TWO-DIMENSIONAL TRANSPORT EQUATION WITH HAMILTONIAN VECTOR FIELDS

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Consider an autonomous bounded planar vector field $b \in L^\infty(\mathbf{R}^2;\mathbf{R}^2)$ whose distributional divergence vanishes, i.e. $\operatorname{div} b = 0$. We are interested to the well-posedness in $L^\infty([0,T]\times\mathbf{R}^2)$ of the Cauchy problem for the *transport equation*

$$\begin{cases} \partial_t u(t,x) + b(x)\cdot\nabla u(t,x) = 0 \\ \\ u(0,x) = \bar{u}(x)\,, \end{cases}$$

where $\bar{u} \in L^\infty(\mathbf{R}^2)$ and the equation is intended in the usual distributional sense.

In this particular setting, stronger well-posedness results are available, compared to the by-now classical theorems by DiPerna–Lions and Ambrosio, which are valid in every space-dimension but require weak differentiability assumptions on the vector field (namely, Sobolev or BV regularity). The starting point in the two-dimensional theory is the remark that (thanks to the zero-divergence assumption) we can find a *Hamiltonian function* $H \in \operatorname{Lip}(\mathbf{R}^2)$ such that

$$b = \nabla^\perp H\,.$$

Formally, the value of H is conserved under the action of the flow, thus it is possible to hope for a kind of dimensional reduction of the problem. This observation has been exploited by various authors (Bouchut, Desvillettes, Colombini, Lerner, Rauch, Hauray) to deduce well-posedness under an additional assumption on the local direction of the vector field, which roughly speaking allows to change variables and to straighten the level curves of H.

However, the meaning and the necessity of this extra assumption were not completely clear. We introduce a different approach to tackle this problem. We replace the change-of-variables argument with a splitting of the equation on the level curves of H, and this allows to determine a sharp condition needed for the well-posedness to hold.

Our heuristic strategy can be summarized in the following four steps:

(a) Localize the equation to each level set, thanks to the invariance under the flow of the level sets;
(b) Understand the structure of the level sets, proving that generically they are "one-dimensional sets";
(c) See the equation on each level set as a one-dimensional problem and show uniqueness for it;
(d) Deduce uniqueness for the problem in $\mathbf{R}^2$ from the uniqueness of all the problems on the level sets.

JOINT WORK WITH GIOVANNI ALBERTI (PISA) AND STEFANO BIANCHINI (SISSA-ISAS TRIESTE)

However, in order to make this project work, we need a condition, which roughly speaking requires that the amount of critical points of H is "small". This *weak Sard property* is reminescent of the property shown in the classical Sard lemma, but it is much weaker. We need to assume that the push-forward via H of the Lebesgue measure restricted to the set of the critical points of H is a singular measure, i.e.

$$H_{\#}\big(\mathcal{L}^2 \llcorner \{\nabla H = 0\}\big) \perp \mathcal{L}^1 \,.$$

This assumption is needed in our proof in order to separate the dynamics in $\{\nabla H \neq 0\}$ from the dynamics in $\{\nabla H = 0\}$. Assuming the weak Sard property we are able to fully implement our strategy and thus to show well-posedness for the transport equation. We are also able to contruct explicit examples of Hamiltonian functions not satisfying the weak Sard property for which uniqueness does not hold: in these examples we precisely see that interactions between the two dynamics in $\{\nabla H \neq 0\}$ and in $\{\nabla H = 0\}$ can occur.

We remark that the weak Sard property is satisfied in various important situations, for instance if the vector field has Sobolev regularity (so we recover the result by DiPerna–Lions), and more generally if it is approximately differentiable almost everywhere in a sufficiently strong sense. This also encompasses the case of vector fields whose distributional curl is a bounded measure.

The main tools in our proof come from geometric measure theory and from real analysis. We remark that some extensions of our result are possible: for instance, we are able to treat the case of bounded divergence, and to allow some mild dependence of the vector field on the time variable. Some points of the proof (for instance, the technique of splitting on the level sets) have multi-dimensional analogues, but we also show that the uniqueness result is strictly two-dimensional.

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