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PRINCIPAL EIGENVALUES, SPECTRAL GAPS AND EXPONENTIAL SEPARATION BETWEEN POSITIVE AND SIGN-CHANGING SOLUTIONS OF PARABOLIC EQUATIONS

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Abstract. We consider the Dirichlet problem for nonautonomous second order parabolic equations with bounded measurable coefficients on bounded Lipschitz cylinders. We discuss the exponential separation between positive and sign changing solutions and its consequences on principal eigenvalues, eigenfunctions in the time-independent case, and principal Lyapunov exponent and principal Floquet bundle in the general case.

1. **Introduction.** The classical concepts of principal eigenfunction and eigenvalue of elliptic operators have played an important role in analysis of linear and nonlinear PDEs. For example, in nonlinear parabolic equations of second order, when investigating solutions near a steady state, the linearization of the equation about the steady state plays a prominent role. Analyzing the linearization, the following information regarding the underlying elliptic operator is very useful. It has a unique eigenvalue λ_1 (the principal eigenvalue) with a positive eigenfunction (the principal eigenfunction); λ_1 is real, algebraically simple and smaller than the real part of any other eigenvalue. In particular, the stability of the steady state is determined by the sign of the principal eigenvalue. Similarly, dealing with linearizations around periodic solutions, one can rely on similar properties of the principal eigenvalue and eigenfunction of the underlying periodic parabolic operator (see [8] for a discussion of principal eigenvalues and some background).

When one is concerned with the behavior of solutions near a compact invariant set, not necessarily a steady state or periodic solution, linearization may still be very useful. This time the linearized equation is a linear parabolic equation with no particular time dependence. It is an interesting problem to determine whether some notions analogous to principal eigenvalues and eigenfunctions are meaningful for general nonautonomous second-order parabolic equations.

It was discovered some time ago (see [15, 23]) that, indeed, there are very natural and useful extensions, the relevant concepts being the principal Floquet bundle, exponential separation and principal Lyapunov exponent. Further studies of these objects and their applications can be found in [9, 10, 12, 16, 17, 18, 19, 20, 22, 25].

In our recent paper [11], we took a new approach to several basic issues in this theory. Dealing with the Dirichlet problem for general second order parabolic

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equations, we were able to significantly improve earlier theorems on the principal Floquet bundle and exponential separation and derive new perturbation and robustness results. As an interesting by-product of these results we have obtained a new information even in the elliptic (time-independent) case. Namely, we can prove that, given the ellipticity condition and a bound on the coefficients, there is a universal gap in the spectrum of the operator between the principal eigenvalue and the rest of the eigenvalues, see Section 2 for a precise formulation.

The purpose of this paper is two-fold. In Section 2, we prove, in a simplified situation, the result on the universal spectral gap for elliptic operators. This will illustrate our techniques, based on parabolic Harnack inequalities, and their relevancy even for the elliptic problem. Interestingly, we do not know a direct "elliptic" proof of the spectral gap theorem for operators in the non-divergence form. In Section 3, we discuss the results in the general case of time-dependent equations. We present theorem that naturally extend standard results on eigenvalues and eigenfunctions of elliptic operators and their perturbations. The proofs of these theorems will appear in [11].

Let us now fix the notations and basic hypotheses. We consider the following problem for a linear nonautonomous parabolic equation

$$\begin{aligned} u_t + Lu &= 0 \quad \text{in} \quad \Omega \times I, \\ u &= 0 \quad \text{on} \quad \partial\Omega \times I. \end{aligned} \tag{1.1}$$

Here Ω is a bounded domain (open connected set) in $\mathbb{R}^N$, I is an open interval in $\mathbb{R}$, and L is a time-dependent second-order elliptic operator of either the divergence form

$$Lu = -\partial_i(a_{ij}(x,t)\partial_j u + a_i(x,t)u) + b_i(x,t)\partial_i u + c_0(x,t)u \tag{D}$$

or the non-divergence form

$$Lu = -a_{ij}(x,t)\partial_i\partial_j u + b_i(x,t)\partial_i u + c_0(x,t)u \tag{ND}$$

(we use the summation convention and the notations $\partial_i = \partial/\partial x_i, u_t = \partial u/\partial t$). The coefficients a_{ij}, a_i, b_i, c_0, $i,j = 1,\dots,N$, are real valued functions in $L^\infty(\Omega\times\mathbb{R})$. In both cases we always consider uniformly parabolic equations: there exists $\alpha_0 > 0$ such that

$$a_{ij}(x,t)\xi_i\xi_j \geq \alpha_0|\xi|^2 \quad ((x,t)\in\Omega\times\mathbb{R},\ \xi\in\mathbb{R}^N). \tag{1.2}$$

Fix a constant $d_0 > 0$ and define

$$\mathcal{B}_{d_0} := \{f \in L^\infty(\Omega\times\mathbb{R}) : \|f\|_{L^\infty(\Omega\times\mathbb{R})} \leq d_0\}. \tag{1.3}$$

We shall assume that L has all its coefficients in $\mathcal{B}_{d_0}$:

$$a_{ij}, a_i, b_i, c_0 \in \mathcal{B}_{d_0} \quad (i,j = 1,\dots,N). \tag{1.4}$$

Concerning Ω, our assumption is that it is a bounded *Lipschitz* domain in $\mathbb{R}^N$. This means that there are positive constants r_Ω and m_Ω such that for each $y \in \partial\Omega$, there is an orthonormal coordinate system centered at y in which

$$\Omega \cap B_{r_\Omega}(y) = \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > \phi(x'), |x| < r_\Omega\}, \tag{1.5}$$

and

$$\|\nabla\phi\|_{L^\infty} \leq m_\Omega. \tag{1.6}$$

Here $B_r(y)$ denotes the ball in $\mathbb{R}^N$ of radius $r > 0$ and center y. Fix positive constants r_0, R_0, M_0. We shall assume that Ω is a domain as above with

$$r_\Omega \geq r_0,\ \ m_\Omega \leq M_0,\ \ \operatorname{diam}\Omega \leq R_0. \tag{1.7}$$

Our notion of a solution is standard in both the divergence and non-divergence cases, see for example [14] or [13]. Namely, in the divergence case, the solution is understood in the integral (weak) sense. Considering the problem (1.1) on the interval $I = (s, \infty)$, we always assume that the initial data $u_0 = u(\cdot, s)$ is defined and contained in $C_0(\bar{\Omega})$, the space of continuous functions on $\bar{\Omega}$ vanishing on the boundary $\partial\Omega$. Under such initial conditions, there exists a unique weak solution u to the problem (1.1) in the cylinder $Q := \Omega \times (s, \infty)$. Moreover, $u \in C(\bar{Q})$, i.e., one can modify u on a set of zero measure in such a way that the resulting function will be continuous on $\bar{Q}$.

In the non-divergence case, we consider *strong solutions* u in $W^{2,1}_{n+1,loc}(Q) \cap C(\bar{Q})$, which means that $u \in C(\bar{Q})$ and all its Sobolev derivatives $\partial_i u, \partial_i \partial_j u, u_t \in L^{n+1}_{loc}(Q)$. However, in order to guarantee the existence of a solutions in this class with the initial condition $u(\cdot, s) = u_0 \in C_0(\bar{\Omega})$, we need to assume in addition that $a_{ij} \in C(Q)$ for all i, j. Sometimes we denote this solution by $u(\cdot, t; s, u_0)$. We say that a function changes sign if it assumes both positive and negative values.

The following statement is a particular case of the well known *comparison principle*. It holds for operators L in both cases (ND) and (D).

Theorem 1.1. *Let u_1, u_2 be two real solutions of* (1.1) *in the cylinder $Q := \Omega \times (s, \infty)$, and let $u_1 \geq u_2$ on $\Omega \times \{s\}$. Then $u_1 \geq u_2$ on Q.*

Corollary 1.2. *Let u_1, u_2 be two real solutions of (1.1) and let $u_1 > 0$ in Q. Then*

$$\sup_Q \frac{u_2}{u_1} = M(s) := \sup_{\Omega \times \{s\}} \frac{u_2}{u_1}, \quad \inf_{\Omega \times \{s\}} \frac{u_2}{u_1} = m(s) := \inf_{\{s\} \times \Omega} \frac{u_2}{u_1}. \tag{1.8}$$

In particular,

$$M(t) \leq M(s), \quad m(t) \geq m(s) \quad \textit{for} \quad t \geq s. \tag{1.9}$$

Proof. Multiplying u_2 by a constant, we can reduce the proof of the first equality here to the case $M(s) = 1$. This case is contained in the previous theorem. The second equality follows by changing sign of u_2. □

2. **A universal spectral gap for the elliptic problem.** In this section we assume that the operator L is as in (ND) or (D) but without lower order terms, that is, all coefficients b_i, c_0, and in the divergence case also a_i, are equal zero.

The following theorem, a Harnack inequality for quotients of solutions plays a crucial role in our analysis.

Theorem 2.1. *Let u_1, u_2 be two positive solutions of* (1.1) *in the cylinder $Q := \Omega \times (s, \infty)$, and let functions M, m be defined as in* (1.8)*. Then*

$$M(t) \leq N_1 m(t) \quad \textit{for} \quad t \geq s + 1. \tag{2.1}$$

with a constant $N_1 \geq 1$ depending only on $N, \alpha_0, d_0, r_0, R_0, M_0$.

Proof. The desired estimate will be obtained by combining together a few results from [5], [6]. Let $x_0 \in \partial\Omega$ and $t_0 \geq s + 1$ be fixed. Set

$$r := \min\left\{r_0, \frac{1}{2}\right\}, \quad \rho := \frac{r}{1 + M_0}, \quad Q_r := (\Omega \cap B_r(x_0)) \times (t_0 - r^2, t_0 + r^2). \tag{2.2}$$

By the Lipschitz property of $\partial\Omega$, there is a point $y_0 \in \Omega \cap B_r(x_0)$, such that $d(y_0) := dist(y_0, \partial\Omega) > \rho$. Then the following estimate holds true:

$$\sup_{Q_r} \frac{u_2}{u_1} \leq N_2 \frac{u_2(y_0, t_0 + 2r^2)}{u_1(y_0, t_0 - 2r^2)} \tag{2.3}$$

with a constant $N_2 \geq 1$ depending only on the prescribed quantities. In the divergence case (D), this estimate is contained in [5], p. 547. For operators of the form (ND), it is proved in [6]. Note that the points $(y_0, t_0 \pm 2r^2)$ belong to the cylinder

$$Q^\rho := \Omega^\rho \times \left(t_0 - \frac{3}{4}, t_0 + \frac{3}{4}\right), \quad \text{where} \quad \Omega^\rho := \{x \in \Omega : d(x) > \rho\}. \tag{2.4}$$

Therefore, the previous estimate implies

$$\sup_{Q_r} \frac{u_2}{u_1} \leq N_2 M_2 m_1^{-1}, \quad \text{where} \quad M_j := \sup_{Q^\rho} u_j, \quad m_j := \inf_{Q^\rho} u_j. \tag{2.5}$$

For an arbitrary $x \in \Omega$, consider separately the two possible cases: (i) $d(x) := dist(x, \partial\Omega) \leq \rho$, and (ii) $d(x) > \rho$. In the case (i), we take $x_0 \in \partial\Omega$ such that $|x - x_0| = d(x) \leq \rho < r$, so that $(x, t_0) \in Q_r$, and at this point, the ratio u_2/u_1 does not exceed $N_2 M_2 m_1^{-1}$. In the remaining case (ii), $(x, t_0) \in Q^\rho$, so that $u_2/u_1 \leq M_2 m_1^{-1} \leq N_2 M_2 m_1^{-1}$. Taking $t_0 = t$, we now have

$$M(t) := \sup_{\Omega \times \{t\}} \frac{u_2}{u_1} \leq \frac{N_2 M_2}{m_1} \quad \text{for} \quad t \geq s + 1. \tag{2.6}$$

Interchanging u_1 and u_2, we also get

$$m(t) := \inf_{\Omega \times \{t\}} \frac{u_2}{u_1} = \left(\sup_{\Omega \times \{t\}} \frac{u_1}{u_2}\right)^{-1} \geq \left(\frac{N_2 M_1}{m_2}\right)^{-1} = \frac{m_2}{N_2 M_1}. \tag{2.7}$$

Now it remains to apply the so-called *elliptic-type Harnack inequalities* (see [5],[6]), which guarantee that $M_j \leq N_3 m_j$ for $j = 1, 2$, with another constant $N_3 \geq 1$ depending only on $N, \alpha_0, d_0, r_0, R_0, M_0$. This gives us the desired estimate with $N_1 := N_2^2 N_3^2$. □

The following proposition deals with complex solutions and it is of independent interest. Recall that the *oscillation* of a real function f on Ω can be defined in the two equivalent ways:

$$\operatorname*{osc}_\Omega f := \sup_{x,y \in \Omega} |f(x) - f(y)| = \sup_\Omega f - \inf_\Omega f.$$

The first equality here makes sense for complex valued functions as well, and we have

$$\operatorname*{osc}_\Omega f := \sup_{x,y \in \Omega} |f(x) - f(y)| = \sup_{0 \leq \phi \leq 2\pi} \operatorname*{osc}_\Omega \operatorname{Re}(e^{i\phi} f). \tag{2.8}$$

Proposition 2.2. *Let u_1 and u be solutions of the problem* (1.1) *in $Q := \Omega \times (s, \infty)$, and let $u_1 > 0$ in Q, while u is allowed to be complex valued. Then*

$$\omega(t) := \operatorname*{osc}_{\Omega \times \{t\}} \frac{u}{u_1} \leq \omega(s) \quad \textit{for} \quad t \geq s, \tag{2.9}$$

and

$$\omega(t) \leq \theta_0 \cdot \omega(s) \quad \textit{for} \quad t \geq s + 1, \quad \textit{where} \quad \theta_0 := 1 - N_1^{-1} \in (0, 1), \tag{2.10}$$

and $N_1 \geq 1$ is the constant in Theorem 2.1.

Proof. We first consider the case when u is real. Using Corollary 1.2 with $u_2 := u$, we immediately get (2.9):

$$\omega(t) = M(t) - m(t) \leq M(s) - m(s) = \omega(s) \quad \text{for} \quad t \geq s.$$

For the proof of (2.10), we use Corollary 1.2 with

$$u_2(x,t) := u(x,t) - m_0(s)\cdot u_1(x,t), \quad \text{where} \quad m_0(s) := \inf_{\Omega\times\{s\}} \frac{u}{u_1}.$$

(Here we are assuming that $m_0(s)$ is finite; if $m_0(s) = -\infty$ (2.9), (2.10) hold trivially.) Since $m(s) = 0$, the corollary gives $u_2 \geq 0$ in $\Omega \times I$, and (2.10) then follows from Theorem 2.1:

$$\omega(t) = M(t) - m(t) \leq (1 - N_2^{-1})\cdot M(t) = \theta_0 M(t) \leq \theta_0 M(s) = \theta_0 \omega(s) \quad \text{for } t \geq s+1.$$

Thus the proposition is proved for real solutions u. In the general case of complex-valued solutions u, the function $\mathrm{Re}(e^{i\phi}u)$ is a real solution for arbitrary constant $\phi \in [0, 2\pi]$. For this solution, the property (2.9) has the form

$$\omega(t,\phi) := \operatorname*{osc}_{\Omega\times\{t\}} \frac{\mathrm{Re}(e^{i\phi}u)}{u_1} \leq \omega(s,\phi) \quad \text{for} \quad t \geq s, \quad 0 \leq \phi \leq 2\pi.$$

Taking the supremum over $0 \leq \phi \leq 2\pi$ and using (2.8), we get (2.9) for complex-valued u. Similarly, the estimate (2.10) is also extended to complex-valued u. □

Next we consider operators L with coefficients independent of time. Our aim is to show how the above theorem implies the existence of a gap in the spectrum of L which depends only the constants specified in the introduction and not on the operator itself (in this sense the gap is "universal"). More specifically, consider the following eigenvalue problem

$$\begin{aligned} Lv &= \lambda v \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.11}$$

It is well known, see [2], that the principal eigenvalue λ_1, defined as the eigenvalue with the smallest real part, is real, algebraically simple, the associated eigenfunction v_1 can be chosen positive, no other eigenvalue has a positive eigenfunction, and one has $\mathrm{Re}\,\lambda > \lambda_1$ for any other eigenvalue λ. We have the following new additional information.

Theorem 2.3. *Let λ_1 be the principal eigenvalue* (2.11) *and let λ be any other eigenvalue of* (2.11)*. Then*

$$\mathrm{Re}(\lambda) - \lambda_1 \geq \gamma > 0, \tag{2.12}$$

where γ is a constant depending only on $N, \alpha_0, d_0, r_0, R_0, M_0$.

Proof. Note that if $v(x)$ is an eigenfunction of (2.11) which corresponds to an eigenvalue λ, then the function $u(x,t) := e^{-\lambda t}v(x)$ satisfies (1.1), with $I := \mathbb{R}^1$. In the case $\lambda = \lambda_1$, $v = v_1$, the function $u_1(x,t) := e^{-\lambda_1 t}v_1(x)$ is a positive solution of (1.1) on $\Omega \times I$. For $\lambda \neq \lambda_1$, we have $v \neq const \cdot v_1$, so that

$$\omega(t) := \operatorname*{osc}_{\Omega\times\{t\}} \frac{u}{u_1} = e^{(\lambda_1 - \mathrm{Re}\lambda)t}\omega(0), \quad \text{where} \quad \omega(0) = \operatorname*{osc}_{\Omega} \frac{v}{v_1} \neq 0.$$

It can also be proved that $\omega(0) < \infty$ (see [11]). Comparing with (2.10), we derive

$$\omega(1) = e^{\lambda_1 - \mathrm{Re}\,\lambda}\omega(0) \leq \theta_0\omega(0), \quad \text{hence} \quad \mathrm{Re}\,\lambda - \lambda_1 \geq c_0 := -\ln\theta_0 > 0.$$

The theorem is proved. □

Remark 2.4. For special operators (e.g. Laplacian on a convex domain) explicit lower bounds on the gap are available (see [26]).

3. **Time-dependent coefficients.** In this section we assume that L is a general time-dependent operator with coefficients satisfying the assumptions from the introduction. Our main objective here is to discuss extensions of the concepts of principal egeinvalues, eigenfunctions and their basic properties to this more general setting.

A basic result is that sign-changing solutions are exponentially dominated by positive solutions.

Theorem 3.1. *Let u_0 be such that the solution $u(\cdot, t, s, u_0)$ changes sign for all $t > s$. Let v be a positive solution (vanishing on $\partial\Omega$) of the same equation on $\Omega \times (s-1, \infty)$. Then there are constants $C, \gamma > 0$, depending only on $N, \alpha_0, d_0, r_0, R_0, M_0$, such that*

$$\frac{||u(\cdot, t; s, u_0)||_{L^\infty(\Omega)}}{||v(\cdot, t)||_{L^\infty(\Omega)}} \leq Ce^{-\gamma(t-s)} \frac{||u_0||_{L^\infty(\Omega)}}{||v(\cdot, s)||_{L^\infty(\Omega)}} \qquad (t \geq s). \tag{3.1}$$

We refer to estimate (3.1) as an *exponential separation* between sign changing and positive solutions of (1.1). The proof of this theorem is given in [11]; it uses Theorem 2.1 which we prove in [11] without the restriction on lower order terms.

As it is shown in [11], these theorems imply the following uniqueness result for positive entire solutions (that is, solutions defined for all $t \in \mathbb{R}$). It corresponds to the property of the principal eigenvalue being simple and the only eigenvalue with a positive eigenfunction.

Proposition 3.2. *If u_1, u_2 are positive entire solutions of* (1.1)*, then there is a constant q such that $u_1 \equiv qu_2$.*

For equations in divergence form, simple proofs of the uniqueness result are given in [22, 10]. The proof of Proposition 3.2 given in [11] is based on Theorem 2.1 and it works for both types of equations. Earlier uniqueness theorems under much more restrictive hypotheses can be found in [17, 18, 21].

In the case when L is a divergence-form operator, we can complement Theorem 3.1 with additional information introducing principal Floquet bundles for (1.1) and its adjoint equation, and the principal spectrum of (1.1).

¿From now on we assume that L is as in (D). Together with (1.1) we consider its adjoint problem:

$$\begin{aligned} -v_t + L^* v &= 0 \quad \text{in} \quad \Omega \times I, \\ v &= 0 \quad \text{on} \quad \partial\Omega \times I, \end{aligned} \tag{3.2}$$

with L^* defined by

$$L^* v = -\partial_j(a_{ij}(x,t)\partial_i v + b_j(x,t)v) + a_j(x,t)\partial_j v + c_0(x,t)v. \tag{3.3}$$

For any continuous $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ define

$$\underline{\lambda}(f) = \liminf_{t-s\to\infty} \frac{\log ||f(\cdot,t)||_{L^2(\Omega)} - \log ||f(\cdot,s)||_{L^2(\Omega)}}{t-s}, \tag{3.4}$$

and

$$\bar{\lambda}(f) = \limsup_{t-s\to\infty} \frac{\log ||f(\cdot,t)||_{L^2(\Omega)} - \log ||f(\cdot,s)||_{L^2(\Omega)}}{t-s}. \tag{3.5}$$

Let $X = L^2(\Omega)$ with the standard norm.

Theorem 3.3. *Assume L is as in* (D) *with coefficients satisfying* (1.2), (1.4).

(i) *There exist uniquely determined positive entire solutions* φ, ψ *of* (1.1), (3.2), *respectively, normalized by* $||\varphi(0)||_{L^2(\Omega)} = ||\psi(0)||_{L^2(\Omega)} = 1$. *They satisfy*

$$-C \le \underline{\lambda}(\varphi) \le \bar{\lambda}(\varphi) \le C,$$
$$-C \le \underline{\lambda}(\psi) \le \bar{\lambda}(\psi) \le C,$$

for some positive constant $C = C(N, \alpha_0, d_0, r_0, R_0, M_0)$.

(ii) *Set*

$$X^1(t) := \operatorname{span}\{\varphi(\cdot, t)\},$$
$$X^2(t) := \{v \in X : \int_\Omega \psi(x,t)v(x)\,dx = 0\} \quad (t \in \mathbb{R}).$$

These spaces are invariant under (1.1) *in the following sense: if* $i \in \{1, 2\}$, $u_0 \in X^i(s)$, *then* $u(\cdot, t; s, u_0) \in X^i(t)$ $(t \ge s)$. *Moreover,* $X^1(t)$, $X^2(t)$ *are complementary subspaces of* X:

$$X = X^1(t) \oplus X^2(t) \quad (t \in \mathbb{R}). \tag{3.6}$$

(iii) *There are constants* $C, \gamma > 0$, *depending only on* N, α_0, d_0, r_0, R_0, M_0, *such that for any* $u_0 \in X^2(s)$ *one has*

$$\frac{||u(\cdot, t; s, u_0)||_{L^2(\Omega)}}{||\varphi(\cdot, t)||_{L^2(\Omega)}} \le Ce^{-\gamma(t-s)} \frac{||u_0||_{L^2(\Omega)}}{||\varphi(\cdot, s)||_{L^2(\Omega)}} \qquad (t \ge s). \tag{3.7}$$

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We refer to the collection of the one-dimensional spaces $X^1(t)$, $t \in \mathbb{R}$, as the *principal Floquet bundle* of (1.1) and to property (*iii*) as an *exponential separation*. Following [19, 20], we call the numbers $\underline{\lambda}(\varphi)$ and $\bar{\lambda}(\varphi)$ *the lower* and *the upper singular exponent*, respectively, and the interval $[\underline{\lambda}(\varphi), \bar{\lambda}(\varphi)]$ *the principal spectrum*.

The interval $[\underline{\lambda}(\varphi), \bar{\lambda}(\varphi)]$ and the positive entire solution φ serve as analogs of the principal eigenvalue and eigenfunction of elliptic or periodic parabolic operators. The simplicity of the principal eigenvalue and its dominance over the real part of all other eigenvalues are paralleled by the uniqueness of φ and the exponential separation property.

Results similar to Theorem 3.3 have been previously proved in [23], [15], and more recently in [22] and [10]. In all those papers, a compactness condition with respect to time shifts of coefficients is needed. This is usually achieved by assuming, say, Hölder condition (in time) on the coefficients or a uniform bound on the modulus of continuity. Our approach in [11] is based on direct estimates and it requires no such additional assumptions.

We next examine robustness properties of the exponential separation and continuity of the principal spectrum and Floquet bundle. The results are modelled on well-known continuity properties of the principal eigenvalue and eigenfunction of elliptic or time-periodic operators under perturbations of the coefficients and the domain (see [1, 2, 3, 4], for example). One can also show that the gap between the principal eigenvalue and the rest of the spectrum is a continuous function of the coefficients and the domain. The following theorem extends these continuity properties to the general nonautonomous case.

Let L be an operator as in (D) with coefficients defined on $\mathbb{R}^{N+1}$ and satisfying the ellipticity and boundedness conditions (1.2) and (1.4) on $\mathbb{R}^{N+1}$. Let $\tilde{L}$ be another such operator with coefficients $(\tilde{a}_{ij})_{i,j=1}^N$, $\tilde{a}_i, \tilde{b}_i, \tilde{c}_0$, $i = 1, \ldots, N$. Let Ω, $\tilde{\Omega}$ be two Lipschitz domains such that their Lipschitz constants satisfy (1.7). Given a

pair L, Ω, Theorem 3.3 yields the unique normalized positive entire solutions φ, ψ of (1.1), (3.2), respectively. We use the notation $\tilde{\varphi}$, $\tilde{\psi}$ for these solutions corresponding to $\tilde{L}$, $\tilde{\Omega}$. We extend all these functions by zero outside their domains so that they can be viewed as elements of $L^2(\mathbb{R}^N)$.

We assume that $\tilde{L}$, $\tilde{\Omega}$ are perturbations of L, Ω in the following sense. For some $\delta > 0$

$$\begin{gathered}\{||a_{ij} - \tilde{a}_{ij}||, ||a_i - \tilde{a}_i||, ||b_i - \tilde{b}_i||, ||c_0 - \tilde{c}_0||\} \subseteq [0, \delta] \quad (i, j = 1, \dots, N), \\ d(\partial\Omega, \partial\tilde{\Omega}) \leq \delta,\end{gathered} \tag{3.8}$$

where $||\cdot||$ is the norm of $L^\infty(\mathbb{R}^{N+1})$ and $d(\partial\Omega, \partial\tilde{\Omega})$ the distance of $\partial\Omega$ and $\partial\tilde{\Omega}$. We say that (1.1) admits exponential separation with constants $C_{\Omega,L}$, $\gamma_{\Omega,L}$ if (3.7) holds with $C = C_{\Omega,L}$, $\gamma = \gamma_{\Omega,L}$.

Theorem 3.4. *With the above notation, assume that* (3.8) *is satisfied for some* $\delta > 0$. *Then the following statements hold.*

(i) *For each* $\varepsilon > 0$ *there exists* $\delta_1 > 0$ *depending only on* ε, N, α_0, d_0, r_0, R_0, M_0 *such that if* $\delta < \delta_1$, *then*

$$\max\{|\underline{\lambda}(\varphi) - \underline{\lambda}(\tilde{\varphi})|, |\bar{\lambda}(\varphi) - \bar{\lambda}(\tilde{\varphi})|\} \leq \varepsilon \tag{3.9}$$

and

$$\left|\left|\frac{\varphi(t)}{||\varphi(t)||_{L^2(\mathbb{R}^N)}} - \frac{\tilde{\varphi}(t)}{||\tilde{\varphi}(t)||_{L^2(\mathbb{R}^N)}}\right|\right|_{L^2(\mathbb{R}^N)} \leq \varepsilon \quad (t \in \mathbb{R}). \tag{3.10}$$

Estimates (3.9) *and* (3.10) *are also valid with* φ *and* $\tilde{\varphi}$ *replaced by* ψ *and* $\tilde{\psi}$, *respectively.*

(ii) *Suppose that* (1.1) *admits exponential separation with constants* $C_{\Omega,L}$, $\gamma_{\Omega,L}$. *Then for each* $\varepsilon > 0$ *there exists* $\delta_2 > 0$ *depending only on* ε, $C_{\Omega,L}$, $\gamma_{\Omega,L}$, N, α_0, d_0, r_0, R_0, M_0 *such that if* $\delta < \delta_2$, *then equation* (1.1) *with* Ω, L *replaced by* $\tilde{\Omega}$, $\tilde{L}$ *admits exponential separation with some constants* $C = C_{\tilde{\Omega},\tilde{L}}$ *and* $\gamma_{\tilde{\Omega},\tilde{L}} \geq \gamma_{\Omega,L} - \varepsilon$.

This theorem follows from Theorems 2.8, 2.9 of [11], where separate continuity properties with respect to the coefficients and the domain are established. Theorems 1.7 actually gives a slightly stronger statement regarding the continuity with respect to the coefficients. The techniques used in the proofs of these results combine various basic estimates of positive solutions with results on exponential dichotomies (see for example [7] for the definition of and a background in exponential dichotomies). Since the principal Floquet bundle is one-dimensional, it is not hard to reveal a relation between the exponential separation as established in Theorem 3.3 and a suitable exponential dichotomy. Standard robustness results on the latter can then be used.

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