

Some evolution equations as gradient flows of functionals in spaces of measures and its geodesic convexity

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(Joint work with J. A. Carrillo, G. Savaré, D. Slepčev)

Università degli Studi di Pavia

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- The case of a new class of metrics

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The (EVI) in Banach spaces was formulated (in integrated form) by Benilan (1972) for m -accretive operators (see also Brezis (1971, 1973)).

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In metric spaces was formulated in Ambrosio-Gigli-Savaré (2006).

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These properties are purely metric and can be taken as definition of gradient flow in a metric space.

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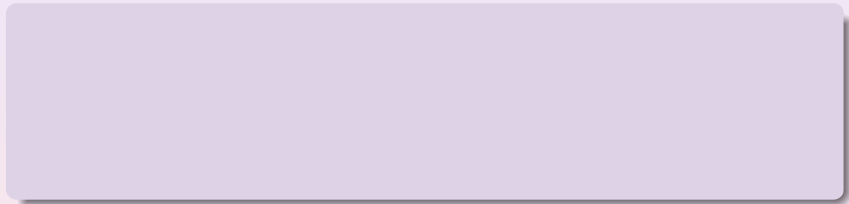
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Regularizing effect

$$\phi(S_t(u)) \leq \phi(v) + \frac{1}{2t}d^2(u, v), \quad \forall u \in X, v \in D(\phi), t > 0.$$

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More precisely: if $u : [0, 1] \rightarrow D(\phi)$ is a constant speed geodesic then

$$\phi(u(s)) \leq (1 - s)\phi(u(0)) + s\phi(u(1)) \quad \forall s \in [0, 1].$$

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For every two points $\mu_0, \mu_1 \in \mathcal{P}(\overline{\Omega})$ there exists a constant speed geodesic $\mu : [0, 1] \rightarrow \mathcal{P}(\overline{\Omega})$ connecting μ_0 to μ_1 ,

$$W(\mu_t, \mu_s) = |t - s| W(\mu_0, \mu_1) \quad \forall s, t \in [0, 1].$$

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Example

For $\beta > 0$ we can define

$$U_{\beta}(\rho) = \begin{cases} \frac{1}{\beta-1} \rho^{\beta} & \beta \neq 1 \\ \rho \log \rho & \beta = 1 \end{cases}.$$

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It is immediate to verify that U_β of the previous example satisfies the condition for $\beta \geq 1 - 1/d$.

Existence of GF

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Mc Cann condition

$s \mapsto s^d U(s^{-d})$ is convex and non-increasing.

Equivalently $P'(r)r - (1 - 1/d)P(r) \geq 0$, $\forall r \in (0, +\infty)$, where $P(r) := \int_0^r sU''(s) ds$.

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Theorem ((Carrillo-L.-Savaré-Slepcev), (Daneri-Savaré), (Ambrosio-Gigli-Savaré))

If the previous assumptions hold, then the functional \mathcal{U} generates a unique metric contraction gradient flow \mathcal{S} in the space $(\mathcal{P}(\bar{\Omega}), W)$.

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where P , pressure, is defined by

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Dynamical formulation of Wasserstein distance (Benamou-Brenier)

Given $\mu_0, \mu_1 \in \mathcal{P}(\Omega)$

$$W^2(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\Omega} |\mathbf{v}_t(x)|^2 \rho_t(x) dx dt : (\rho, \mathbf{v}) \in A(\mu_0, \mu_1) \right\},$$

$A(\mu_0, \mu_1)$ denotes the set of $\rho : (0, 1) \times \Omega \rightarrow [0, +\infty)$ and $\mathbf{v} : (0, 1) \times \Omega \rightarrow \mathbb{R}^d$ such that the continuity equation holds

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{in } \Omega \times (0, 1)$$

and

$$\rho_0 \mathcal{L}^d = \mu_0, \quad \rho_1 \mathcal{L}^d = \mu_1.$$

New distance, non linear mobility

New distance, non linear mobility

We consider a modification of the dynamical formulation of the Wasserstein distance

New distance, non linear mobility

Given a (possibly) non linear function (mobility function)

$m : [0, M) \rightarrow [0, +\infty)$, where $0 < M \leq +\infty$, we define

$$W_m^2(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\Omega} |\mathbf{v}_t(x)|^2 m(\rho_t(x)) dx dt : (\rho, \mathbf{v}) \in A_m(\mu_0, \mu_1) \right\},$$

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This distance was studied in Dolbeault-Nazaret-Savaré (2008), in the present paper and in Lisini-Marigonda (2009).

Convexity of the integrand

The functional to be minimized in the previous “definition” of the distance is

$$\int_{\Omega} |\mathbf{v}_t(x)|^2 m(\rho_t(x)) dx.$$

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$$\partial_t \rho + \nabla \cdot \mathbf{w} = 0$$

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Using the identity $\mathbf{v}m(\rho) = \mathbf{w}$, the functional can be rewritten as

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In this case the function $(\rho, \mathbf{w}) \mapsto |\mathbf{w}|^2/m(\rho)$ is convex if and only if m is concave.

Precise definition

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The natural framework for considering the variational problems in the definition of the new distance is that of time dependent families of Radon measures in \mathbb{R}^d .

Continuity equation

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μ is a weak* continuous curve $\mu : [0, 1] \rightarrow \mathcal{M}^+ \mathbb{R}^d$,

ν is a Borel family of vector measures $\nu_t \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$, satisfying

$$\int_0^1 |\nu_t|(\mathbb{R}^d) dt < +\infty,$$

and, denoting $\nu_t = \mathbf{w}_t |\nu_t|$,

$$\int_0^1 \int_{\mathbb{R}^d} \partial_t \varphi d\mu_t dt + \int_0^1 \int_{\mathbb{R}^d} \mathbf{w}_t \cdot \nabla \varphi d|\nu_t| dt = 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d \times (0, 1)).$$

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Given $\mu^0, \mu^1 \in \mathcal{M}^+(\overline{\Omega})$, we denote by $CE(\mu^0, \mu^1; \Omega)$ the set of (μ, ν) satisfying the continuity equation and $\mu_0 = \mu^0$, $\mu_1 = \mu^1$ and $\text{supp}(\mu_t) \subset \overline{\Omega}$, $\text{supp}(\nu_t) \subset \overline{\Omega}$, $\forall t \in [0, 1]$.

Action function

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Given the mobility function satisfying

$$m \in C^0([0, M)), \text{ concave, } m(0) = 0, \quad 0 < M \leq +\infty$$
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Example

$$M = +\infty, \quad m(r) = r^\alpha, \quad \alpha \in (0, 1].$$

$$M < +\infty, \quad m(r) = r(M - r).$$

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We define the *action density function*

$$\phi_m : \mathbb{R} \times \mathbb{R}^d \rightarrow [0, +\infty], \\ \phi_m(\rho, \mathbf{w}) = \begin{cases} \frac{|\mathbf{w}|^2}{m(\rho)} & 0 \leq \rho \leq M \\ +\infty & \rho < 0 \text{ or } \rho > M \end{cases}$$

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Under our assumptions on m we have ϕ_m convex and lower semi continuous.

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Definition and basic properties of the distance

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With the action functional $\Phi_{m,\Omega}$ we can define

$$W_{m,\Omega}^2(\mu^0, \mu^1) = \inf \left\{ \int_0^1 \Phi_{m,\Omega}(\mu_t, \nu_t) dt : (\mu, \nu) \in CE(\mu^0, \mu^1; \Omega) \right\}.$$

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Given $\sigma \in \mathcal{M}^+(\overline{\Omega})$, we denote by

$$\mathcal{M}_m[\sigma](\Omega) = \{\mu \in \mathcal{M}^+(\overline{\Omega}) : W_m(\mu, \sigma) < +\infty\}.$$

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Theorem

The space $\mathcal{M}_m[\sigma](\Omega)$ endowed with the distance $W_{m,\Omega}$ is a complete metric space.

Moreover for every $\mu_0, \mu_1 \in \mathcal{M}_m[\sigma](\Omega)$ there exists a constant speed geodesic for $W_{m,\Omega}$, i.e. satisfying

$$W_{m,\Omega}(\mu_t, \mu_s) = |t - s| W_{m,\Omega}(\mu_0, \mu_1) \quad \forall s, t \in [0, 1].$$

Condition for the existence of the contractive gradient flow

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We proceed formally computing the second derivative of the functional along a geodesic.

Condition for geodesic convexity

The geodesics of the distance W_m between $\mu_0 = \rho_0 \mathcal{L}^d$ and $\mu_1 = \rho_1 \mathcal{L}^d$ can be formally obtained as solution of the system

$$\begin{cases} \partial_t \rho + \nabla \cdot (m(\rho) \nabla \psi) = 0 & \text{in } \Omega \times (0, 1), \\ \partial_t \psi + \frac{1}{2} m'(\rho) |\nabla \psi|^2 = 0 & \text{in } \Omega \times (0, 1) \end{cases}$$

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Defining

$$P(r) = \int_0^r m(s) U''(s) ds, \quad H(r) = \int_0^r P'(s) m'(s) ds + H_0 \geq 0,$$

(P is increasing) we obtain after some computations

$$\begin{aligned} \frac{d^2}{dt^2} \mathcal{U}(\mu_t) &\geq \int_{\Omega} (P'(\rho) m(\rho) - (1 - 1/d) H(\rho)) (\Delta \psi)^2 dx \\ &\quad - \int_{\Omega} P'(\rho) m''(\rho) |\nabla \rho|^2 |\nabla \psi|^2 dx. \end{aligned}$$

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The condition for the geodesic convexity of \mathcal{U} with respect to W_m can be stated as follows:

$$P'(r)m(r) - (1 - 1/d)H(r) \geq 0, \quad \forall r \in (0, +\infty).$$

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We observe that for $m(r) = r$ we have $H = P$ and the condition coincides with the McCann condition for displacement convexity.

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Mc Cann generalized condition

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$$X[\sigma] := \left\{ \mu \in \mathcal{M}^+(\Omega) : \mu \ll \mathcal{L}|_{\Omega}, W_{m,\Omega}(\mu, \sigma) < +\infty, \mathcal{U}(\mu) < +\infty \right\}.$$

endowed with the distance $W_{m,\Omega}$.

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Theorem (Carrillo-L.-Savaré-Slepcev)

If the previous assumptions hold, then for every reference measure $\sigma \in \mathcal{M}^+(\Omega)$ with finite energy $\mathcal{U}(\sigma) < +\infty$ the functional \mathcal{U} generates a unique metric contraction gradient flow \mathcal{S} in the space $X[\sigma]$ with the distance $W_{m,\Omega}$.

Moreover, denoting by $\mathcal{S}_t \mu_0 = \rho_t \mathcal{L}_{|\Omega}^d$ we have that ρ is a weak solution of the Neumann boundary value problem

$$\begin{cases} \partial_t \rho - \Delta P(\rho) = 0 & \text{in } (0, +\infty) \times \Omega \\ \nabla P(\rho) \cdot \mathbf{n} = 0 & \text{on } (0, +\infty) \times \partial\Omega \\ \rho(0, x) = \rho_0(x) & \text{in } \Omega \end{cases}$$

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The topology induced by the distance $W_{m,\Omega}$ is equivalent to the usual weak* one, and the metric space $\overline{X[\sigma]}$ is compact.

In the case $m(r) = r^\alpha$ the condition is satisfied if and only if $\alpha > 1 - \frac{1}{d}$.

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If U has a superlinear growth, then $\mathcal{S}_t(\mu_0) = \rho_t \mathcal{L}^d \ll \mathcal{L}^d_{|\Omega}$ for every $t > 0$ and ρ is a weak solution of the Neumann problem

$$\begin{cases} \partial_t \rho - \Delta P(\rho) = 0 & \text{in } (0, +\infty) \times \Omega \\ \nabla P(\rho) \cdot \mathbf{n} = 0 & \text{on } (0, +\infty) \times \partial\Omega \\ \rho(0, \cdot) = \mu_0 \end{cases}$$

with $P(r) := \int_0^r m(s)U''(s) ds$.

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$$\mathcal{U}(\mu_t) - \mathcal{U}(\mu_s) = - \int_s^t \int_{\Omega} \frac{|\nabla P(\rho(\tau, x))|^2}{m(\rho(\tau, x))} \, dx \, d\tau,$$

$$\forall s, t \in (0, +\infty), \quad s < t.$$

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- Energy estimates

Main convexity result

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Theorem (Carrillo-L.-Savaré-Slepcev)

Let us assume that Ω is a bounded convex open set, and the functions U, m satisfy the generalized McCann condition. For every $\mu_0, \mu_1 \in \mathcal{M}^+(\Omega) \cap D(\mathcal{U})$ with finite distance $W_{m,\Omega}(\mu_0, \mu_1) < +\infty$ there exists a constant speed minimizing geodesic $\mu : [0, 1] \rightarrow \mathcal{M}^+(\Omega)$ connecting μ_0 to μ_1 such that

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Remark: weak and strong convexity

If $M < +\infty$ or $M = +\infty$ and $m'(\infty) = 0$ and m strictly concave then the geodesics are unique and \mathcal{U} is strongly convex.

If U has superlinear growth there exist a geodesic such that $\mu_s \ll \mathcal{L}^d$ and the convexity hold along every geodesic of this kind.

Example: porous medium, heat equation, fast diffusion

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We consider, for $\beta > 0$,

$$U_\beta(\rho) = \begin{cases} \frac{1}{\beta-1}\rho^\beta & \beta \neq 1 \\ \rho \log \rho & \beta = 1 \end{cases}.$$

and

$$m(\rho) = \rho^\alpha, \quad 0 < \alpha \leq 1.$$

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The equation $\partial_t \rho - \Delta P(\rho) = 0$ is

$$\partial_t \rho - \frac{\beta}{\gamma} \Delta \rho^\gamma = 0,$$

and is the standard porous medium equation when $\alpha + \beta > 2$, the heat equation when $\alpha + \beta = 2$ and the fast diffusion equation when $\alpha + \beta < 2$.

Example: porous medium, heat equation, fast diffusion

Moreover, by an elementary computation we obtain

$$P'(r)m(r) - (1 - 1/d)H(r) = \beta r^{2\alpha + \beta - 2} \left(1 - (1 - 1/d) \frac{\alpha}{2\alpha + \beta - 2} \right),$$

hence the condition for the generation of the contraction gradient flow and the convexity is satisfied if and only if

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- The minimum admissible exponent for the fast diffusion equation is $1 - \frac{\alpha}{d}$ which increase when α decrease.
- A fast diffusion equation can be the gradient flow of a superlinear functional: There are several choices of α and β . If $1 - \frac{1}{d} < \alpha \leq \frac{d}{d+1}$ and $1 \leq \beta < 1 + \frac{1}{d+1}$ the corresponding γ satisfies $1 > \gamma \geq 1 - \frac{\alpha}{d}$.

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- First step: prove the EVI (in the integral form) in the smooth setting
- Second step: use an approximation argument (by convolution and convex combination, smoothing of the mobility m and of U) and pass to the limit by compactness.

Future developments

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$$\mathcal{U}(\rho) = \int V(x)\rho(x) dx.$$

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- The MM approach for the functional





$$\mathcal{U}(\rho) = \frac{1}{2} \int |\nabla \rho(x)|^2 dx + \int U(\rho) dx.$$

The related equation is a thin film type equation

$$\partial_t \rho + \nabla \cdot (m(\rho)\nabla \Delta \rho) - \Delta P(\rho) = 0$$

(Work in progress: Lisini, Matthes, Savaré).

Referencies

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