Unspecified Book Proceedings Series

Dedicated to Nina N. Ural'tseva

Non-divergence Elliptic Equations of Second Order with Unbounded Drift

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Abstract. We consider uniformly elliptic equations and inequalities of second order in the *non-divergence* form $Lu = a_{ij}D_{ij}u + b_iD_iu = 0$ $(\le 0,\ \ge 0)$, in a domain $\Omega \subset \mathbb{R}^n$. We derive an interior Harnack inequality, a boundary Harnack inequalities and a comparison theorem for Lipschitz boundaries in the case $b_i \in L^n$, and a Hopf-Oleinik type estimate near flat boundary in the case $b_i \in L^q$, $q > n$. Our main tool are special Landis-type growth lemmas.

1. Introduction

In this paper, we derive interior and boundary pointwise estimates for solutions to the equation

$$Lu := a_{ij}D_{ij}u + b_iD_iu = 0 \quad \text{in a domain} \quad \Omega \subset \mathbb{R}^n, \tag{1.1}$$

and the inequalities $Lu \le 0$ or $Lu \ge 0$, under minimal assumptions on the functions a_{ij} and b_i. We use the summation convention over repeated indices and the notations $D_i := \partial/\partial x_i,\ D_{ij} := D_iD_j$. We assume that a_{ij} and b_i are measurable functions on $\mathbb{R}^n$, a_{ij} satisfy the *uniform ellipticity condition*

$$a_{ij}(x) = a_{ji}(x), \qquad \nu\,|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \nu^{-1}|\xi|^2 \quad \forall \xi, x \in \mathbb{R}^n, \tag{1.2}$$

with a constant $\nu \in (0, 1]$, and $|\mathbf{b}| := (|b_i|^2)^{1/2}$, $f \in L^n(\Omega)$. Throughout the paper, the operator L in (1.1) is applied to functions u in the class $W(\Omega) := W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$, which implies, in particular, that $u,\ D_iu,\ D_{ij}u$ belong to the Lebesgue space $L^n(\Omega')$ for any bounded open set $\Omega' \subset \overline{\Omega'} \subset \Omega$.

The following estimate is crucial for our considerations. It is usually referred to as the Aleksandrov-Bakel'man-Pucci (ABP) maximum principle. This estimate, in a more general setting, which allows L to be non-uniformly elliptic, is contained in [**A66**], Theorem A (see also [**GT83**], (9.14)).

Theorem 1.1 (ABP maximum principle). *Let Ω be a bounded open set in $\mathbb{R}^n$, and let u be a function in $W(\Omega)$ such that $Lu \ge f$ in Ω. Suppose that the coefficients a_{ij} satisfy* (1.2)*, and $|\mathbf{b}|$, $f \in L^n(\Omega)$. Then*

$$\sup_\Omega u \le \sup_{\partial\Omega} u + N \operatorname{diam}\Omega \cdot e^{NS}\,||f||_{n,\Omega}, \tag{1.3}$$

1991 *Mathematics Subject Classification.* Primary 35J15, 35J67; Secondary 35B05.

Key words and phrases. Second-order elliptic equations, Harnack inequality, boundary Hopf lemma, measurable coefficients.

where

$$S := S(\Omega) := \int_\Omega |\mathbf{b}|^n dx, \qquad ||f||_{n,\Omega} := \left(\int_\Omega |f|^n dx \right)^{1/n} \tag{1.4}$$

is the norm of the function f in $L^n(\Omega)$, and N is a positive constant depending only on n and ν.

Corollary 1.2. *Let Ω be a bounded open set in $\mathbb{R}^n$, and let u be a function in $W(\Omega)$ such that $Lu \geq 0$ in Ω. Then the maximum of u on $\overline{\Omega}$ is attained on $\partial\Omega$.*

Remark 1.3. Note that the equality $Lu = f$ in (1.1) and the inequality $Lu \geq f$ in Theorem 1.1 hold true almost everywhere in Ω with respect to the Lebesgue measure on $\mathbb{R}^n$. It is easy to see that the function $u(x) := 1 - |x|^2$ satisfies

$$\Delta u + b_i D_i u = 0 \qquad \text{in} \quad B_1 := \{x \in \mathbb{R}^n : |x| < 1\},$$

where $b_i(x) := -n x_i \cdot |x|^{-2} \in L^{n-\varepsilon}(B_1)$ for arbitrary small $\varepsilon > 0$, and the maximum of u on $\overline{B_1}$ is not attained on ∂B_1. Therefore, the assumption $b_i \in L^n$ is essential for the whole theory, which is based on the maximum principle.

Remark 1.4. Note that the quantity $S(\Omega)$ in (1.4) is scaling invariant in the following sense. If $u = u(x)$ satisfies the equation $Lu = 0$ in Ω, then for any constant $k > 0$, the function $\tilde{u}(y) := u(ky)$ satisfies $D_i\tilde{u} = kD_i u$, $D_{ij}\tilde{u} = k^2 D_{ij} u$, hence

$$\tilde{L}\tilde{u} := \tilde{a}_{ij} D_{ij}\tilde{u} + \tilde{b}_i D_i \tilde{u} = 0 \quad \text{in} \quad \tilde{\Omega} := \{y \in \mathbb{R}^n : ky \in \Omega\},$$

where $\tilde{a}_{ij}(y) := a_{ij}(ky)$, $\tilde{b}_i(y) := k b_i(ky)$. Therefore, $|\tilde{b}| := \left(\sum \tilde{b}_i^2\right)^{1/2}$ satisfies

$$\tilde{S}(\tilde{\Omega}) := \int_{\tilde{\Omega}} |\tilde{b}(y)|^n dy = \int_{\tilde{\Omega}} |b(ky)|^n k^n dy = \int_\Omega |b(x)|^n dx =: S(\Omega).$$

Obviously, this argument also works for inequalities $Lu \leq 0$ or $Lu \geq 0$ in Ω.

In the next section, we will use the scaling invariance of $S(\Omega)$, together with the maximum principle, in order to prove some special *growth lemmas* (Lemmas 2.1 and 2.4) and a *doubling property* of solutions (Lemma 2.2) in the case $|\mathbf{b}| \in L^n$, i.e. $S(\Omega) < \infty$. The growth lemmas were first introduced by E.M. Landis [**La67, La71**]. Among other applications of these lemmas, Landis gave alternative proofs of results by De Giorgi and Moser on Hölder regularity and Harnack inequalities for solutions to second order elliptic equations in the *divergence* form. These results were extended to the equations in the *non-divergence* form with $|\mathbf{b}| \in L^\infty$ in our joint work with N.V. Krylov [**KS80, S80**] (see also the book [**K85**]). For this purpose, we also used some variants of growth lemmas. Now this technique became standard, see the paper by H. Aimar, L. Forzani, and R. Toledano [**AFT01**], in which they treat Hölder and Harnack properties form an abstract point of view.

In Section 3, we use the growth lemmas and the doubling property of solutions in order to derive the interior and boundary Harnack inequalities and the Hölder estimates for solutions of equations (1.1) with $|\mathbf{b}| \in L^n$. Note that the Hölder regularity in the case $|\mathbf{b}| \in L^n$ was proved earlier by O.A. Ladyzhenskaya and N.N. Ural'tseva [**LU85**]. However, the Hölder constants in that paper depend on the modulus of continuity of $|\mathbf{b}|$ in L^n, or more precisely, on the constant $\rho > 0$ for which the norms of $|\mathbf{b}|$ in $L^n(\Omega \cap B_\rho(x))$ are small enough for all $x \in \Omega$. In our estimates, all the constants depend on b only through the quantity $S(\Omega)$ in (1.4).

Finally, in Section 4, we prove a Hopf-Oleinik type estimate near flat boundary for positive solutions of the equation $a_{ij}D_{ij}u + b_iD_i = 0$ with $b_i \in L^q$, $q > n$.

Notations. $x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n)$ are vectors or points in $\mathbb{R}^n$, $(x, y) := x_iy_i$ – the scalar product of $x, y \in \mathbb{R}^n$ (the summation convention is implied), and $|x| := (x, x)^{1/2}$ – the length of $x \in \mathbb{R}^n$. The standard orthonormal basis in $\mathbb{R}^n$ consists of n vectors $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$, so that $x = x_i\mathbf{e_i}$ for all $x \in \mathbb{R}^n$. $\operatorname{tr} a := \sum_i a_{ii}$ – the trace of a $n \times n$ matrix $a = [a_{ij}]$. For $c \in \mathbb{R}^1$, $[c]$ denotes its integer part (the maximal integer $\leq c$), $c_+ := \max\{c, 0\}$.

$B_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$ – a ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$.

$B'_r(x') := \{y' \in \mathbb{R}^n : |y' - x'| < r\}$ – a similar ball in $\mathbb{R}^{n-1}$ centered at $x \in \mathbb{R}^{n-1}$. $B_r := B_r(0)$, $B'_r := B'_r(0)$.

$\partial\Omega$ is the boundary of an open set $\Omega \subset \mathbb{R}^n$, $|\Omega|$ – its Lebesgue measure.

The notation $A := B$, or $B =: A$, means "$A = B$ by definition". Throughout the paper, N, c (with indices or without) denote different constants depending only on the prescribed quantities, such that n, ν, etc. This dependence is indicated in the parentheses: $N = N(n, \nu, \ldots)$, $c = c(n, \nu, \ldots)$.

2. Growth Lemmas

The first growth lemma can be treated as a growth lemma for "narrow" domains.

Lemma 2.1 (First growth lemma). *Let Ω be an open set in $\mathbb{R}^n$, and let $u \in W(\Omega)$, $x_0 \in \mathbb{R}^n$, and $r > 0$ be such that*

$$u \geq 0, \quad Lu \geq 0 \quad \text{in} \quad \Omega; \quad \text{and} \quad u = 0 \quad \text{on} \quad (\partial\Omega) \cap B_{2r}(x_0).$$

We claim that for arbitrary constant $\beta_1 \in (0, 1)$, there is a constant $\mu_1 \in (0, 1)$, depending only on n, ν, S, and $\beta_1) \in (0, 1)$, such that from the estimate for the Lebesgue measure

$$|\Omega \cap B_{2r}(x_0)| \leq \mu_1 \cdot |B_{2r}| \tag{2.1}$$

it follows

$$M_r := \sup_{\Omega \cap B_r(x_0)} u \leq \beta_1 \cdot M_{2r}. \tag{2.2}$$

Proof. Using translation in $\mathbb{R}^n$, and multiplying u by a constant, we can assume $x_0 = 0$, $M_r > 0$, and $M_{2r} = 1$. Moreover, since $S(\Omega)$ is invariant with respect to rescaling (Remark 1.4), we can choose any convenient value $r > 0$.

(a) We first take $r = 1$ and assume that $S := S(\Omega) \leq 1$. By Corollary 1.2, the maximum of u on $\overline{\Omega \cap B_1}$ is attained at some point $y_0 \in \Omega \cap (\partial B_1)$. Consider the function $v(x) := u(x) - |x - y_0|^2$ on the set $\Omega' := \Omega \cap B_1(y_0)$. Since $v \leq u = 0$ on $(\partial\Omega) \cap B_2$, and $v = u - 1 \leq 0$ on $\overline{\Omega} \cap (\partial B_1(y_0))$, we have $v \leq 0$ on $\partial\Omega'$. Moreover,

$$Lv = Lu - 2\operatorname{tr} a - 2\,(\mathbf{b}, x - y_0) \geq f := -2n\nu^{-1} - 2|\mathbf{b}| \quad \text{in} \quad \Omega'.$$

Applying Theorem 1.1 to the function v in Ω', and having in mind that $S(\Omega) \leq 1$ and $|\Omega'| \leq \mu_1 \cdot |\Omega \cap B_2| \leq \mu_1 \cdot |B_2|$, we get

$$M_1 := \sup_{\Omega \cap B_1} u = u(y_0) = v(y_0) \leq N_0 \cdot ||f||_{n,V} \leq N_1 \cdot \left(\mu_1^{1/n} + S^{1/n}\right), \tag{2.3}$$

with constants N_0, $N_1 \geq 1$, depending only on n and ν.

Now fix an arbitrary constant $\beta_1 \in (0,1)$, and set $\mu_0 := S_0 := (\beta_1/2N_1)^n$. If $S \le S_0$, we can take $\mu_1 = \mu_0$, and then the desired estimate $M_1 \le \beta_1$ follows from (2.3).

(b) In the remaining case $S > S_0$, we set $m := [S/S_0] + 1 > S/S_0 > 1$. It is convenient to take $r := 4m$. Then we divide $\Omega \cap (B_{8m} \setminus \overline{B_{4m}})$ into m disjoint sets

$$\Omega_k := \Omega \cap \left(B_{4m+4k} \setminus \overline{B_{4m+4k-4}}\right) \quad \text{for} \quad k = 1, 2, \ldots, m.$$

We have

$$\sum_{k=1}^{m} \int_{\Omega_k} |\mathbf{b}|^n dx \le \int_{\Omega} |\mathbf{b}|^n dx =: S.$$

Since $S < mS_0$, at least one of the integrals on the left side $\le S_0$. Fix the corresponding index $k = k_0$ for which this is the case. Finally, we set $\mu_1 := (4m)^{-n}\mu_0$. By the maximum principle,

$$M_r := \sup_{\Omega \cap B_{4m}} u \le \sup_{\Omega \cap B_{4m+4k_0-2}} u = u(y) \tag{2.4}$$

for some $y \in \Omega \cap (\partial B_{4m+4k_0-2})$. Note that the set $\Omega \cap B_2(y)$ is contained in Ω_{k_0} hence the integral of $|\mathbf{b}|^n$ over this set $\le S_0$, and moreover, from (2.1) it follows

$$|\Omega \cap B_2(y)| \le |\Omega \cap B_{8m}| \le \mu_1 \cdot |B_{8m}| = \mu_0 \cdot |B_2|.$$

From the previous part (a) of the proof it follows that

$$u(y) \le \sup_{\Omega \cap B_1(y)} u \le \beta_1 \cdot \sup_{\Omega \cap B_2(y)} u \le \beta_1 \cdot M_{2r}.$$

Together with (2.4), this completes the proof. □

LEMMA 2.2 (Doubling property). *Let $x_0 \in \mathbb{R}^n$, $r > 0$, and a function v in $W(B_{4r}(x_0))$ be such that*

$$v \ge 0, \quad Lv \le 0 \quad \text{in} \quad B_{4r}(x_0); \quad \text{and} \quad v \ge 1 \quad \text{in} \quad B_r(x_0). \tag{2.5}$$

Then

$$v \ge \lambda = \lambda(n, \nu, S) > 0 \quad \text{in} \quad B_{2r}(x_0). \tag{2.6}$$

PROOF. Without loss of generality, we assume $x_0 = 0$, $r = 1$. In the part (a) below, we consider the case $n = 1$, part(b) deals with $n \ge 2$ and small S, and part (c) – with general $S < \infty$. We need to treat the case $n = 1$ separately, because in our approach, we use the fact that the spheres $\partial B_r \subset \mathbb{R}^n$ are connected. This fails if $n = 1$, because in this case ∂B_r consists of two disjoint point $\pm r$.

(a) In the case $n = 1$, we have

$$v \ge 0, \quad av'' + bv' \le 0 \quad \text{in} \quad (-4, 4); \quad \text{and} \quad v \ge 1 \quad \text{on} \quad [-1, 1],$$

where $0 < \nu \le a \le \nu^{-1}$, and $b \in L^1(-4, 4)$. Using properties of weak derivatives (the *product formula* (7.18) and the *chain rule* in Lemma 7.5, [**GT83**]), one can rewrite the above differential inequality as follows:

$$(Av')' \le 0, \quad \text{where} \quad A(x) := \exp\left[\int_0^x \frac{b(y)}{a(y)} dy\right]. \tag{2.7}$$

The function $A(x)$ satisfies

$$N_0^{-1} \le A(x) \le N_0 := \exp(\nu^{-1} S) \quad \text{in} \quad (-4, 4), \qquad S := \int_{-4}^{4} |b(x)| dx. \tag{2.8}$$

If suffices to get the estimate $v \ge \lambda = \lambda(n,\nu,S) > 0$ on the interval $[-2,-1]$, because a similar estimate on the symmetric interval $[1,2]$ can be obtained by replacing x with $-x$. Denote $\kappa := \min\limits_{[-2,-1]} v$. We can assume $\kappa \le 1/2$, because otherwise the estimate $v \ge \lambda$ holds true with $\lambda = 1/2$. By the maximum principle, i.e. Corollary 1.2 applied to the function $-v$, we have $\kappa = v(-2)$. Since v is continuous and $v(-2) = \kappa \le 1/2 < 1 \le v(-1)$, there is $h \in (0,1]$ such that

$$v(-2+h) = 1, \quad \text{and} \quad v(x) < 1 \quad \text{for all} \quad x \in [-2,-2+h).$$

Furthermore, for each $y \in (-4,-2+h)$ the minimum of $v(x)$ on $[y,-2+h]$ is attained at the point y. This means that the function $v(x)$ is non-decreasing on $(-4,-2+h]$, and therefore $v' \ge 0$ in $(-4,-2+h)$. From (2.7) it follows that the function Av' is non-increasing. Together with (2.8), this implies

$$\begin{aligned}\kappa = v(-2) &\ge \int_{-2-h}^{-2} v'dx \ge \frac{1}{N_0}\int_{-2-h}^{-2} Av'dx \ge \frac{1}{N_0}\int_{-2}^{-2+h} Av'dx\\ &\ge \frac{1}{N_0^2}\int_{-2}^{-2+h} v'dx = \frac{1}{N_0^2}[v(-2+h)-v(-2)] \ge \frac{1}{2N_0^2} =: \lambda.\end{aligned}$$

This proves the estimate (2.6) in the case $n = 1$.

(b) In the rest of the proof of this lemma, we assume $n \ge 2$. First consider the case when $S := S(\Omega) \le S_0 = S_0(n,\nu)$ – a small positive constant to be chosen below in (2.10). Note that

$$\begin{aligned}L_0\big(|x|^{-\gamma_0}\big) := a_{ij}D_{ij}\big(|x|^{-\gamma_0}\big) &= \gamma_0 \cdot \left[\frac{(\gamma_0+2)a_{ij}x_ix_j}{|x|^2} - \operatorname{tr} a\right] \cdot |x|^{-\gamma_0-2}\\ &\ge \gamma_0 \cdot \big[(\gamma_0+2)\nu - n\nu^{-1}\big] \cdot |x|^{-\gamma_0-2} \ge 0 \quad \text{for} \quad x \ne 0,\end{aligned}$$

provided $\gamma_0 + 2 \ge n\nu^{-2}$. Fix such $\gamma_0 = \gamma_0(n,\nu) > 0$, and consider the function

$$w(x) := \frac{|x|^{-\gamma_0} - 4^{-\gamma_0}}{1 - 4^{-\gamma_0}} \quad \text{in} \quad \Omega_1 := \{x \in \mathbb{R}^n : 1 < |x| < 4\}.$$

We have $L_0 w \ge 0$ in Ω_1, $w \le v$ on $\partial\Omega_1$. Therefore,

$$L(w-v) \ge Lw = L_0 w + (b, Dw) \ge (b, Dw) \quad \text{in} \quad \Omega_1,$$

and by Theorem 1.1,

$$\sup_{\Omega_1}(w-v) \le N_0 \cdot ||(b,Dw)||_{n,\Omega_1} \le N_1 \cdot \|\,|\mathbf{b}|\,\|_{n,\Omega} = N_1 \cdot S^{1/n}, \tag{2.9}$$

with some constants N_0, $N_1 \ge 1$ depending only on n and ν. Next, fix the constants

$$\lambda_0 := \frac{2^{-\gamma_0} - 4^{-\gamma_0}}{2\cdot\big(1-4^{-\gamma_0}\big)} \in \Big(0, \frac{1}{2}\Big), \quad \text{and} \quad S_0 := \left(\frac{\lambda_0}{N_1}\right)^n, \tag{2.10}$$

which also depend only on n and ν. Then from $S \le S_0$ and (2.9) it follows $w - v \le \lambda_0$ in Ω_1. On the other hand, obviously $w \ge 2\lambda_0$ on the set $\{1 \le |x| \le 2\} \subset \Omega_1$. Therefore, $v(x) \ge \lambda_0$ for $1 \le |x| \le 2$. Since also $v(x) \ge 1$ for $|x| < 1$, the desired estimate (2.6) holds true with $\lambda = \lambda_0(n,\nu) > 0$, provided $S \le S_0 = S_0(n,\nu)$.

(c) Now it remains to consider the case $n \ge 2$, $S > S_0$. As in the proof of Lemma 2.1, we set $m := [S/S_0] + 1 > S/S_0 > 1$. Choose $\rho_0 \in (0,1/2)$ of order 1/2 and $y_1, \dots, y_m \in \partial B_{1/2}$ such that the balls $B_{\rho_0}(y_k)$ are disjoint for $k = 1, \dots, m$. Then the "tubes"

$$T_k := B_4 \bigcap \Big(\bigcup_{t \ge 1} B_{\rho_0}(t y_k) \Big), \qquad k = 1, \dots, m,$$

are also disjoint. Since $S < m S_0$, we can fix $k = k_1$ such that the integral of $|\mathbf{b}|^n$ over T_{k_1} does not exceed S_0. Further, divide $B_4 \setminus \overline{B_2}$ into m disjoint shells

$$\Omega_k := B_{r_k} \setminus \overline{B_{r_{k-1}}}, \quad \text{where} \quad r_k := 2 + \frac{2k}{m}; \quad k = 1, \dots, m,$$

and choose $k = k_2$ such that the integral of $|\mathbf{b}|^n$ over Ω_{k_2} does not exceed S_0. Finally, set

$$R := 2 + \frac{2k_2 - 1}{m} \in \Big(2 + \frac{1}{m}, 4 - \frac{1}{m} \Big), \qquad \rho := \frac{1}{4} \min \Big\{ \rho_0, \frac{1}{m} \Big\}.$$

Fix an arbitrary $y \in \partial B_R$. One can move the ball $B_{4\rho}(z)$ continuously inside T_{k_1} from $z = y_{k_1} \in \partial B_{1/2}$ to $z = 2R \cdot y_{k_1} \in \partial B_R$, and then inside Ω_{k_2} to $z = y$. Therefore, there is a sequence

$$z_0 := y_{k_1}, z_1, \dots, z_{j_0}, z_{j_0+1}, \dots, z_{m_0} := y,$$

such that

$$B_{4\rho}(z_j) \subset T_{k_1} \quad \text{for} \quad j \le j_0; \quad B_{4\rho}(z_j) \subset \Omega_{k_2} \quad \text{for} \quad j \ge j_0 + 1, \tag{2.11}$$

$|z_{j+1} - z_j| \le \rho$ for all $j = 0, 1, \dots, m_0 - 1$, and m_0 is of order m.

We claim that

$$v \ge \lambda_0^j \quad \text{in} \quad B_\rho(z_j) \quad \text{for} \quad j = 0, 1, \dots, m_0, \tag{2.12}$$

where $\lambda_0 = \lambda_0(n, \nu) > 0$ is a constant in part (b), i.e. in (2.10), which corresponds to $S = S_0$. For $j = 0$, this inequality is contained in (2.5), because $B_\rho(z_0) = B_\rho(y_{k_1}) \subset B_{1/2}(y_{k_1}) \subset B_1$. Moreover, if (2.12) is true for some j, then the function $v_j := \lambda_0^{-j} v$ satisfies (2.5) with $x_0 = z_j$ and $r = \rho$. By our construction of the sets T_{k_1} and Ω_{k_2}, from (2.11) it follows $S(B_{4\rho}(z_j)) \le S_0$ for all j. Therefore, we can apply the preceding part (b) of this proof, which yields $v_j \ge \lambda_0$ in $B_{2\rho}(z_j)$. Since $|z_{j+1} - z_j| \le \rho$, we have $B_\rho(z_{j+1}) \subset B_{2\rho}(z_j)$, hence $v \ge \lambda_0^{j+1}$ on $B_\rho(z_{j+1})$. By induction, (2.12) holds true for all $j = 0, 1, \dots, m_0$. Here m_0 does not exceed a constant m_1 depending only on n, ν, and S. Therefore,

$$v(y) = v(z_{m_0}) \ge \lambda_0^{m_0} \ge \lambda_0^{m_1} =: \lambda = \lambda(n, \nu, S) > 0.$$

Here y is an arbitrary point in ∂B_R. From the inequalities $Lv \le 0$ in B_R and $v \ge \lambda$ on ∂B_R it follows $v \ge \lambda$ on B_R. Since $R > 2 = 2r$, the desired estimate (2.6) follows. □

Corollary 2.3. *Let $x_0 \in \mathbb{R}^n$, $r > 0$, and a function v in $W(B_r(x_0))$ be such that*

$$v \ge 0, \quad Lv \le 0 \quad \text{in} \quad B_r(x_0); \quad \text{and} \quad v \ge 1 \quad \text{in} \quad B_{\varepsilon r}(x_0), \tag{2.13}$$

where $0 < \varepsilon \le 1/2$. Then

$$v \ge \varepsilon^\gamma \quad \text{in} \quad B_{r/2}(x_0), \tag{2.14}$$

where $\gamma = \gamma(n, \nu, S) = -\log_2 \lambda > 0$, and λ is the constant in the previous lemma.

Proof. We assume $x_0 = 0$. Set $r_j := 2^{-j}r$ for $j = 1, 2, \cdots$, and choose natural k such that $2^{-k-1} < \varepsilon \le 2^{-k}$, so that $r_{k+1} < \varepsilon r \le r_k$. By our assumptions, $v \ge 1$ on $B_{\varepsilon r} \supset B_{r_{k+1}}$. The previous lemma with $r = r_{k+1}$ yields $v \ge \lambda$ in B_{r_k}. Repeatedly using this lemma again, we get

$$v \ge \lambda^2 \quad \text{in} \quad B_{r_{k-1}}, \quad \ldots, \quad v \ge \lambda^k \quad \text{in} \quad B_{r_1} = B_{r/2}.$$

Since $\lambda^k = 2^{-k\gamma} \ge \varepsilon^\gamma$, the corollary is proved. □

The following technical lemma will help us to deal with functions defined on a ball B_r rather than on a general open set $\Omega \subset \mathbb{R}^n$.

Lemma 2.4 (Extension lemma). *Let Ω be an open set in $\mathbb{R}^n$, and let u be a function in $W(\Omega)$, such that*

$$u \ge 0, \quad Lu \ge 0 \quad \text{in} \quad \Omega; \quad \text{and} \quad u = 0 \quad \text{in} \quad (\partial\Omega) \cap B_r,$$

where $B_r := B_r(x_0)$ for some $r > 0$ and $x_0 \in \mathbb{R}^n$. We claim that there are functions $u_\varepsilon \in W(B_r)$ defined for each $\varepsilon > 0$, such that

$$u_\varepsilon \ge 0, \quad Lu_\varepsilon \ge 0 \quad \text{in} \quad B_r; \qquad u_\varepsilon \equiv 0 \quad \text{in} \quad \overline{B_r} \setminus \Omega,$$

and $u_\varepsilon \to u$ as $\varepsilon \to 0^+$ uniformly on $\Omega \cap B_r$.

Proof. We partially follow [**CS07**], pp. 6–7. Fix a standard function

$$0 \le \eta \in C^\infty(\mathbb{R}^1), \quad \text{such that} \quad \eta(t) \equiv 0 \quad \text{for} \quad |t| \ge 1, \quad \text{and} \quad \int_{\mathbb{R}^1} \eta(t)\, dt = 1;$$

and set $\eta_\varepsilon(t) := \varepsilon^{-1}\eta(\varepsilon^{-1}t - 2)$ for $\varepsilon > 0$, $t \in \mathbb{R}^1$. These are smooth functions vanishing on $\mathbb{R}^1 \setminus [\varepsilon, 3\varepsilon]$. Further, by repeated integration of η_ε, define the functions $G_\varepsilon \in C^\infty(\mathbb{R}^1)$ satisfying the properties

$$G_\varepsilon \equiv 0 \ \text{ on } (-\infty, \varepsilon], \ \ G''_\varepsilon \equiv \eta_\varepsilon \ \text{ on } \mathbb{R}^1.$$

Now we set

$$u_\varepsilon := G_\varepsilon(u) \ \text{ in } \Omega, \ \ u_\varepsilon \equiv 0 \ \text{ on } \overline{B_r} \setminus \Omega.$$

From the properties of η_ε it follows $G'_\varepsilon \ge 0$, $G''_\varepsilon \ge 0$, and $(u - 3\varepsilon)_+ \le u_\varepsilon \le (u - \varepsilon)_+$. Since $u = 0$ on the set $(\partial\Omega) \cap B_r$, the functions u_ε vanish near this set. Hence we have $u_\varepsilon \in W(B_r)$, $u_\varepsilon \ge 0$, and $|u_\varepsilon - u| \le 3\varepsilon$ on Ω. Finally,

$$Lu_\varepsilon = LG_\varepsilon(u) = G'_\varepsilon(u) \cdot Lu + G''_\varepsilon(u) \cdot a_{ij} D_i u \, D_j u \ge 0 \ \text{ in } \Omega.$$

Lemma is proved. □

In comparison with the first growth lemma (Lemma 2.1), the next lemma states that, roughly speaking, from (2.1) with $\mu_1 < 1$ it follows (2.2) with $\beta_1 < 1$.

Lemma 2.5 (Second growth lemma). *Let Ω be an open set in $\mathbb{R}^n$, and let $u \in W(\Omega)$, $x_0 \in \mathbb{R}^n$, and $r > 0$ be such that*

$$u \ge 0, \quad Lu \ge 0 \quad \text{in} \quad \Omega; \quad \text{and} \quad u = 0 \quad \text{on} \quad (\partial\Omega) \cap B_{2r}(x_0).$$

We claim that for arbitrary $\mu_2 \in (0, 1)$, there is a constant $\beta_2 = \beta_2(n, \nu, S, \mu_2) \in (0, 1)$, such that from

$$|\Omega \cap B_r(x_0)| \le \mu_2 \cdot |B_r| \tag{2.15}$$

it follows

$$M_r := \sup_{\Omega \cap B_r(x_0)} u \le \beta_2 \cdot M_{2r}. \tag{2.16}$$

Replacing Ω by $\Omega\cap B_{2r}(x_0)$, and u by $\mathrm{const}\cdot u$ we can assume that $0\le u\le M_{2r}=1$ in Ω. Taking

$$v:=1-u,\quad \mu:=1-\mu_2\in(0,1),\quad\text{and}\quad \beta:=1-\beta_2\in(0,1),$$

we see that this lemma follows from the following one.

LEMMA 2.6. *Let Ω be an open set in $\mathbb{R}^n$, and let $v\in W(\Omega)$, $x_0\in\mathbb{R}^n$, and $r>0$ be such that*

$$v\ge 0,\quad Lv\le 0\quad\text{in}\quad\Omega;\quad\text{and}\quad v\ge 1\quad\text{on}\quad(\partial\Omega)\cap B_{2r}(x_0).$$

We claim that for arbitrary $\mu\in(0,1)$, there is a constant $\beta=\beta(n,\nu,S,\mu)\in(0,1)$ such that from $|B_r(x_0)\setminus\Omega|\ge\mu\cdot|B_r|$ it follows $v\ge\beta$ on $\Omega\cap B_r(x_0)$.

PROOF. We follow the lines of the proof of Lemma 2.3 in [**S80**], though some details are different. As before, we assume $x_0=0$. Moreover, applying the extension lemma (Lemma 2.4) to the function $u:=1-v$, we can also assume that the function v belongs to $W(B_{2r})$ and satisfies $v\ge 0$, $Lv\le 0$ in B_{2r}. One needs to show that from $|B_r\cap\{v\ge 1\}|\ge\mu\cdot|B_r|$ with $0<\mu<1$ it follows $v\ge\beta=\beta(n,\nu,S,\mu)\in(0,1)$ on B_r.

First consider the case

$$|B_r\cap\{v\ge 1\}|\ge\mu_0\cdot|B_r|,\quad\text{where}\quad\mu_0:=1-\mu_1(n,\nu,S,1/2),\tag{2.17}$$

i.e. μ_1 is the constant in Lemma 2.1 corresponding to $\beta_1=1/2$. Then

$$\begin{aligned}|B_r\cap\{u>0\}|=|B_r\cap\{v<1\}|&=|B_r|-|B_r\cap\{v\ge 1\}|\\&\le(1-\mu_0)\cdot|B_r|=\mu_1\cdot|B_r|.\end{aligned}$$

By Lemma 2.1 applied to $\Omega':=\Omega\cap B_r\cap\{u>0\}$ and $r/2$ in place of r, we get $u\le 1/2$, and $v=1-u\ge 1/2$ on $B_{r/2}$. Now by the doubling property (Lemma 2.2), $v\ge\beta_0=\beta_0(n,\nu,S):=\lambda/2>0$ on B_r. We have proved that from (2.17) it follows $v\ge\beta_0=\beta_0(n,\nu,S)>0$ on B_r, so that the estimate $v\ge\beta>0$ holds true for $\mu\ge\mu_0$ with $\beta=\beta_0$.

Now consider the remaining case when the set $\Gamma:=B_r\cap\{v\ge 1\}$ satisfies $\mu\cdot|B_r|\le|\Gamma|<\mu_0\cdot|B_r|$. Almost every point x in the set Γ is its density point, which implies that $B_\rho(x)\subset B_r$ and $|\Gamma\cap B_\rho(x)|>\mu_0\cdot|B_\rho|$ for small $\rho>0$. One can include $B_\rho(x)$ into a monotone continuous family of balls B^θ, $0\le\theta\le 1$, such that $B^0=B_\rho(x)$ and $B^1=B_r$. Then $\varphi(\theta):=|\Gamma\cap B^\theta|/|B^\theta|$ is a continuous function on $[0,1]$ with the boundary values $\varphi(0)>\mu_0>\varphi(1)$. Therefore, for some intermediate value $\theta_0\in(0,1)$ we have $\varphi(\theta_0)=\mu_0$, and the corresponding ball $B:=B^{\theta_0}$ satisfies

$$B\subset B_r,\qquad|B\cap\Gamma|=|B\cap\{v\ge 1\}|=\mu_0\cdot|B|.\tag{2.18}$$

Denote by Γ_1 the union of all the balls B satisfying (2.18). By the previous argument $v\ge\beta_0>0$ on Γ_1. Obviously $|\Gamma\setminus\Gamma_1|=0$, because Γ_1 contains all the density points of Γ. Further, by the *simple Vitali lemma*, there is a finite collection of *disjoint* balls $B^{(1)},B^{(2)},\dots,B^{(m)}$, satisfying (2.18), such that

$$\sum_{j=1}^m|B^{(j)}|\ge c_0\cdot|\Gamma_1|,\quad\text{where}\quad c_0=c_0(n)>0.$$

This implies

$$|\Gamma_1 \setminus \Gamma| \ge \sum_{j=1}^m |B^{(j)} \cap (\Gamma_1 \setminus \Gamma)| = \sum_{j=1}^m |B^{(j)} \setminus \Gamma|$$

$$= (1-\mu_0) \sum_{j=1}^m |B^{(j)}| \ge (1-\mu_0)c_0 \cdot |\Gamma_1| \ge (1-\mu_0)c_0 \cdot |\Gamma|,$$

and

$$|B_r \cap \{v \ge \beta_0\}| \ge |\Gamma_1| \ge c_1 \cdot |\Gamma| = c_1 \cdot |B_r \cap \{v \ge 1\}| \ge c_1\mu \cdot |B_r|, \tag{2.19}$$

where $c_1 = c_1(n,\nu,S) := 1 + (1-\mu_0)c_0 > 1$. If the left side is $< \mu_0 \cdot |B_r|$, then we can apply this estimate again with the function $\beta_0^{-1}v$ in place of v, then to $\beta_0^{-2}v$, etc, which gives us

$$|B_r \cap \{v \ge \beta_0^k\}| \ge c_1^k \mu \cdot |B_r| \quad \text{for} \quad k = 1, 2, \dots.$$

This iteration stops when the left side becomes $\ge \mu_0 \cdot |B_r|$. Since $c_1^k\mu \le 1$, we must have $k \le k_0 = k_0(n,\nu,S,\mu) := [-\ln\mu/\ln c_1]$. Finally, since the function $\beta_0^{-k_0}v$ satisfies (2.17), it follows $\beta_0^{-k_0}v \ge \beta_0 > 0$ on B_r, and the estimate $v \ge \beta > 0$ on B_r holds true with $\beta = \beta(n,\nu,S,\mu) := \beta_0^{k_0+1} > 1$. Lemma is proved. □

Hölder regularity of solutions to non-homogeneous equations $Lu = f$ with $f \in L^n$ follows from Theorem 1.1 and Lemma 2.5 in the same way as Theorem IV.2.5 in [**K85**], or Theorem 4.1 in [**S80**], are derived from the corresponding statements. The next theorem is quite similar to these results, therefore we formulate it without proof. We only note that the proof uses approximation of u by solutions of equations with regular coefficients, so that some auxiliary boundary value problems have solutions. This part is provided be the approximation Lemma 4.2 below.

Theorem 2.7. *Let u be a function in $W(B_{2r})$, $r > 0$, such that $Lu = f$ in B_{2r}, where $f \in L^n(B_{2r})$. Then there are constants $\alpha \in (0,1)$ and $N > 0$, depending only on n, ν, and S, such that*

$$\sup_{x,y \in B_r} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \le \frac{N}{r^\alpha} \cdot \Big(\sup_{B_{2r}} |u| + r \cdot ||f||_{n,B_{2r}} \Big). \tag{2.20}$$

3. Interior and boundary Harnack inequalities

Theorem 3.1 (Interior Harnack inequality). *Let u be a function in $W(B_{8r})$ satisfying $u > 0$, $Lu := a_{ij}D_{ij}u + b_iD_iu = 0$ in B_{8r} for some $r > 0$. Then*

$$\sup_{B_r} u \le N_1 \cdot \inf_{B_r} u, \quad \text{where} \quad N_1 = N_1(n,\nu,S) \ge 1, \quad S := \int_{B_{8r}} |\mathbf{b}|^n dx. \tag{3.1}$$

Proof. We partially follow the proof of Theorem 3.1 in [**S80**].

Without loss of generality, we assume $r = 1$. Let $\gamma = \gamma(n,\nu,S) > 0$ be the constant in Corollary 2.3. Since $2 - |x| \ge 1$ in the ball $B_1 := B_1(0)$, we have

$$\sup_{B_1} u \le M := \sup_{B_2} (2-|x|)^\gamma u = (2-|x_0|)^\gamma u(x_0) \tag{3.2}$$

for some $x_0 \in B_2$. Further, consider the function

$$u_0 := u - \frac{u(x_0)}{2} \quad \text{in the ball} \quad B_\rho(x_0), \quad \text{where} \quad \rho := \frac{2-|x_0|}{2}.$$

Since $2-|x|\ge\rho$ in $B_\rho(x_0)$, we also have

$$\sup_{B_\rho(x_0)} u_0 < \sup_{B_\rho(x_0)} u \le \sup_{B_\rho(x_0)} \left(\frac{2-|x|}{\rho}\right)^\gamma u \le \rho^{-\gamma}M = 2^\gamma u(x_0) = 2^{\gamma+1}u_0(x_0),$$

and

$$\sup_{B_{\rho/2}(x_0)} u_0 \ge u_0(x_0) > \beta_1\cdot \sup_{B_\rho(x_0)} u_0, \quad \text{where} \quad \beta_1=\beta_1(n,\nu,S):=2^{-\gamma-1}>0.$$

Now we can use Lemma 2.1 in an equivalent form "if (2.2) fails, then (2.1) fails", with $\Omega := B_8\cap\{u_0>0\}$, $r:=\rho/2$, and u_0 in place of u. By this lemma, there is a constant $\mu_1>0$ depending only on n,ν, and S, such that

$$|B_\rho(x_0)\cap\{u_0>0\}| > \mu_1\cdot|B_\rho|. \tag{3.3}$$

Next, the function $v:=u/u_0(x_0)$ satisfies

$$v>0, \quad Lv=0 \quad \text{in} \quad \Omega' := B_8\cap\{v<1\} = B_8\setminus\overline{\Omega},$$

and $v=1$ on $(\partial\Omega')\cap B_8$. Moreover, by virtue of (3.3),

$$|B_\rho(x_0)\setminus\Omega'| = |B_\rho(x_0)\cap\{v\ge1\}| = |B_\rho(x_0)\cap\{u_0\ge0\}| > \mu_1\cdot|B_\rho|.$$

Applying Lemma 2.6 to the function v in Ω', with $r:=\rho$, we obtain the estimate

$$v\ge\beta=\beta(n,\nu,S,\mu_1)>0 \quad \text{in} \quad B_\rho(x_0).$$

By the choice of x_0 and ρ,

$$u=\frac{u(x_0)}{2}\cdot v\ge c_1 := \frac{\beta}{2}\cdot(2\rho)^{-\gamma}M \quad \text{in} \quad B_\rho(x_0).$$

Finally, we apply Corollary 2.3 to the function $c_1^{-1}u$ with $r=6$ and $\varepsilon:=\rho/6\in(0,1/6)$. In our case, $B_3(x_0)=B_{r/2}(x_0)\subset B_6(x_0)\subset B_8$, so that the estimate (2.14) implies $c_1^{-1}u\ge\varepsilon^\gamma$ in the ball $B_3(x_0)$, which contains $B_1:=B_1(0)$. Therefore,

$$\inf_{B_1} u \ge \inf_{B_3(x_0)} u \ge \varepsilon^\gamma c_1 = \left(\frac{\rho}{6}\right)^\gamma\cdot\frac{\beta}{2}\cdot(2\rho)^{-\gamma}M = \frac{\beta M}{2\cdot 12^\gamma}.$$

This estimate together with (3.2) imply the Harnack inequality (3.1) with $N_1 = N_1(n,\nu,S) := 2\cdot12^\gamma\cdot\beta^{-1}\ge1$. □

As a standard consequence of the interior Harnack inequality, we have

THEOREM 3.2 (Liouville). *Let u be a bounded from above or from below function in $W(\mathbb{R}^n) := W^{2,n}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ such that $u>0$, and $Lu := a_{ij}D_{ij}u + b_iD_iu = 0$ in the entire space $\mathbb{R}^n$, with a_{ij} satisfying* (1.2) *and $|\mathbf{b}|\in L^n(\mathbb{R}^n)$. Then $u=\mathrm{const}$ in $\mathbb{R}^n$.*

Indeed, replacing u by const $\pm\, u$ is necessary, we can reduce the proof to the case $\inf_{\mathbb{R}^n} u = 0$. In this case, taking the limit in (3.1) as $r\to\infty$, we get $u\equiv0$ in $\mathbb{R}^n$.

The following theorem is a more general form of the interior Harnack inequality, which is convenient for applications.

THEOREM 3.3. *Let Ω be a bounded open set in $\mathbb{R}^n$, and let u be a function in $W(\Omega)$ satisfying $u>0$, $Lu=0$ in Ω. Then for arbitrary constant $\delta>0$, such that the set $\Omega^\delta := \{x\in\Omega : \operatorname{dist}(x,\partial\Omega)>\delta\}$ is nonempty and* connected, *we have*

$$\sup_{\Omega^\delta} u \le N_2\cdot \inf_{\Omega^\delta} u, \quad \text{where} \quad N_2 = N_2(n,\nu,S,\delta/\operatorname{diam}\Omega), \quad S := \int_\Omega |\mathbf{b}|^n dx. \tag{3.4}$$

PROOF. Fix points $x,y\in\Omega^\delta$. One can choose a sequence $x^{(0)},x^{(1)},\dots,x^{(m)}$ in Ω^δ, such that

$$x^{(0)}=x, \quad x^{(m)}=y, \quad \text{and} \quad |x^{(k)}-x^{(k-1)}|<\delta/8 \quad \text{for all} \quad k=1,\dots,m;$$

and m does not exceed a number m_0 which depends only on n and $\delta/\operatorname{diam}\Omega$. Applying Theorem 3.1 to the balls $B_r(x^{(k)})$, $k=1,\dots,m\le m_0$, we obtain

$$u(x)=u(x^{(0)})\le N_1u(x^{(1)})\le \dots N_1^m u(x^{(m)}) = N_1^m u(y)\le N_1^{m_0}u(y),$$

where $N_1=N_1(n,\nu,S)$ is the constant in (3.1). Since the points $x,y\in\Omega^\delta$ can be selected in an arbitrary way, the inequality (3.4) follows with $N_2:=N_1^{m_0}$. □

We now proceed to the boundary estimates for positive solutions vanishing at the "bottom" of a Lipschitz cylinder. Here ψ is a Lipschitz function on $\mathbb{R}^{n-1}$ with a Lipschitz constant $K\ge 0$, i.e.

$$|\psi(x')-\psi(y')|\le K\cdot|x'-y'| \quad \text{for all} \quad x',y'\in\mathbb{R}^{n-1}. \tag{3.5}$$

For $r>0$, denote

$$\begin{aligned} Q_r &:= \{x=(x',x_n)\in\mathbb{R}^n : \ |x'|<r,\ 0<x_n-\psi(x')<r\},\\ \Gamma_r &:= \{x=(x',x_n)\in\mathbb{R}^n : \ |x'|\le r,\ x_n=\psi(x')\}\subset\partial Q_r. \end{aligned} \tag{3.6}$$

The following theorem contains a Carleson type estimate for equations (1.1) with $|\mathbf{b}|\in L^n$.

THEOREM 3.4 (Boundary Harnack inequality). *Let ψ be a function on $\mathbb{R}^{n-1}$ satisfying the Lipschitz condition* (3.5)*, $\psi(0)=0$, and let u be a function in $W(Q_{2r})$ for some $r>0$, such that*

$$u>0, \quad Lu=0 \quad \text{in} \quad Q_{2r}; \quad \text{and} \quad u=0 \quad \text{on} \quad \Gamma_{2r}, \tag{3.7}$$

where Q_{2r} and Γ_{2r} are defined according to (3.6)*. Then*

$$\sup_{Q_r} u\le N_3u(0,r), \quad \text{where} \quad N_3=N_3(n,\nu,S,K)\ge 1. \tag{3.8}$$

PROOF. Since all the conditions here are invariant with respect to rescaling (see Remark 1.4), we can assume that $r=1$.

(a) First we show that there is a constant $\gamma=\gamma(n,\nu,S,K)>0$, such that

$$M:=\sup_{Q_2} d^\gamma u\le N_1u(P_1), \quad \text{where} \quad d=d(x):=\operatorname{dist}(x,\partial Q_2), \tag{3.9}$$

$N_1=N_1(n,\nu,s)\ge 1$ is the constant in Theorem 3.1, and $P_1:=(0,1)$.

Fix $x\in Q_2$. The ball of radius $d(x)$ centered at x touches ∂Q_2 at some point y. A simple geometrical consideration shows that there is a smooth curve parameterized by the arc length $C:=\{z=z(s),\ 0\le s\le s_0\}\subset\overline{Q_2}$, connecting the points $y=z(0)$ and $P_1=z(s_0)$, passing through the point x, i.e. $z(s_1)=x$ for some $s\in(0,s_0]$, and such that

$$cs\le d(z(s))\le s \quad \text{for all} \quad s\in[0,s_0], \tag{3.10}$$

with a constant $c \in (0,1)$ depending only on K. (This property means that Q_2 is a *John domain*)

Set $\theta := 1 - c/8$. Then for arbitrary $s \in (0, s_0]$ and $t \in [\theta s, s]$, we have

$$|z(t) - z(s)| \le |t - s| \le (1-\theta)s = \frac{cs}{8} \le \frac{d(z(s))}{8}.$$

From Theorem 3.1, with $r = d(z(s))/8)$, it follows

$$N_1 u(z(s)) \ge u(z(t)), \qquad 0 < \theta s \le t \le s \le s_0. \tag{3.11}$$

Fix a constant $\gamma = \gamma(n, \nu, S, K) > 0$ such that $1 \ge \theta^\gamma N_1$. Then

$$\varphi(s) := s^\gamma u(z(s)) \ge N_1(\theta s)^\gamma u(z(s)) \ge (\theta s)^\gamma u(z(\theta s)) = \varphi(\theta s), \qquad 0 < s \le s_0.$$

For each $s \in (0, s_0]$, there is an integer $k \ge 0$, such that $\theta s_0 < \theta^{-k} s \le s_0$. Using the previous inequalities, including (3.11) with $s = s_0$, we get

$$\varphi(s) \le \varphi(\theta^{-1}s) \le \cdots \le \varphi(\theta^{-k}s) \le N_1 u(z(s_0)) = N_1 u(P_1).$$

Therefore, at the point $x = z(s_1)$,

$$d^\gamma u(x) = d^\gamma u(z(s_1)) \le s_1^\gamma u(z(s_1)) = \varphi(s_1) \le N_1 u(P_1).$$

Since $x \in Q_2$ can be chosen in an arbitrary way, the estimate (3.9) follows.

(b) Our next step is to prove the estimate

$$M_0 := \sup_{Q_2} d_0^\gamma u \le N_0 M, \quad \text{where} \quad d_0 = d_0(x) := \operatorname{dist}(x, (\partial Q_2) \setminus \Gamma_2), \tag{3.12}$$

with a constant $N_0 = N_0(n, \nu, S, K) \ge 1$. Note that $d_0^\gamma u = 0$ on ∂Q_2, hence the *supremum* in (3.12) is attained at some point $x_0 \in Q_2$, i.e. $d_0^\gamma u(x_0) = M_0$. We claim that

$$d(x_0) \ge \varepsilon_0 d_0(x_0) \quad \text{with} \quad \varepsilon_0 = \varepsilon_0(n, \nu, S, K) \in (0, 1/4]. \tag{3.13}$$

The constant ε_0 will be specified below. Suppose (3.13) fails, i.e.

$$\rho := d(x_0) < \varepsilon_0 \rho_0, \quad \text{where} \quad \rho_0 := d_0(x_0), \quad 0 < \varepsilon_0 \le 1/4.$$

Since $4\rho < 4\varepsilon_0 d_0(x_0) < d_0(x_0)$, the intersection $(\partial Q_2) \cap B_{4\rho}(x_0)$ lies in Γ_2, so that $u = 0$ on this set. Further, the ball $B_\rho(x_0)$ touches ∂Q_2 at some point $y_0 \in \Gamma_2$. By (3.6), Γ_2 is the graph of a Lipschitz function $x_n = \psi(x')$ restricted to $|x'| \le 2$. It is easy to see that the measure

$$|B_{2\rho}(x_0) \setminus Q_2| \ge |B_\rho(y_0) \setminus Q_2| \ge \mu \rho^n \quad \text{with} \quad \mu = \mu(n, K) \in (0,1).$$

Now we can apply Lemma 2.5 with $r := 2\rho$ and $\mu_2 := 1 - 2^{-n}\mu$. By this lemma,

$$u(x_0) \le \sup_{Q_2 \cap B_{2\rho}(x_0)} u \le \beta \cdot \sup_{Q_2 \cap B_{4\rho}(x_0)} u, \quad \text{where} \quad \beta = \beta(n, \nu, S, K) \in (0,1).$$

By the triangle inequality,

$$d_0(x) \ge \rho_0 - 4\rho > (1 - 4\varepsilon_0)\rho_0 \quad \text{on} \quad Q_2 \cap B_{4\rho}(x_0).$$

Combining these inequalities, we obtain

$$M = \rho_0^\gamma u(x_0) \le (1 - 4\varepsilon_0)^{-\gamma} \beta \cdot \sup_{Q_2 \cap B_{4\rho}(x_0)} d_0^\gamma u \le (1 - 4\varepsilon_0)^\gamma \beta \cdot M.$$

For small enough $\varepsilon_0 = \varepsilon_0(n, \nu, S, K) \in (0,1)$, the right side is strictly less that M, and we get the desired contradiction.

The above argument proves the estimate (3.13), which in turn implies (3.12) with $N_0 := \varepsilon_0^{-\gamma} \ge 1$ as follows:

$$M_0 = d_0^\gamma u(x_0) \le \varepsilon_0^{-\gamma} d^\gamma u(x_0) \le \varepsilon_0^{-\gamma} M.$$

(c) Both "top" and "bottom" portions of ∂Q_2 are graphs of Lipschitz functions $x_n = \psi(x') + c$, with $c = 0$ or 2. An elementary geometric reasoning shows that $d_0(x) \ge (1+K^2)^{-1/2}$ on Q_1. Hence

$$\sup_{Q_1} u \le (1+K^2)^{\gamma/2} \sup_{Q_1} d_0^\gamma u \le (1+K^2)^{\gamma/2} M_0.$$

This estimate together with (3.12) and (3.9) yield the desired estimate estimate (3.4). Lemma is proved. □

In the following Theorem 3.6, which is preceded with a technical Lemma 3.5, we deal with ratios u_1/u_2 of positive solutions. Note that only the numerator u_1 vanishes on Γ_{2r}, while u_2 is just a positive solution. In particular, in the case $u_2 \equiv 1$, Theorem 3.6 is reduced to Theorem 3.4.

Let $\psi = \psi(x')$ be a function on $\mathbb{R}^{n-1}$ satisfying the Lipschitz condition (3.5), and $\psi(0) = 0$. For $r > 0$ and $h \ge 0$, denote

$$\begin{aligned} Q_{r,h} &:= \{x = (x', x_n) \in \mathbb{R}^n : |x'| < r, \quad 0 < x_n - \psi(x') < h\}, \\ Q^+_{r,h} &:= \{x = (x', x_n) \in \mathbb{R}^n : |x'| < r, \quad h/2 < x_n - \psi(x') < h\}, \\ \Gamma_{r,h} &:= \{x = (x', x_n) \in \mathbb{R}^n : |x'| \le r, \quad x_n = \psi(x')\}, \\ S_{r,h} &:= \{x = (x', x_n) \in \mathbb{R}^n : |x'| = r, \quad 0 < x_n - \psi(x') < h\}. \end{aligned} \tag{3.14}$$

Comparing these notations with (3.6), we see that $Q_r = Q_{r,r}$, $\Gamma_r = \Gamma_{r,0}$. For $r, h > 0$, the boundary $\partial Q_{r,h}$ of the "cylinder" $Q_{r,h}$ is the union of three disjoint sets: the "top" $\Gamma_{r,h}$, the "bottom" Γ_r, and the "lateral side" $S_{r,h}$. If $\psi \equiv 0$, this terminology is understood in the usual sense.

LEMMA 3.5. *Let w be a function in $W(Q_{r,h})$ for some $0 < h \le r$, such that*

$$Lw = 0 \quad \text{in} \quad Q_{r,h}, \qquad w \ge 0 \quad \text{on} \quad \Gamma_r, \tag{3.15}$$

and

$$\inf_{\Gamma_{r,h}} w \ge \sup_{S_{r,h}} (-w)_+. \tag{3.16}$$

We claim that there is a constant $\varepsilon_1 = \varepsilon_1(n, \nu, S, K) \in (0, 1/4]$, such that from $h \le \varepsilon_1 r$ it follows

$$w(0, x_n) \ge 0 \quad \text{for} \quad 0 \le x_n \le h. \tag{3.17}$$

PROOF. Without loss of generality, we assume $r = 1$, and the left side of (3.16) is equal to 1. Then

$$w \ge 1 \quad \text{on} \quad \Gamma_{1,h}, \qquad w \ge 0 \quad \text{on} \quad \Gamma_1, \qquad \text{and} \quad w \ge -1 \quad \text{on} \quad S_{1,h}. \tag{3.18}$$

(a) Consider the function

$$u := -w \quad \text{on the set} \quad \Omega := Q_{1,h} \cap \{w < 0\}.$$

Obviously, it satisfies

$$0 < u \le 1, \quad Lu = 0 \quad \text{in} \quad \Omega; \qquad u = 0 \quad \text{on} \quad (\partial\Omega) \cap Q_{1,h}.$$

For each $x_0 \in \Omega \cap S_{3/4,h}$, the measure

$$|\Omega \cap B_{1/4}(x_0)| \le |Q_{1,h}| \le N_0 \cdot h \quad \text{with} \quad N_0 = N_0(n) > 0,$$

so that we can apply Lemma 2.1 with $x_0 \in \Omega \cap S_{3/4,h}$ and $r = 1/8$ to the function $u := -w$. Since also $u \le 0$ on the remaining part of $\partial Q_{3/4,h}$, we obtain the estimate

$$\sup_{Q_{3/4,h}} (-w)_+ = \sup_{S_{3/4,h}} (-w)_+ \le \beta_1 = \beta_1(n, \nu, S, h) \to 0^+ \quad \text{as} \quad h \to 0^+. \tag{3.19}$$

(b) Next, set $w_1 := w + \beta_1$. By the maximum principle, (3.18), and (3.19), it follows

$$w_1 \ge 0, \quad Lw_1 = 0 \quad \text{in} \quad Q_{3/4,h}; \qquad w_1 \ge 0 \quad \text{on} \quad \Gamma_{3/4}, \quad w_1 \ge 1 \quad \text{on} \quad \Gamma_{3/4,h}.$$

For an arbitrary $x_0 = (x_0', x_{0n}) \in \Gamma_{1/2,h}$ and $\rho := (1 + K^2)^{-1/2} h/2$, the intersection $(\partial Q_{3/4,h}) \cap B_{2\rho}(x_0)$ lies in the set $\Gamma_{3/4,h}$, which is the graph of a Lipschitz function. Therefore, the measure $|B_\rho(x_0) \setminus Q_{3/4,h}| \ge \mu \cdot |B_\rho|$ with a constant $\mu = \mu(n, K) \in (0, 1)$. Using Lemma 2.6 with $Q_{3/4,h}$, w_1, ρ in place of Ω, v, r respectively, we get the estimate

$$w_1 \ge \beta = \beta(n, \nu, S, K) > 0 \quad \text{on} \quad Q_{3/4,h} \cap B_\rho(x_0).$$

Further, by Theorem 3.3, applied to the function w_1 on the intersection $Q_{3/4,h} \cap \{|x' - x_0'| < \rho\}$, it follows

$$w_1(x_0 - t\mathbf{e_n}) \ge \beta_0 = \beta_0(n, \nu, S, K) > 0 \quad \text{for} \quad 0 \le t \le \frac{h}{2}.$$

Since x_0 is an arbitrary point in $\Gamma_{1/2,h}$, we get the estimate

$$w_1 \ge \beta_0 > 0 \quad \text{in} \quad Q^+_{1/2,h}. \tag{3.20}$$

(c) It is important that the constant β_0 in (3.20) does not depend on h. By virtue of (3.19), one can choose the constant $\varepsilon_1 = \varepsilon_1(n, \nu, s, K) \in (0, 1/4]$ in such a way that from $h \in (0, \varepsilon_1]$ it follows $2\beta_1 \le \beta_0$. Then the estimates (3.20) and (3.19) imply

$$\inf_{Q^+_{1/2,h}} w = \inf_{Q^+_{1/2,h}} w_1 - \beta_1 \ge \beta_1 \ge \sup_{Q_{1/2,h}} (-w)_+. \tag{3.21}$$

Note that $\partial Q^+_{1/2,h}$ contains the set $\Gamma_{1/2,h/2} = \Gamma_{r/2,h/2}$, and $Q_{1/2,h}$ contains $S_{1/2,h/2} = S_{r/2,h/2}$. Therefore

$$\inf_{\Gamma_{r/2,h/2}} w \ge \sup_{S_{r/2,h/2}} (-w)_+.$$

This simply means that the inequality (3.16) remains true with r, h being replaced by $r/2, h/2$. Iterating this procedure, we can replace r, h by $2^{-k}r, 2^{-k}h$ for $k = 1, 2, \ldots$. Correspondingly, the first inequality in (3.21) implies

$$w \ge 0 \quad \text{on} \quad Q^+_{2^{-k-1}r, 2^{-k}h} \quad \text{for} \quad k = 0, 1, 2, \ldots,$$

and (3.17) follows by continuity of w. Lemma is proved. □

Theorem 3.6 (Comparison theorem). *Let ψ be a function on $\mathbb{R}^{n-1}$ satisfying the Lipschitz condition (3.5), $\psi(0) = 0$, and let let u_1 and u_2 be functions in $W(Q_{3r})$, $r > 0$, such that*

$$u_{1,2} > 0, \quad Lu_{1,2} = 0 \quad \text{in} \quad Q_{3r}; \quad \text{and} \quad u_1 = 0 \quad \text{on} \quad \Gamma_{3r}. \tag{3.22}$$

Then

$$\sup_{Q_r} \frac{u_1}{u_2} \le N_4 \cdot \frac{u_1(0,r)}{u_2(0,r)}, \quad \text{where} \quad N_4 = N_4(n,\nu,S,K) \ge 1. \tag{3.23}$$

PROOF. Multiplying $u_{1,2}$ by appropriate constants if necessary, we can assume $u_1(0,r) = u_2(0,r) = 1$. By Theorem 3.4

$$u_1 \le N_3 \quad \text{in} \quad Q_r, \tag{3.24}$$

where $N_3 = N_3(n,\nu,S,K) \ge 1$ is the constant in this theorem. Then set $h := \varepsilon_1 r$, where $\varepsilon_1 = \varepsilon_1(n,\nu,S,K) \in (0,1)$ is the constant in the previous lemma. Applying the Harnack inequality, Theorem 3.3, to the function u_2, we obtain

$$u_2 \ge c_0 = c_0(n,\nu,S,K) > 0 \quad \text{on} \quad Q_r \setminus Q_{r,h}. \tag{3.25}$$

Finally, we set

$$N_4 := 2N_3/c_0, \qquad \text{and} \qquad w := N_4 u_2 - u_1.$$

Then

$$w \ge N_4 c_0 - N_3 \ge N_3 \quad \text{on} \quad Q_r \setminus Q_{r,h}, \qquad \text{and} \qquad -w \le u_1 \le N_3 \quad \text{in} \quad Q_r.$$

Therefore, the function w satisfies all the assumption of the previous lemma, which implies $w(0,x_n) \ge 0$ for $0 \le x_n \le r$.

This construction remains valid if we move the origin $0 \in \mathbb{R}^n$ to any point $y = (y', y_n) \in \Gamma_r$. Under this translation, the set Q_{2r} will be replaced by

$$Q_{2r}(y) := \{x = (x', x_n) \in \mathbb{R}^n : \ |x' - y'| < 2r, \ \ 0 < x_n - \psi(x') < 2r\},$$

which is a subset of Q_{3r}. As a result, we have $w \ge 0$, or equivalently, $u_1/u_2 \le N_4$, on the whole set Q_1. Theorem is proved. □

COROLLARY 3.7. *Let u_1 and u_2 be functions in $W(Q_{3r})$, $r > 0$, satisfying* (3.22)*, and in addition, $u_2 = 0$ on Γ_{3r}. Then*

$$\sup_{Q_r} \frac{u_1}{u_2} \le N_4^2 \cdot \inf_{Q_r} \frac{u_1}{u_2}, \tag{3.26}$$

where $N_4 = N_4(n,\nu,S,K) \ge 1$ is the constant in (3.23).

PROOF. Since both functions u_1 and u_2 vanish on Γ_{3r}, we can interchange u_1 and u_2 in (3.23), so that

$$\Big(\inf_{Q_r} \frac{u_1}{u_2}\Big)^{-1} = \sup_{Q_r} \frac{u_2}{u_1} \le N_4 \cdot \frac{u_2(0,r)}{u_1(0,r)}.$$

Multiplying both sides of this inequality by the corresponding sides of (3.23), we get the desired estimate (3.26). □

4. Boundary Hopf-Oleinik estimates

In a particular case $\psi \equiv 0$ and $b \equiv 0$, the function $u_2(x) = u_2(x', x_n) := x_n$ satisfies $Lu = 0$ and vanishes on the $\{x_n = 0\}$. In this case $Q_r := \{|x'| < r, 0 < x_n < r\}$, and the estimate (3.26) provides both upper and lower bounds for the ratio $u_1(x)/x_n$ near the origin in $\mathbb{R}^n$. In 1952, the lower bounds of such kind for solutions of uniformly elliptic equations $Lu = 0$ with $|\mathbf{b}| \in L^\infty$ were independently obtained by E. Hopf [**H52**] and O.A. Oleinik [**O52**]. They considered domains Ω satisfying the *interior sphere condition* at a point $y_0 \in \partial\Omega$, i.e. there exists a ball $B_r(x_0) \subset \Omega$ such that $(\partial\Omega) \cap \partial B_r(x_0) = \{y_0\}$. The Hopf-Oleinik estimates state that for any function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $u > 0$, $Lu \le 0$ in Ω, and $u(y_0) = 0$, and any vector $\mathbf{v} \in \mathbb{R}^n$ such that $(\mathbf{v}, x_0 - y_0) > 0$, we have $t^{-1}u(y_0 + t\mathbf{v}) \ge \text{const} > 0$ for small $t > 0$. See the books by M.H. Protter and H.F. Weinberger [**PW67**], and by D. Gilbarg and N.S. Trudinger [**GT83**], for further references on this subject. Here we only mention that in a majority of sources, such kind of estimates are obtained by means of more or less standard *barrier technique*, which requires the boundedness (at least locally) of coefficients.

Our approach uses special iterations based on Theorem 1.1 and Lemma 2.1. It allows us to derive a Hopf-Oleinik type estimate in the case $|\mathbf{b}| \in L^q$, $q > n$. On the other hand, this estimate fails if $|\mathbf{b}| \in L^n$, as the following example shows.

EXAMPLE 4.1. Consider the functions

$$u_1(x) := \frac{x_n}{\big|\ln|x|\big|} \quad \text{and} \quad u_2(x) := x_n \cdot \big|\ln|x|\big|$$

in the cylinder $Q := \{x = (x', x_n) \in \mathbb{R}^n : \ |x'| < 1/2, \ 0 < x_n < 1/2\}$, extended as $u_1 = u_2 = 0$ on $(\partial Q) \cap \{x_n = 0\}$. Each of these two functions can be considered as a positive solution to the equation

$$\Delta u + (b, Du) := \Delta u + b_i D_i u = 0 \quad \text{in} \quad Q,$$

where the vector function $b := -\Delta u \cdot |Du|^{-2} Du$ satisfies

$$|\mathbf{b}| = \frac{|\Delta u|}{|Du|} \le \frac{\text{const}}{|x| \cdot \big|\ln|x|\big|} \in L^n(Q) \quad \textit{for} \quad n \ge 2.$$

However, in any neighborhood of $0 \in \mathbb{R}^n$, $\inf(u_1/x_n) = 0$ and $\sup(u_2/x_n) = \infty$.

In fact, if $|\mathbf{b}| \in L^q$ with $q > n$, then any positive solution of the equation $Lu = 0$ in Q_{2r} vanishing on Γ_{2r} (we follow notations (3.6) with $\psi \equiv 0$) satisfies a two-sided estimate

$$0 < \inf_{Q_r} \frac{u}{x_n} \le \sup_{Q_r} \frac{u}{x_n} < \infty. \tag{4.1}$$

The first inequality here is contained in Theorem 4.3 below, the second one can be derived in a similar manner. The following lemma helps to reduce the proofs of such kind of results to the case when the coefficients of L are smooth.

LEMMA 4.2 (Approximation lemma). *Let Ω be a bounded open set in $\mathbb{R}^n$ satisfying an interior cone condition, i.e. there a constants $K > 0$ and $r_0 > 0$ such that for each $y \in \partial\Omega$, there is a Cartesian coordinate system centered at the point $y = 0$, such that the cone*

$$C := \{x = (x', x_n) \in \mathbb{R}^n : \ K \cdot |x'| < x_n < K r_0\} \subset \Omega^c := \mathbb{R}^n \setminus \Omega. \tag{4.2}$$

Let u be a function in $W^{2,n}(\Omega) \cap C(\overline{\Omega})$ satisfying the inequality

$$Lu := a_{ij} D_{ij} u + b_i D_i u \le f \quad \text{in} \quad \Omega, \tag{4.3}$$

where a_{ij} satisfy (1.2), $|\mathbf{b}| \in L^n(\Omega)$, $f \in L^n(\Omega)$.

We claim that there are approximations of functions a_{ij}, b_i, f by functions $a^\varepsilon_{ij}, b^\varepsilon_i, f^\varepsilon \in C^\infty(\overline{\Omega})$, $\varepsilon > 0$, such that a^ε_{ij} satisfy (1.2),

$$a^\varepsilon_{ij} \to a_{ij} \quad \text{a.e. in} \quad \Omega; \qquad b^\varepsilon_i \to b_i, \quad f^\varepsilon \to f \quad \text{in} \quad L^n(\Omega) \tag{4.4}$$

as $\varepsilon \to 0^+$, and solutions $u^\varepsilon \in C^\infty(\Omega) \cap C(\overline{\Omega})$ of the Dirichlet problems

$$L^\varepsilon u^\varepsilon := a^\varepsilon_{ij} D_{ij} u^\varepsilon + b^\varepsilon_i D^\varepsilon_i u = f^\varepsilon \quad \text{in} \quad \Omega, \qquad u^\varepsilon = u \quad \text{on} \quad \partial\Omega, \tag{4.5}$$

satisfy

$$\sup_\Omega (u^\varepsilon - u) \to 0 \quad \text{as} \quad \varepsilon \to 0^+. \tag{4.6}$$

The solvability of the problems (4.5) in Lipschitz domains for equations with smooth, or even Hölder or just continuous, coefficients is well known; see e.g. [**M67**], Theorem 3.

Proof. **(a)** We first consider the case $|\mathbf{b}| \in L^\infty(\Omega)$. The coefficients a_{ij} are defined on the whole space $\mathbb{R}^n$, and also b_i, f can be extended from Ω to $\mathbb{R}^n$ by setting $b_i, f \equiv 0$ on $\mathbb{R}^n \setminus \Omega$. One can approximate a_{ij}, b_i, f by convolutions $a_{ij}^\varepsilon := a_{ij} * \eta^\varepsilon$, $b_i^\varepsilon := b_i * \eta^\varepsilon$, $f^\varepsilon := f * \eta^\varepsilon$ with standard kernels η^ε, $\varepsilon > 0$, satisfying

$$0 \le \eta^\varepsilon \in C^\infty(\mathbb{R}^n), \quad \eta^\varepsilon \equiv 0 \quad \text{on} \quad \mathbb{R}^n \setminus B_\varepsilon, \quad \text{and} \quad \int_{\mathbb{R}^n} \eta^\varepsilon(x)\, dx = 1.$$

Then $a_{ij}^\varepsilon, b_i^\varepsilon, f^\varepsilon \in C^\infty(\mathbb{R}^n)$, a_{ij}^ε satisfy (1.2), and

$$a_{ij}^\varepsilon \to a_{ij}, \quad b_i^\varepsilon \to b_i \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{a.e. in} \quad \Omega. \tag{4.7}$$

This convergence follows from the properties of the Lebesgue sets (see [**St70**], Sec. I.1.8). Moreover, $f^\varepsilon \to f$ in $L^n(\mathbb{R}^n)$ as $\varepsilon \to 0^+$ (see [**GT83**], Sec. 7.2). Having in mind the application of Theorem 1.1 to $u^\varepsilon - u$, we need to estimate from below the functions

$$L^\varepsilon(u^\varepsilon - u) = L^\varepsilon u^\varepsilon - Lu + (L - L^\varepsilon)u \ge F^\varepsilon := f^\varepsilon - f + (L - L^\varepsilon)u.$$

We have

$$(L - L^\varepsilon)u := (a_{ij} - a_{ij}^\varepsilon)D_{ij}u + (b_i - b_i^\varepsilon)D_i u \to 0 \quad \text{in} \quad L^n(\Omega) \quad \text{as} \quad \varepsilon \to 0^+.$$

(Here we use our additional assumption $b_i \in L^\infty$). By Theorem 1.1, from the property (4.7) it follows:

$$\sup_\Omega (u^\varepsilon - u) \le N \cdot ||F^\varepsilon||_{n,\Omega} \to 0 \quad \text{as} \quad \varepsilon \to 0^+.$$

(b) In the general case $|\mathbf{b}| \in L^n(\Omega)$, fix a small constant $\delta > 0$, and choose an open subset $\Omega_\delta \subset \Omega$, such that $|\mathbf{b}| \in L^\infty(\Omega \setminus \overline{\Omega_\delta})$, and the norms in $L^n(\Omega_\delta)$,

$$||a_{ij}D_{ij}u||_{n,\Omega_\delta}, \quad ||b_i||_{n,\Omega_\delta}, \quad ||f||_{n,\Omega_\delta} \le \delta.$$

Define the functions $\overline{b_i}, \overline{f}$ as follows

$$\overline{b_i} \equiv 0, \quad \overline{f} := a_{ij}D_{ij}u \quad \text{on} \quad \Omega_\delta, \qquad \overline{b_i} \equiv b_i, \quad \overline{f} \equiv f \quad \text{on} \quad \Omega \setminus \Omega_\delta.$$

The inequality (4.3) remains valid with the modified b_i and f:

$$\overline{L}u := a_{ij}D_{ij}u + \overline{b_i}D_i u \le \overline{f}.$$

By the choice of Ω_δ, we also have

$$||\overline{b_i} - b_i||_{n,\Omega} = ||b_i||_{n,\Omega_\delta} \le \delta, \qquad ||\overline{f} - f||_{n,\Omega} = ||a_{ij}D_{ij}u - f||_{n,\Omega_\delta} \le 2\delta. \tag{4.8}$$

Since $\overline{b_i} \in L^\infty$, we can apply the argument in (a) correspondingly with

$$b_i^{\delta,\varepsilon} := \overline{b_i} * \eta^\varepsilon, \ f^{\delta,\varepsilon} := \overline{f} * \eta^\varepsilon, \ u^{\delta,\varepsilon}, \quad \text{in place of} \quad b_i^\varepsilon, \ , f^\varepsilon, \ u^\varepsilon.$$

One can go through this construction for each $\delta_k := 2^{-k}$, $k = 1, 2, \ldots$. Therefore, there is a sequence $\varepsilon_k \searrow 0$ such that

$$||b_i^{\delta_k,\varepsilon} - \overline{b_i}||_{n,\Omega} \le \delta_k, \quad ||f^{\delta_k,\varepsilon} - \overline{f}||_{n,\Omega} \le \delta_k; \quad \text{and} \quad \sup_\Omega (u^{\delta_k,\varepsilon} - u) \le \delta_k$$

for all $\varepsilon \in (0, \varepsilon_k]$. Here $\overline{b_i}$ and $\overline{f}$ are defined above for $\delta = \delta_k$. Together with (4.8), the first two of these inequalities imply

$$||b_i^{\delta_k,\varepsilon} - b_i||_{n,\Omega} \le 2\delta_k, \quad ||f^{\delta_k,\varepsilon} - f||_{n,\Omega} \le 3\delta_k.$$

Finally, for each $\varepsilon > 0$, we take $a_{ij}^\varepsilon := a_{ij} * \eta^\varepsilon$, as in the part (a), and define $b_i^\varepsilon, f^\varepsilon$ in two steps: (i) find k from the relations $\varepsilon_{k+1} < \varepsilon \le \varepsilon_k$; then (ii) take $b_i^\varepsilon := b_i^{\delta_k,\varepsilon}$, $f^\varepsilon := f^{\delta_k,\varepsilon}$. This choice of $a_{ij}^\varepsilon, b_i^\varepsilon, f^\varepsilon$ satisfies all the requirements of our lemma. Lemma is proved. □

Theorem 4.3. *Let Ω be an open set in $\mathbb{R}^n$, and let u be a function in $W(\Omega)$ satisfying $u > 0$ and $Lu = 0$ in Ω. Let $y_0 \in \partial\Omega$ and $r > 0$ be such that in a Cartesian coordinate system centered at $y_0 = 0$, the cylinder*

$$Q_{2r} := B'_{2r} \times (0, 2r) = \{x = (x', x_n) \in \mathbb{R}^n : \ |x'| < 2r, \ \ 0 < x_n < 2r\} \tag{4.9}$$

is contained in Ω, and $u(0) = 0$. Suppose that

$$S := \int_{Q_{2r}} |\mathbf{b}|^n dx < \infty, \qquad \text{and} \quad S_1 := r^{q-n} \int_{Q_{2r}} |b_n|^q dx < \infty, \tag{4.10}$$

where $q = \text{const} > n$. Then

$$x_n^{-1} u(0, x_n) \ge c_1 \cdot r^{-1} u(0, r) \quad \text{for} \quad 0 < x_n \le r, \tag{4.11}$$

where $c_1 = c_1(n, \nu, S, S_1, q) \in (0, 1]$.

Remark 4.4. This theorem provides a Hopf-Oleinik type estimate for equations with $|q| \in L^q$, $q > n$. One can modify this formulation in different directions.

(i) One can replace the condition $Q_{2r} \subset \Omega$ with $y_0 \in (\partial Q_{2r}) \cap (\partial\Omega)$ by the interior sphere condition $B_r \subset \Omega$ with $y_0 \in (\partial B_r) \cap (\partial\Omega)$, in both cases $u(y_0) = 0$. One case is reduced to another by an appropriate C^2-transformation of coordinates.

(ii) Using Lemma 4.2, and the comparison principle, one can replace the equality $Lu = 0$ in Ω by the inequality $Lu \le 0$ in Ω.

(iii) By the interior Harnack inequality, from (4.11) it follows that for any vector $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ such that $v_n > 0$, we have $t^{-1} u(t\mathbf{v}) \ge \text{const} > 0$ for small $t > 0$.

Proof of Theorem 4.3. All the quantities in the above formulation are invariant with respect to transformations $x \to \text{const} \cdot x$ and $u \to \text{const} \cdot u$. Therefore, without loss of generality, we assume $r = 1$ and $u(0, 1) = 1$. By the previous lemma, we can approximate the function u from below by solutions u^ε of equations with smooth coefficients, so we can also assume that $a_{ij}, b_i \in C^\infty(\overline{\Omega})$.

Set $\beta := 1 - n/q \in (0, 1)$, $\alpha := \beta/2$, and for $k = 1, 2, \dots,$ $r_k := 2^{-k}$, $h_k := r_k^{1+\alpha}$,

$$\Omega_k := Q_{r_k} := B'_{r_k} \times (0, r_k), \quad \Omega_k^+ := \Omega_k \cap \{x_n > h_k\}, \quad \Omega_k^- := \Omega_k \cap \{x_n < h_k\}.$$

By the Harnack inequality, the quantities

$$m_k := \inf_{\Omega_k^+} x_n^{-1} u > 0, \quad k = 1, 2, \dots, \tag{4.12}$$

are estimated from below by a positive constants depending on the prescribed quantities and k. Our goal is to eliminate the dependence on k. From the definition of m_k it follows

$$v_k := m_k x_n - u \le 0 \quad \text{on} \quad \overline{\Omega_k} \cap \big(\{x_n = 0\} \cup \{x_n = h_k\}\big).$$

We can "split" v_k into two functions: $v_k = w_k + z_k$ on Ω_k^-, where w_k and z_k are solutions of the problems

$$
\begin{aligned}
Lw_k = Lv_k = m_k b_n \quad &\text{in} \quad \Omega_k^-, \qquad w_k = 0 \quad \text{on} \quad \partial\Omega_k^-; \\
Lz_k = 0 \quad &\text{in} \quad \Omega_k^-, \qquad z_k = v_k \quad \text{on} \quad \partial\Omega_k^-.
\end{aligned} \tag{4.13}
$$

These problems have classical solutions because of our smoothness assumptions on the coefficients a_{ij}, b_i. By Theorem 1.1,

$$\sup_{\Omega_k^-} w \le N r_k m_k \cdot ||b_n||_{n,\Omega_k^-} \quad \text{with} \quad N = N(n,\nu,S).$$

The last factor is estimated by Hölder's inequality ($1/n = 1/q + 1/p$):

$$||b_n||_{n,\Omega_k^-} \le ||b_n||_{q,\Omega_k^-} \cdot ||\,1\,||_{p,\Omega_k^-} \le N S_1^{1/q} r_k^{n/p}, \qquad N = N(n).$$

Since $n/p = 1 - n/q = \beta > 0$, we get

$$w \le N r_k^{1+\beta} m_k \quad \text{in} \quad \Omega_k^-. \tag{4.14}$$

Here and in the rest of the proof, N denotes different constants depending only on n, ν, S, S_1, q. Our next step is to evaluate, for $0 < \rho \le r_k$ and fixed k,

$$M(\rho) := \sup_{D(\rho)} (z_k)_+, \quad \text{where} \quad D(\rho) := B'_\rho \times (0, h_k).$$

Consider the case $M(\rho) > 0$, and $0 < \rho \le r_k - h_k$. Since $z_k = v_k \le 0$ on $\{x_n = 0\} \cup \{x_n = h_k\}$, by the maximum principle, the value $M(\rho)$ is attained by z_k on the lateral side of $D(\rho)$:

$$M(\rho) = z_k(x(\rho)) = z_k(x'(\rho), x_n(\rho)), \quad \text{where} \quad |x'(\rho)| = \rho, \quad 0 \le x_n(\rho) \le h_k.$$

In a subcase $0 \le x_n(\rho) \le h_k/2$, one can apply Lemma 2.5 to the function z_k with $\Omega := \Omega_k^- \cap \{z_k > 0\}$, $x_0 := (x'(\rho), 0)$, and $r := h_k/2$. Obviously, the measure condition (2.15) holds true with $\mu_2 = 1/2$. From this lemma and the maximum principle, it follows

$$M(\rho) = z_k(x(\rho)) \le \sup_{\Omega \cap B_{h_k/2}(x_0)} z_k \le \beta \cdot \sup_{\Omega \cap B_{h_k}(x_0)} z_k \le \beta \cdot M(\rho + h_k),$$

where $\beta = \beta(n,\nu,S) \in (0,1)$. If $h_k/2 \le x_n \le h_k$, then these inequalities remain valid with $x_0 := (x'(\rho), h_k)$.

Let k be large enough, so that the integer part $j := [r_{k+1}/h_k] = [2^{k\alpha-1}] \ge 1$. Then $r_{k+1} + jh_k \le 2r_{k+1} = r_k$, and iterating the previous estimate, we get

$$M(r_{k+1}) \le \beta \cdot M(r_{k+1} + h_k) \le \ldots \le \beta^j \cdot M(r_{k+1} + jh_k) \le M(r_k) \le \beta^j m_k h_k.$$

Here

$$\beta^j \le \beta^{-1} \cdot \beta^{2^{k\alpha-1}} = \beta^{-1} \cdot \exp\big(-c r_k^{-\alpha}\big), \quad \text{where} \quad c := -2^{-1} \ln \beta > 0.$$

This estimate together with (4.14) yield the estimate for $m_k x_n - u =: v_k = w_k + z_k$:

$$m_k x_n - u \le N \cdot \Big(r_k^{1+\beta} + \exp\big(-c r_k^{-\alpha}\big)\Big) \cdot m_k \quad \text{in} \quad D(r_{k+1}). \tag{4.15}$$

From the definition (4.12) of m_k it follows that if $m_k > m_{k+1}$, then

$$m_{k+1} = \inf_{\Delta_k} x_n^{-1} u, \quad \text{where} \quad \Delta_k := \big(\Omega_{k+1}^+ \setminus \overline{\Omega_k^+}\big) \subset D(r_{k+1}), \tag{4.16}$$

Since $x_n \ge h_{k+1} = r_{k+1}^{1+\alpha} = 2^{-1-\alpha} r_k^{1+\alpha}$, and $\beta = 2\alpha$, the properties (4.15) and (4.16) imply

$$m_k - m_{k+1} \le N \cdot \Big(r_k^\alpha + r_k^{-1-\alpha} \exp\big(-c r_k^{-\alpha} \big) \Big) \cdot m_k.$$

We proved the last inequality in the case $m_k > m_{k+1}$. In the opposite case $m_k \le m_{k+1}$ it is obvious, so this inequality holds true without any restrictions.

Since the exponential term converges to 0 as $k \to \infty$ faster than any positive power of r_k, there are large natural numbers k_0 and N_0, depending only on n, ν, S, S_1, q, such that

$$m_k - m_{k+1} \le N_0 r_k^\alpha m_k, \quad \text{and} \quad N_0 r_k^\alpha \le 1/2 \quad \text{for all} \quad k \ge k_0.$$

From $m_{k+1} \ge \big(1 - N_0 r_k^\alpha\big) m_k$, together with an elementary inequality $\ln(1-t) \ge -2t$ for $0 \le t \le 1/2$, we obtain

$$\ln m_{k+1} - \ln m_k \ge \ln\big(1 - N_0 r_k^\alpha\big) \ge -2N_0 r_k^\alpha, \qquad k \ge k_0.$$

For any integer $k_1 > k_0$,

$$\ln m_{k_1} - \ln m_{k_0} = \sum_{k=k_0}^{k_1-1} \big(\ln m_{k+1} - \ln m_k \big) \ge -2N_0 \sum_{k=k_0}^{\infty} r_k^\alpha =: N_1.$$

This implies the estimate

$$m_k \ge c_0 = c_0(n, \nu, S, S_1, q) > 0 \quad \text{for all} \quad k > k_0.$$

In particular, $x_n^{-1} u(0, x_n) \ge c_0$ for all $x_n \in (0, r_{k_0+1})$. By the Harnack inequality, this estimate holds true for $x_n \ge r_{k_0+1}$ as well, possibly with a different constant, because k_0 depends only on the prescribed quantities, and the desired estimate (4.11) follows. □

Remark 4.5. In our recent paper [**S08**], the Oleinik-Hopf type estimate (4.11) was extended to the case when Q_r is represented in the form (3.6) with

$$\psi(x') := \psi_0(|x'|), \quad \text{where} \quad \psi_0 \ge 0,\ \psi_0' \ge 0, \quad \text{and} \quad \int_0^1 t^{-2} \psi(t)\, dt < \infty. \tag{4.17}$$

This condition is sharp in a sense that if the integral in (4.17) diverges, then (4.11) fails. However, in [**S08**] we assumed $b_i \equiv 0$. In the present paper, we impose the “complementary” conditions $\psi \equiv 0$, and $|\mathbf{b}| \in L^q$, $q > n$. In our forthcoming work, we plan to combine the results and techniques of these two papers in order to cover the general case: ψ satisfying (4.17), and $|\mathbf{b}| \in L^q$, $q > n$.

Acknowledgements. The author is thankful to Nicolai V. Krylov for stimulating conversations. A part of this paper was completed during June 2009, when the author took part in the INdAM intensive period in Milan, Italy, organized by Ugo Gianazza, Sandro Salsa, and Vincento Vespri. The author thanks the organizers for invitation and useful discussions. Special thanks are due to Nina N. Uraltseva, who raised some open problems which motivated the whole research related to this paper.

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