+1)_{\text{mod}\,N}}}$. Therefore, by Equation (4), $q_j > q_{(j+1)_{\text{mod}\,N}}$, which contradicts the initial assumptions. ■

Proof of Lemma 2: Following Andreoni (1988), let us rewrite Equation (4) (without considering the nonnegativity constraint) for an individual who belongs to a BBV community of size z_j as $\bar{q}(z_j) = I - g(z_j\bar{q}(z_j))$, where $g(z_j\bar{q}(z_j)) = f^{-1}(z_j\bar{q}(z_j)) - z_j\bar{q}(z_j)$. By the normality assumption, $0 < \frac{\partial g(z_j\bar{q}(z_j))}{\partial(z_j\bar{q}(z_j))} < \infty$ which implies that $g(.)$ is invertible. It follows that $z_j\bar{q}(z_j) = g^{-1}(I - \bar{q}(z_j))$. By the normality assumption, $g^{-1}(I) > 0$, $\frac{\partial g^{-1}(I-\bar{q}(z_j))}{\partial(I-\bar{q}(z_j))}$ is strictly positive and finite while $\frac{\partial g^{-1}(I-\bar{q}(z_j))}{\partial\bar{q}(z_j)}$ is strictly negative and finite. Therefore, the previous equation has a unique strictly positive solution in $\bar{q}(z_j)$. Now, suppose $1 \leq z_1 < z_2 \leq k+1$. It follows that $q_j = \bar{q}(z_1)$ and $q_y = \bar{q}(z_2)$ are the unique and strictly positive solutions of $z_1\bar{q}(z_1) = g^{-1}(I - \bar{q}(z_1))$ and $z_2\bar{q}(z_2) = g^{-1}(I - \bar{q}(z_2))$, respectively. Notice that (a) the left hand side of the two equations linearly increase in $\bar{q}(.)$ and (b) by assumption, the slope of the latter is strictly greater than that of the former. Given the properties of $g^{-1}(.)$, it follows that $q_y = \bar{q}(z_2) < \bar{q}(z_1) = q_j$. ■

Proof of Proposition 1: Let $\underline{q}^{BR}$ be the vector containing the best responses functions of the N contributors in the economy. $\underline{q}^{BR}$ defines a continuous function from the compact and convex set $\mathbf{I} = \{\underline{q} \in \Re^N : 0 \leq q_j \leq I, \forall j \in P\}$ to itself. Therefore, by the Brouwer's Fixed Point Theorem there must exist a fixed point $\Lambda\ (k, N) = (q_1^e, \dots, q_N^e)$, whichis a Nash equilibrium vector of contributions. ■

Proof of Proposition 2: In a symmetric Nash equilibrium $q_j = \bar{q}(N) \geq 0, \forall j \in P$, where $\bar{q}(N)$ is the strictly positive solution of $(2k+1)\bar{q}(N) = g^{-1}(I - \bar{q}(N))$ (see lemma 2). By the normality assumption, $g^{-1}(I) > 0$ and $\frac{\partial g^{-1}(I-\bar{q}(N))}{\partial \bar{q}(N)}$ is strictly negative and finite. Therefore, the previous equation has a unique strictly positive solution in $\bar{q}(N)$. ■

Proof of Proposition 3: Given system (6), the following circulant matrix of size N is the Jacobian matrix associated with the symmetric Nash equilibrium:

$$A[k, N] = -circ\left[1, \overbrace{1 - f'(\bar{W}), \dots, 1 - f'(\bar{W})}^{k}, \overbrace{0, \dots, 0}^{N-2k-1}, \overbrace{1 - f'(\bar{W}), \dots, 1 - f'(\bar{W})}^{k}\right], \tag{A1}$$

where $\bar{W} = 2k\bar{q}(N)$. The circulant nature[14] of matrix $A[k, N]$ is important for two reasons. First, since $A[k, N]$ is real and symmetric, its eigenvalues are real. Second, since $A[k, N]$ is circulant, its eigenvalues can be easily obtained by applying a discrete Fourier transform. By doing this, we have that the generic eigenvalue λ_{j+1} of $A[k, N]$ can be written as follows:

$$\lambda_{j+1} = -1 - 2(1 - f'(\bar{W}))\,\mathrm{Re}\left(\sum_{s=1}^{k} e^{\frac{2js\pi i}{N}}\right), \quad j = 0, \dots, N-1, \tag{A2}$$

where $\mathrm{Re}[d]$ is the real part of the complex number d. Let $O[k, N]$ be the set of values of $(1 - f'(\bar{W}))$ included between 0 and 1 such that at least one eigenvalue of $A[k, N]$ is zero. Then, given (A2), $O[k, N]$ is nonempty and finite. By setting $\beta = \frac{2\pi j}{N}$, using the properties of the geometric series, applying the Euler's formula $\cos\beta s = \mathrm{Re}[e^{i\beta s}]$ and using proper trigonometric identities, we can rewrite the term $\mathrm{Re}[\sum_{s=1}^{k} e^{\frac{2js\pi i}{N}}]$ of Equation (A2) as follows:

[14] For further references on circulant matrices, see Davis (1979).

$$\mathrm{Re}\left[\sum_{s=1}^{k} e^{i\beta s}\right] = \mathrm{Re}\left[\frac{1-e^{i\beta(k+1)}}{1-e^{i\beta}}\right] - 1 = \frac{\sin\beta\left(k+\frac{1}{2}\right)}{2\sin\frac{\beta}{2}} - \frac{1}{2}. \quad \text{(A3)}$$

Therefore, by combining Equation (A3) with the equation of the generic eigenvalue $j+1$ given by (A2), we obtain:

$$\lambda_{j+1} = -1 + (1 - f'(\bar{W})) - (1 - f'(\bar{W}))\gamma(\beta), \quad \text{(A4)}$$

where we have set $\gamma(\beta) = \frac{\sin\beta(k+\frac{1}{2})}{\sin\frac{\beta}{2}}$. By imposing λ_{j+1} to be equal to zero, we obtain the following condition on $(1 - f'(\bar{W}))$.

$$(1 - f'(\bar{W})) = \frac{1}{1-\gamma(\beta)}. \quad \text{(A5)}$$

From Equation (A5) we have that $0 < (1 - f'(\bar{W})) < 1$ if and only if $\gamma(\beta) < 0$. By considering only the first positive interval in which $\sin x$ is negative, we have that

$$\sin\frac{2\pi j}{N}\left(k+\frac{1}{2}\right) < 0 \Longleftrightarrow \frac{N}{2k+1} < j < \frac{2N}{2k+1}. \quad \text{(A6)}$$

Note that the length of the interval for j is $\frac{N}{2k+1} > 1$. The previous consideration implies that $\exists j \in (\frac{N}{2k+1}, \frac{2N}{2k+1})$ such that $\gamma(\beta)$ is negative. Therefore, $O[k, N]$ is nonempty. Moreover, given the properties of $\gamma(\beta)$ and the fact that $A[k, N]$ has a discrete number of eigenvalues, $O[k, N]$ is finite. By defining $1 - \alpha_{ns} = \min O[k, N]$, it follows that when $0 < 1 - \alpha_{ns} < (1 - f'(\bar{W})) < 1$ and $(1 - f'(\bar{W})) \notin O[k, N]$, there exists at least one eigenvalue of $A[k, N]$ which is strictly positive and, therefore, the symmetric Nash equilibrium is not stable. Finally, when $(1 - f'(\bar{W})) \in O[k, N]$, system (6) has an infinite number of unstable equilibria, one of which is the symmetric Nash equilibrium. ■

Proof of Proposition 4: For any $N \geq (2k+2)$ and for any $k \geq 1$, the following table can be used to determine a combination of *BBV communities* only, C, such that it exhausts N.

Let $\Phi[C]$ be the set of null contributors in C. For any combination C in Table A1, there must be an individual $y \in \Phi[C]$, such that P_y is composed of herself, an individual who belongs to a BBV_1, an individual who belongs to a BBV_z with $z \in [1, k+1]$, and $2(k-1)$ null contributors. We have that $Q_{P_{-y}} \leq Q_{P_{-j}}, \forall j \in \Phi[C]$ with $j \neq y$. Consider the case in which $z = 1$. By using Equation (4), we have that $q_y^e = 0$ if and only if $f(I + 2\bar{q}(1)) \leq 2\bar{q}(1)$, where $\bar{q}(1) = f(I)$ is the contribution of an individual belonging to a BBV_1. By the intermediate value theorem, the previous condition becomes $2\bar{q}(1) f'(W) \leq \bar{q}(1)$, for some $W \in (0, \bar{q}(1)]$. Therefore, if

Table A1: Population sizes and BBV communities

N	Combination of BBV Communities
$sk + s$	$sBBV_1$
$sk + s + 1$	$(s - 1)$ BBV_1 with 1 BBV_2
$sk + s + 2$	$(s - 1)$ BBV_1 with 1 BBV_3
...	...
$sk + s + k$	$(s - 1)$ BBV_1 with 1 BBV_{k+1}
$s = 2, 3, 4, \ldots$	

$0 < f'(W) \leq \frac{1}{2}, \forall W > 0, q_y^e = 0$. Consider the case in which $z \in [2, k + 1]$. By using Equation (4), we have that $q_y^e = 0$ if and only if $f(I + \bar{q}(1) + \bar{q}(z)) \leq \bar{q}(1) + \bar{q}(z)$, where $\bar{q}(z)$ is the contribution of a generic memberof a BBV_z. By the intermediate value theorem, the previous condition can be rewritten as $[\bar{q}(1) + \bar{q}(z)] f'(W) \leq \bar{q}(z)$, for some $W \in (0, \bar{q}(1) + \bar{q}(z)]$. Therefore, if $0 < f'(W) \leq \frac{\bar{q}(z)}{\bar{q}(1)+\bar{q}(z)} < 1, \forall W > 0, q_y^e = 0$. ■

Proof of Proposition 5: We show that for any pair of consecutive *BBV communities* such that at least one of them has size $z < k + 1$, it is always possible to find conditions on $f'(W)$ such that they satisfy the assumption of normality and imply that the null contributors located between the *BBV communities* effectively contribute nothing. Let C be a generic combination of *BBV communities* only, such that it exhausts N and it presents two consecutive *BBV communities*, B_1 and B_2, of sizes z_1 and z_2 respectively. Let $\Phi[C; (B_1, B_2)] = \{\phi_1, \phi_2, \ldots, \phi_k\}$ be the set of the k null contributors located in clockwise order between B_1 and B_2. Let us consider all the possible cases.

Case 1. $z_1 = z_2 = z$. If $1 \leq z < k + 1$, then $\forall \phi_j \in \Phi[C; (B_1, B_2)]$, P_{ϕ_j} is composed of herself, $z + 1$ individuals who belong to two BBV_z, and $2k - z - 1$ null contributors. By using Equation (4), we have that $q_{\phi_j}^e = 0$ if and only if $f(I + (z + 1)\bar{q}(z)) \leq (z + 1)\bar{q}(z)$. By the intermediate value theorem, the previous condition can be rewritten as $f(I + (z - 1)\bar{q}(z)) + 2\bar{q}(z) f'(W) \leq (z + 1)\bar{q}(z)$, for some $W \in (0, 2\bar{q}(z)]$. Notice that, by Equation (4), the first term on the left-hand side of the previous condition is equal to $z\bar{q}(z)$. Therefore, if $0 < f'(W) \leq \frac{1}{2}, \forall W > 0, q_{\phi_j}^e = 0$. If $z = k + 1$, then $\forall \phi_j \in \Phi[C; (B_1, B_2)]$, P_{ϕ_j} is composed of herself, $k + 1$ individuals who belong to two BBV_{k+1} and $k - 1$ null contributors. By using Equation (4), we have that $q_{\phi_j}^e = 0$ if and only if $f(I + (k + 1)\bar{q}(k + 1)) \leq (k + 1)\bar{q}(k + 1)$. By the intermediate value theorem, the previous condition can be rewritten as $f(I + k\bar{q}(k + 1)) + \bar{q}(k + 1) f'(W) \leq (k + 1)\bar{q}(k + 1)$, for some $W \in (0, \bar{q}(k + 1)]$. Notice that, by Equation (4), the first term on the left hand side of the previous condition is equal to $(k + 1)\bar{q}(k + 1)$. Therefore, $q_{\phi_j}^e = 0$ if $f'(W) \leq 0$, for some W in $(0, \bar{q}(k + 1)]$, which violates the assumption of normality.

Case 2. $z_1 = z_2 - s$, with $1 \leq s < z_2$ and $z_2 \in [2, k+1]$. Let us analyze the possible subcases.

2.i. Suppose $s = z_2 - 1$. It follows that P_{ϕ_1} is composed of herself, an individual who belongs to a BBV_1, an individual who belongs to a BBV_{z_2}, and $2(k-1)$ null contributors. It follows that $Q_{P_{-\phi_1}} \leq Q_{P_{-\phi_j}}, \forall \phi_j \in \Phi[C; (B_1, B_2)]$, with $j \neq 1$. By the proof of Proposition 4, $q^e_{\phi_1} = 0$ if and only if $0 < f'(W) \leq \frac{\bar{q}(z)}{\bar{q}(1)+\bar{q}(z)} < 1, \forall W > 0$.

2.ii. Suppose $1 \leq s < z_2 - 1$ and $2 \leq z_2 \leq k$ and let $m = z_2 - s - 1$. It follows that each of $\phi_1, \phi_2, \ldots, \phi_{k-m}$ has a neighborhood composed of herself, $z_2 - s$ individuals who belong to a BBV_{z_2-s}, respectively 1, 2, $\min[3; z_2], \ldots, \min[(k-m); z_2]$ individuals who belong to a BBV_{z_2}, and respectively $(2k-m-2)$, $(2k-m-3)$, $(2k-m-1) - \min[3; z_2], \ldots, (2k-m-1) - \min[(k-m); z_2]$ null contributors. Therefore, $Q_{P_{-\phi_1}} < Q_{P_{-\phi_2}} \leq \cdots \leq Q_{P_{-\phi_{k-m}}}$. Similarly, it follows that each of ϕ_{k-m+1}, $\phi_{k-m+2}, \ldots, \phi_k$ has a neighborhood composed of herself, respectively m, $(m-1), \ldots, 1$ individuals who belong to a BBV_{z_2-s}, respectively $\min[(k-m+1); z_2]$, $\min[(k-m+2); z_2], \ldots, z_2$ individuals who belong to a BBV_{z_2}, and respectively $2k - m - \min[(k-m+1); z_2]$, $2k - (m-1) - \min[(k-m+2); z_2], \ldots, 2k - 1 - z_2$ null contributors. Consider two neighbors in $\Phi[C;(B_1, B_2)]$, ϕ_{k-m+j} and $\phi_{k-m+j+1}$, with $1 \leq j \leq m-1$. Given their negihborhoods, we can have two possibilities: (a) $Q_{P_{-\phi_{k-m+j}}} - Q_{P_{-\phi_{k-m+j+1}}} = \bar{q}(z_2 - s) > 0$; (b) $Q_{P_{-\phi_{k-m+j}}} - Q_{P_{-\phi_{k-m+j+1}}} = \bar{q}(z_2 - s) - \bar{q}(z_2) > 0$ (by Lemma 2). It follows that $Q_{P_{-\phi_{k-m+1}}} > Q_{P_{-\phi_{k-m+2}}} > \cdots > Q_{P_{-\phi_k}}$. Now we prove that there are two values $\bar{\alpha}$ and $\bar{\bar{\alpha}}$, with $0 < \bar{\alpha}, \bar{\bar{\alpha}} < 1$, such that, if $0 < f'(W) \leq \bar{\alpha}, \forall W > 0$, then $q^e_{\phi_1} = 0$ and if $0 < f'(W) \leq \bar{\bar{\alpha}}, \forall W > 0$, then $q^e_{\phi_k} = 0$. By using Equation (4), we have that $q^e_{\phi_1} = 0$ if and only if $f(I + (z_2 - s)\bar{q}(z_2 - s) + \bar{q}(z_2)) \leq (z_2 - s)\bar{q}(z_2 - s) + \bar{q}(z_2)$. By the intermediate value theorem, the previous condition can be rewritten as $f(I + (z_2 - s - 1)\bar{q}(z_2 - s)) + [\bar{q}(z_2 - s) + \bar{q}(z_2)]f'(W) \leq (z_2 - s)\bar{q}(z_2 - s) + \bar{q}(z_2)$, for some $W \in (0, \bar{q}(z_2 - s) + \bar{q}(z_2)]$. Notice that, by Equation (4), the first term on the left-hand side of the previous condition is equal to $(z_2 - s)\bar{q}(z_2 - s)$. Therefore, if $0 < f'(W) \leq \bar{\alpha} = \frac{\bar{q}(z_2)}{\bar{q}(z_2-s)+\bar{q}(z_2)} < 1, \forall W > 0$, $q^e_{\phi_1} = 0$. In the same way, by using Equation (4), we have that $q^e_{\phi_k} = 0$ if and only if $f(I + \bar{q}(z_2 - s) + z_2\bar{q}(z_2)) \leq \bar{q}(z_2 - s) + z_2\bar{q}(z_2)$. By the intermediate value theorem, the previous condition can be rewritten as $f(I + (z_2 - 1)\bar{q}(z_2)) + [\bar{q}(z_2 - s) + \bar{q}(z_2)]f'(W) \leq \bar{q}(z_2 - s) + z_2\bar{q}(z_2)$, for some $W \in (0, \bar{q}(z_2 - s) + \bar{q}(z_2)]$. Notice that, by Equation (4), the first term on the left-hand side of the previous condition is equal to $z_2\bar{q}(z_2)$. Therefore, if $0 < f'(W) \leq \bar{\bar{\alpha}} = \frac{\bar{q}(z_2-s)}{\bar{q}(z_2-s)+\bar{q}(z_2)} < 1, \forall W > 0$, $q^e_{\phi_k} = 0$. It follows that if $0 < f'(W) \leq \min(\bar{\alpha}; \bar{\bar{\alpha}}), \forall W > 0$, $q^e_{\phi_1} = q^e_{\phi_k} = 0$.

2.iii. Suppose $1 \leq s < z_2 - 1$ and $z_2 = k + 1$. It follows that each of $\phi_1, \phi_2, \ldots, \phi_s$ has a neighborhood composed of herself, $(k + 1 - s)$

individuals who belong to a BBV_{k+1-s}, respectively $1, 2, \ldots, s$ individuals who belong to a BBV_{k+1}, and respectively $(k+s-2), (k+s-3), \ldots, (k-1)$ null contributors. Therefore, $Q_{P_{-\phi_1}} < Q_{P_{-\phi_2}} < \cdots < Q_{P_{-\phi_s}}$. Similarly, it follows that each of $\phi_{s+1}, \phi_{s+2}, \phi_{s+3} \ldots, \phi_k$ has a neighborhood composed of herself, respectively $(k-s), (k-s-1), \ldots, 1$ individuals who belong to a BBV_{k+1-s}, respectively $(s+1), (s+2), (s+3), \ldots, k$ individuals who belong to a BBV_{k+1} and $k-1$ null contributors. Consider two neighbors, ϕ_{s+j} and ϕ_{s+j+1}, with $1 \leq j \leq k-s-1$. Given their neighborhoods, by lemma 2, we have that $Q_{P_{-\phi_{s+j}}} - Q_{P_{-\phi_{s+j+1}}} = \bar{q}(k-s+1) - \bar{q}(k+1) > 0$. Therefore, $Q_{P_{-\phi_{s+1}}} > Q_{P_{-\phi_{s+2}}} > \cdots > Q_{P_{-\phi_k}}$. Now we prove that there are two values $\bar{\alpha}$ and $\bar{\bar{\alpha}}$, with $0 < \bar{\alpha}, \bar{\bar{\alpha}} < 1$, such that, if $0 < f'(W) \leq \bar{\alpha}, \forall W > 0$, then $q^e_{\phi_1} = 0$ and if $0 < f'(W) \leq \bar{\bar{\alpha}}, \forall W > 0$, then $q^e_{\phi_k} = 0$. By using Equation (4), we have that $q^e_{\phi_1} = 0$ if and only if $f(I + (k+1-s)\bar{q}(k+1-s) + (s+1)\bar{q}(k+1)) \leq (k+1-s)\,\bar{q}(k+1-s) + (s+1)\,\bar{q}(k+1)$. By the intermediate value theorem, the previous condition can be rewritten as $f(I + (k-s)\bar{q}(k+1-s)) + [\bar{q}(k+1-s) + (s+1)\bar{q}(k+1)]f'(W) \leq (k+1-s)\bar{q}(k+1-s) + (s+1)\bar{q}(k+1)$, for some $W \in (0, \bar{q}(k+1-s) + (s+1)\bar{q}(k+1)]$. Notice that, by Equation (4), the first term on the left hand side of the previous condition is equal to $(k+1-s)\bar{q}(k+1-s)$. Therefore, if $0 < f'(W) \leq \bar{\alpha} = \frac{(s+1)\bar{q}(k+1)}{\bar{q}(k+1-s)+(s+1)\bar{q}(k+1)} < 1, \forall W > 0$, $q^e_{\phi_1} = 0$. In the same way, by using Equation (4), we have that $q^e_{\phi_k} = 0$ if and only if $f(I + \bar{q}(k+1-s) + k\bar{q}(k+1)) \leq \bar{q}(k+1-s) + k\bar{q}(k+1)$. By the intermediate value theorem, the previous condition can be rewritten as $f(I + k\bar{q}(k+1)) + \bar{q}(k+1-s)f'(W) \leq (k+1-s)\bar{q}(k+1-s) + (s+1)b\bar{q}(k+1)$, for some $W \in (0, \bar{q}(k+1-s)]$. Notice that, by Equation (4), the first term on the left hand side of the previous condition is equal to $(k+1)\bar{q}(k+1)$. Therefore, by Lemma 2, if $0 < f'(W) \leq \bar{\bar{\alpha}} = 1 - \frac{\bar{q}(k+1)}{\bar{q}(k+1-s)} < 1, \forall W > 0$, $q^e_{\phi_k} = 0$. It follows that if $0 < f'(W) \leq \min(\bar{\alpha}; \bar{\bar{\alpha}}), \forall W > 0$, $q^e_{\phi_1} = q^e_{\phi_k} = 0$. ■

Proof of Proposition 6: Suppose that the generic asymmetric equilibrium $\Lambda_C(k, N)$ is composed of a combination C of s *BBV communities*, $B_1, B_2, \ldots, B_s$, such that the generic BBV community B_j has size z_j, with $z_j \leq k+1$. Let $J[\Lambda_C(k, N)] \in \Re^{2N}$ be the Jacobian matrix associated with $\Lambda_C(k, N)$. It follows that (i) if individual g is a null contributor then the g-th row of $J[\Lambda_C(k, N)]$ presents -1 on the main diagonal and 0 elsewhere, (ii) if individual h belongs to B_j then the h-th row of $J[\Lambda_C(k, N)]$ coincides with the h-th row of $A[k, N]$ (see the proof of Proposition 3). By applying the Cramer rule, we have that the set of the eigenvalues of $J[\Lambda_C(k, N)]$ consists of

- $N - \sum_j z_j$ eigenvalues all equal to -1,
- the eigenvalues of s matrices $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_s$ where the generic matrix $\mathbf{B}_j$ is such that (a) it is a square matrix of size z_j and (b) it presents

-1 on the main diagonal and $-(1 - f'(\bar{W}_j))$ elsewhere, with $\bar{W}_j = I + (j-1)\bar{q}(j)$.

The eigenvalues of the generic $\mathbf{B}_j$ are obtained by solving the following expression:

$$\begin{vmatrix} \lambda+1 & 1-f'(\bar{W}_j) & 1-f'(\bar{W}_j) & \cdots & 1-f'(\bar{W}_j) \\ 1-f'(\bar{W}_j) & \lambda+1 & 1-f'(\bar{W}_j) & \cdots & 1-f'(\bar{W}_j) \\ 1-f'(\bar{W}_j) & 1-f'(\bar{W}_j) & \lambda+1 & & \vdots \\ \vdots & \vdots & & \ddots & 1-f'(\bar{W}_j) \\ 1-f'(\bar{W}_j) & 1-f'(\bar{W}_j) & \cdots & 1-f'(\bar{W}_j) & \lambda+1 \end{vmatrix} = 0, \tag{A7}$$

where the size of the argument square matrix is z_j.

If we first subtract the second row from the first and then we add the the second column to the first, the determinant does not chance and (A7) can be rewritten as follows:

$$(\lambda + f'(\bar{W}_j)) \begin{vmatrix} 2(1-f'(\bar{W}_j)) + \lambda + f'(\bar{W}_j) & 1-f'(\bar{W}_j) & 1-f'(\bar{W}_j) & \cdots & 1-f'(\bar{W}_j) \\ 2(1-f'(\bar{W}_j)) & \lambda+1 & 1-f'(\bar{W}_j) & \cdots & 1-f'(\bar{W}_j) \\ 2(1-f'(\bar{W}_j)) & 1-f'(\bar{W}_j) & \lambda+1 & & \vdots \\ \vdots & \vdots & & \ddots & 1-f'(\bar{W}_j) \\ 2(1-f'(\bar{W}_j)) & 1-f'(\bar{W}_j) & 1-f'(\bar{W}_j) & \cdots & \lambda+1 \end{vmatrix} = 0, \tag{A8}$$

where the size of the new argument matrix is $(z_j - 1)$. By repeating iteratively the previous procedure for $(z_j - 2)$ times, the determinant does not change and can be rewritten as follows:

$$(\lambda + f'(\bar{W}_j))^{z_j-2} \begin{vmatrix} (z_j-1)(1-f'(\bar{W}_j)) + \lambda + f'(\bar{W}_j) & 1-f'(\bar{W}_j) \\ (z_j-1)(1-f'(\bar{W}_j)) & \lambda+1 \end{vmatrix} = 0. \tag{A9}$$

Therefore, the characteristic polynomial of $\mathbf{B}_j$ can be rewritten as follows:

$$(\lambda + f'(\bar{W}_j))^{z_j-1}\{\lambda + 1 + (z_j - 1)(1 - f'(\bar{W}_j))\} = 0. \tag{A10}$$

From expression (10) we conclude that the eigenvalues of the generic matrix $\mathbf{B}_j$ are $-f'(\bar{W}_j)$ with moltiplicity $z_j - 1$ and $f'(\bar{W}_j)(z_j - 1) - z_j$ with moltiplicity 1. The last value is negative, as we are assuming $0 < f'(\bar{W}_j) < 1$, and we can then conclude that all the eigenvalues of $J[\Lambda_C(k, N)]$ are strictly negative. ■

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