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Growth Theorems and Harnack Inequality for Second Order Parabolic Equations

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Dedicated to N. V. Krylov, on the occasion of his 60^{th} birthday.

ABSTRACT. A general approach to both divergence (D) and non-divergence (ND) second order parabolic equations is presented, which is based on three growth theorem. These growth theorems look identical in both cases (D) and (ND). They allow to prove the Harnack inequality and other related facts by general arguments, which do not depend on the structure (divergence or non-divergence) of equations. In turn, the growth theorems are established on the grounds of the classical maximum principle and some integral estimates: the energy estimate in the case (D) and the Aleksandrov-Krylov estimate in the case (ND).

Introduction

There are many facts in the theory of second order elliptic and parabolic PDE, which look similar for equations in the divergence and non-divergence forms. These facts include different versions of the Harnack inequality, boundary estimates for solutions, doubling properties for caloric measures, etc. Such kind of results for second order parabolic equations were recently obtained by Fabes, Safonov and Yuan in [**FS**], [**FSY**], [**SY**]. In those papers, different methods were employed for divergence and non-divergence equations. Some of methods, in particular those based on the properties of fundamental solutions, are available only for divergence equations. However, it turns out, the main results there (backward Harnack inequalities, doubling properties for L-caloric measures, and others) can be obtained on the grounds of two principles: the classical maximum principle and the interior Harnack inequality, without using any other specific properties of divergence or non-divergence equations (see [**S98**]).

Unlike the classical maximum principle, the interior Harnack inequality is far from obvious. For elliptic and parabolic equations with measurable coefficients in the *divergence* form, it was proved by Moser in the papers [**M61**],[**M64**]. Many other mathematicians have contributed to this area as well (see monographs [**F**],[**GT**],[**K**], [**LE**], [**L**],[**LSU**]; papers [**A68**], [**AS**], [**DG**], [**FSt**], [**I**], [**LSW**], [**N**], [**PE**], [**SJ**], [**T**], and references therein). However, a similar result for *non-divergence* equations was obtained only 15 years later after Moser's papers. This was done by Krylov and Safonov [**KS**], [**S80**] in 1978-80, who in turn relied on some improved versions of *growth theorems* from the book by Landis [**LE**]. These growth theorems

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control the behavior of (sub- , super-) solutions of second order elliptic and parabolic equations in terms of the Lebesgue measure of areas in which solutions are positive or negative.

In the present paper, we use growth theorems as a common background for both divergence (D) and non-divergence (ND) equations. We prove three such theorems and derive the interior Harnack inequality as their consequence. For the sake of simplicity, we assume that all functions (coefficients and solutions) are smooth enough. This allows us to treat the cases (D) and (ND) simultaneously. It is easy to get rid of extra smoothness assumptions by means of standard approximation procedures, which are briefly discussed at the end of paper, in Remark 6.1. These procedures, and also minimal smoothness of coefficients and solutions, are different in the cases (D) and (ND). In both cases, it is important to have appropriate estimates for solutions with constants depending only on the prescribed quantities, such as dimension n, parabolicity constant ν, etc., and not depending on “additional” smoothness. We denote such constants $N = N(n, \nu, \cdots)$.

Harnack inequalities have many important applications not only in differential equations, but also in other areas, such as diffusion processes, geometry, etc. For the reader’s convenience, we prove most of auxiliary results in their “weak” form, which is sufficient for our purposes. We concentrate on the *parabolic* equations, because the corresponding results for the *elliptic* equations follow automatically.

In Section 1, we introduce our basic assumptions, prove a weak version of the classical *maximum principle* (Theorem 1.4), and formulate the *interior Harnack inequality* (Theorem 1.5). Then we show that even in the one-dimensional case, the Harnack inequality fails for equations of a “joint” structure, which combine both divergence and non-divergence parts. Therefore, it is inevitable that a part of the proof of the Harnack inequality relies on specific properties of equations in the cases (D) and (ND). These are integral estimates in Section 2: the energy estimate in the case (D), and the Aleksandrov-Krylov estimate in the case (ND). In the next three Sections 3,4 and 5, we formulate and prove three growth theorems. Finally, in Section 6 we prove the interior Harnack inequality and describe some possible generalizations.

Basic notations. “$A := B$” or “$B =: A$” is the definition of A by means of the expression B.

$\mathbb{R}^n$ is the n-dimensional Euclidean space , $n \geq 1$, with points $x = (x_1, \cdots, x_n)^t$, where x_i are real numbers. Here the symbol t stands for the transposition of vectors, which indicates that vectors in $\mathbb{R}^n$ are treated as column vectors. For $x = (x_1, \cdots, x_n)^t$ and $y = (y_1, \cdots, y_n)^t$ in $\mathbb{R}^n$, the *scalar product* $(x, y) := \sum x_i y_i$, the *length* of x is $|x| := (x, x)^{1/2}$.

For a Borel set $\Gamma \subset \mathbb{R}^m$, $\partial\Gamma$ is the *boundary* of Γ in $\mathbb{R}^m$, $\overline{\Gamma} := \Gamma \cup \partial\Gamma$ is the *closure* of Γ, $|\Gamma|$ is the m-dimensional Lebesgue measure of Γ. Sometimes we use same notation $|\Gamma|$ for the surface measure of a subset Γ of a smooth surface S.

For real numbers c, we denote $c_+ := \max(c, 0)$, $c_- := \max(-c, 0)$.

1. Maximum principle and Harnack inequality

We discuss basic properties of solutions to second order linear parabolic equations $Lu = 0$ or inequalities, which do not depend on the smoothness of coefficients, and also do not depend on the structure of equations. Namely, we treat simultaneously equations $Lu = 0$ in the *divergence* form, where

$$Lu := a_0 u_t - \sum_{i,j=1}^{n} D_i \left(a_{ij} D_j u\right) = a_0 u_t - (D, aDu)\,, \tag{D}$$

and in the *non-divergence* form, where

$$Lu := u_t - \sum_{i,j=1}^{n} a_{ij} D_{ij} u = u_t - (aD, Du)\,. \tag{ND}$$

Here all the functions u, a_0 and a_{ij} depend on $X = (x,t) \in \mathbb{R}^{n+1}$, $u_t = \partial u/\partial t$, Du is the vector in $\mathbb{R}^n$ with components $D_i u = \partial u/\partial x_i$, $D_{ij} u = D_i D_j u$, and $a = [a_{ij}]$ is a $n \times n$ matrix with entries $a_{ij} = a_{ij}(X) = a_{ij}(x,t)$, which are defined for all $X = (x,t) \in \mathbb{R}^{n+1}$ and satisfy the *uniform parabolicity* condition

$$\min_{\xi \in \mathbb{R}^n, |\xi|=1} (a\xi, \xi) \geq \nu |\xi|^2, \qquad ||a||^2 := \sum_{i,j=1}^{n} |a_{ij}|^2 \leq \nu^{-2}, \tag{U}$$

with a constant $\nu \in (0,1]$. In addition, in the divergence case we assume that $a_0 = a_0(x)$ does not depend on t and satisfies

$$\nu \leq a_0(x) \leq \nu^{-1}. \tag{U$_0$}$$

In our considerations, functions u, a_0 and a_{ij} are smooth enough, so that all derivatives in Lu are understood in the usual classical sense. More general assumptions require different techniques in the cases (D) and (ND); we discuss these matters in Remark 6.1 at the end of paper. Unless otherwise stated, we always assume that in a given open set Q, functions u and coefficients of L satisfy the following smoothness assumptions.

Assumptions 1.1. Function $u \in C^{2,1}(Q)$, i.e. u is continuous together with all the derivatives $D_i u, D_{ij} u, u_t$ in an open set $Q \subset \mathbb{R}^{n+1}$. Functions $a_0 = a_0(x)$ and a_{ij} are continuous and satisfy (U), (U$_0$) with a constant $\nu \in (0,1]$. In addition, in the case (D), a_{ij} have continuous derivatives $D_k a_{ij}$ for $i,j,k = 1, \cdots, n$.

It is easy to see these assumptions guarantee that Lu in continuous in Q.

Remark 1.2. In the non-divergence case (ND), the additional coefficient a_0 in front of u_t is not needed, because this case is easily reduced to $a_0 = 1$: one can simply divide by a_0 and replace the matrix a by $\tilde{a} = a_0^{-1} a$, which satisfies (U) with $\tilde{\nu} = \nu^2$ instead of ν. Moreover, in the case (ND) we can always replace a_{ij} by $\frac{1}{2}(a_{ij} + a_{ji})$, so that the matrix a becomes symmetric. Furthermore, under parallel translations in $\mathbb{R}^{n+1}$ and rotations with respect to the space variable $x \in \mathbb{R}^n$, the equations $Lu = 0$ are transformed to similar equations with new coefficients, which satisfy the conditions (U) and (U_0) with the same constant $\nu \in (0,1]$.

For formulation of our results, we need some standard definitions and notations.

Definition 1.3. Let Q be an open connected set in $\mathbb{R}^{n+1}$, $n \geq 1$. The *parabolic boundary* $\partial_p Q$ of Q is the set of all points $X_0 = (x_0, t_0) \in \partial Q$, such that there exists a continuous function $x = x(t)$ on an interval $[t_0, t_0 + \delta)$ with values in $\mathbb{R}^n$, such that

$x(t_0) = x_0$ and $(x(t), t) \in Q$ for all $t \in (t_0, t_0 + \delta)$. Here $x = x(t)$ and $\delta > 0$ depend on X_0.

In particular, for cylinders $Q = \Omega \times (0, T)$, the parabolic boundary $\partial_p Q := (\partial_x Q) \cup (\partial_t Q) \cup (\partial_{xt} Q)$, where

$$\partial_x Q := (\partial\Omega) \times (0, T), \ \partial_t Q := \Omega \times \{0\}, \ \partial_{xt} Q := (\partial\Omega) \times \{0\}.$$

We will also use a notation for "standard" parabolic cylinders: for $Y = (y, s)$ and $r > 0$,

$$C_r(Y) := B_r(y) \times \left(s - r^2, s\right), \text{ where } B_r(y) := \{x \in \mathbb{R}^n : \ |x - y| < r\}.$$

The following *maximum principle* is well known. We give its short proof, which uses our simplifying assumptions.

THEOREM 1.4 (Maximum Principle). *Let Q be a bounded open set in $\mathbb{R}^{n+1}$, and let a function $u \in C^{2,1}\left(\overline{Q} \setminus \partial_p Q\right) \cap C\left(\overline{Q}\right)$ satisfy the inequality $Lu \le 0$ in Q. Then*

$$\sup_Q u = \sup_{\partial_p Q} u. \tag{1.1}$$

PROOF. Since a_{ij} are smooth, we can write

$$Lu = a_0 u_t - (aD, Du) + (b, Du) \qquad \text{in} \quad Q,$$

where the vector function b has components $b_j = \sum_i D_i a_{ij}$ in the case (D), and $a_0 = 1, b = 0$ in the case (ND). Suppose (1.1) is not true. Replacing u by $u - const$, we may assume

$$\sup_{\partial_p Q} u < 0 < \sup_Q u = u(Y) \qquad \text{for some} \quad Y = (y, s) \in \overline{Q} \setminus \overline{\partial_p Q}.$$

In addition, replacing u by $u - \varepsilon t$ with a small constant $\varepsilon > 0$, we have $Lu(Y) < 0$. Moreover, replacing a_{ij} by $\frac{1}{2}(a_{ij} + a_{ji})$, we can assume that the matrix $a = [a_{ij}]$ is symmetric. Then by rotating the coordinate axes, we can reduce the matrix of coefficients to the diagonal matrix $diag[\lambda_1, \ldots, \lambda_n]$, where $\lambda_i \in \left[\nu, \nu^{-1}\right]$ are eigenvalues of a. Further, since $Y = (y, s) \in \overline{Q} \setminus \overline{\partial_p Q}$, we also have $\overline{C_r(Y)} \subset \overline{Q} \setminus \overline{\partial_p Q}$ for some small $r > 0$, so that

$$u \in C^{2,1}\left(\overline{C_r(Y)}\right) \qquad \text{and} \qquad \sup_{C_r(Y)} u = u(Y).$$

Since $u(\cdot, s)$ attains its maximum at y, we have $D_i(Y) = 0, \ D_{ii}(Y) \le 0$ for all i, and similarly, $u_t(Y) \ge 0$. Combining these relations, we get

$$Lu(Y) = a_0 u_t(Y) - \sum_i \lambda_i D_{ii}(Y) + \sum_i b_i D_i(Y) \ge 0.$$

On the other hand, by our construction $Lu(Y) < 0$. This contradictions proves the desired equality (1.1). □

In the next theorem, we formulate the *interior Harnack inequality*. The proof of this theorem will be given in Section 6.

THEOREM 1.5 (Interior Harnack Inequality). *Let u and the coefficients of L satisfy Assumptions 1.1 in a cylinder $C_{2r} = C_{2r}(Y)$, where $Y = (y, s) \in \mathbb{R}^{n+1}, r > 0$, and let u satisfy $u \ge 0, \ Lu = 0$ in C_{2r}. Then*

$$\sup_{C^0} u \le N \inf_{C_r} u, \tag{1.2}$$

where $C^0 := B_r(y) \times \left(s - 3r^2, s - 2r^2\right)$ *and the constant* $N = N(n, \nu)$.

For second order elliptic equations, this theorem is reduced to the following

COROLLARY 1.6. *Let* $u = u(x)$ *and* $a_{ij} = a_{ij}(x)$ *satisfy Assumptions 1.1 in a ball* $B_{2r} = B_{2r}(y)$, *where* $y \in \mathbb{R}^n, r > 0$, *and let* u *satisfy*

$$u \geq 0 \quad \text{and} \quad (D) \quad (D, aDu) = 0 \quad \text{or} \quad (ND) \quad (aD, Du) = 0 \quad \text{in} \quad B_{2r}.$$

Then

$$\sup_{B_r} u \leq N \inf_{B_r} u, \tag{1.3}$$

where $N = N(n, \nu)$.

The proof of the maximum principle, Theorem 1.4, with minor changes remains valid for more general "united" equations $u_t - (D, a_1 Du) - (a_2 D, Du) = 0$, which include both divergence and non-divergence equations. However, for such equations the Harnack inequality (1.3) fails even in the one-dimensional elliptic case, as the following example shows.

EXAMPLE 1.7. For arbitrary small $\varepsilon \in \left(0, \frac{1}{2}\right)$, we will construct smooth functions a_1, a_2 and u on $[-1, 1]$, such that $|a_{1,2} - 1| \leq \varepsilon,\ 0 \leq u \leq 1$,

$$u' > 0, \quad (a_1 u')' + a_2 u'' = 0 \quad \text{in} \quad (-1, 1), \tag{1.4}$$

and

$$0 \leq u \leq \varepsilon \quad \text{on} \quad [-1, -\varepsilon], \qquad 1 - \varepsilon \leq u \leq 1 \quad \text{on} \quad [\varepsilon, 1]. \tag{1.5}$$

Obviously, from these properties it follows that the estimate (1.3) fails for $y = 0$ and $r = \frac{1}{2}$.

For fixed $\varepsilon \in \left(0, \frac{1}{2}\right)$, take an arbitrary smooth function $\eta = \eta(x)$ on $[-1, 0]$, satisfying the properties $|\eta| \leq \frac{\varepsilon}{2},\ |\eta'| \leq 1$ on $[-1, 0]$;

$$\eta > 0 \quad \text{in} \quad \left(-\varepsilon, -\frac{\varepsilon}{2}\right), \quad \text{and} \quad \eta(x) \equiv 0 \quad \text{on} \quad [-1, -\varepsilon] \cup \left[-\frac{\varepsilon}{2}, 0\right].$$

For a constant $\omega > 0$ to be chosen later, we first define the functions

$$a_1(x) = 1 - \eta \sin \omega x, \quad a_2(x) = 1 + \eta \cdot (\sin \omega x - \cos \omega x) \qquad \text{on} \quad [-1, 0],$$

and then extend them as even functions to the interval $[-1, 1]$. We can rewrite the equality (1.4) in the form

$$(a_1 + a_2) u'' + a_1' u' = 0, \qquad \text{or} \quad f := (\ln u')' = \frac{u''}{u'} = -\frac{a_1'}{a_1 + a_2}.$$

Therefore, the function

$$u(x) = \frac{1}{A} \int_{-1}^{x} e^{F(z)} dz, \qquad \text{where} \quad F(x) = \int_{-1}^{x} f(z)\, dz, \quad A = \int_{-1}^{1} e^{F(z)} dz > 0,$$

satisfies (1.4) and $u(-1) = 0, u(1) = 1$. Now it remains to show that for large $\omega > 0$ (depending on ε), this function satisfies (1.5) as well.

Since $a_{1,2}$ are even functions, $a_{1,2}(-x) \equiv a_{1,2}(x)$, we also have

$$f(-x) \equiv -f(x), \quad F(-x) \equiv F(x), \quad u(-x) + u(x) \equiv 1 \qquad \text{on} \quad [-1, 1].$$

By the last equality here, $0 \le u \le \varepsilon$ on $[-1, -\varepsilon]$ if and only if $1 - \varepsilon \le u \le 1$ on $[\varepsilon, 1]$, so that it suffices to prove the estimate on the interval $[-1, -\varepsilon]$. We know that

$$f(x) = \frac{(\eta \sin \omega x)'}{2 - \eta \cos \omega x} \qquad \text{on} \quad [-1, 0].$$

Integrating by parts and using the properties $|\eta| \le \frac{\varepsilon}{2} < \frac{1}{4}$, $|\eta'| \le 1$, we estimate

$$\begin{aligned} F(x) &= \int_{-1}^{x} f(z)\, dz = \int_{-1}^{x} \frac{(\eta \sin \omega z)'}{2 - \eta \cos \omega z} dz \\ &= -\int_{-1}^{x} \eta \sin \omega z \left(\frac{1}{2 - \eta \cos \omega z} \right)' dz + O(\varepsilon) \\ &= \omega \int_{-1}^{x} \left(\frac{\eta \sin \omega z}{2 - \eta \cos \omega z} \right)^2 dz + O(\varepsilon) \\ &\ge \frac{\omega}{6} \int_{-1}^{x} \eta^2 \sin^2 \omega z \, dz + O(\varepsilon) \\ &= \frac{\omega}{12} \int_{-1}^{x} \eta^2 dz - \frac{1}{24} \int_{-1}^{x} \eta^2 (\sin 2\omega z)' \, dz + O(\varepsilon) \\ &= \frac{\omega}{12} \int_{-1}^{x} \eta^2 dz + O(\varepsilon) \qquad \text{on} \quad [-1, 0]. \end{aligned}$$

Since $\eta > 0$ in $\left(-\varepsilon, -\frac{\varepsilon}{2}\right)$, the values of $F(x)$ in this interval, and hence the constant $A > 0$, become arbitrary large for large $\omega > 0$. Choose ω such that $A \ge \varepsilon^{-1}$. Then the desired estimate $0 \le u \le \varepsilon$ on $[-1, -\varepsilon]$ follows immediately, because

$$\eta \equiv 0, \quad f \equiv 0, \quad F \equiv 0, \quad u(x) = \frac{1}{A}(x+1) \qquad \text{on} \quad [-1, -\varepsilon].$$

2. Integral estimates

In this sections, we prove basic integral estimates for solutions of second order parabolic equations: the energy estimate in the divergence case (D) and the Aleksandrov-Krylov estimate in the non-divergence case (ND). For formulation of these results, we need further notations.

Let Ω be an open set in $\mathbb{R}^n$. For $0 < p < \infty$, $L^p(\Omega)$ denotes the linear space of measurable functions with the finite

$$\|u\|_p = \|u\|_{p,\Omega} := \left(\int_\Omega |u|^p \, dx \right)^{1/p}. \tag{2.1}$$

For $1 \le p < \infty$, $L^p(\Omega)$ is a Banach space with the norm $\|\cdot\|_p$. For $p = \infty$, $L^\infty(\Omega)$ is the Banach space of bounded functions on Ω with the norm

$$\|u\|_\infty = \|u\|_{\infty,\Omega} := \sup_\Omega |u|. \tag{2.2}$$

We will also use a similar notation $L^p(Q)$ for functions $u(X) = u(x,t)$ on an open set $Q \subset \mathbb{R}^{n+1}$. These notations make sense for vector functions u as well, with the understanding that $|u|$ in (2.1) and (2.2) stands for the length of vector $u = u(x)$.

Let Q be an open set in $\mathbb{R}^{n+1}$. For functions $u = u(x,t)$ on Q, which have derivatives $D_i u$, we denote

$$||u||^2_{V(Q)} := \int\limits_Q |Du|^2 \, dX + \sup_t \int\limits_{Q(t)} |u(x,t)|^2 \, dx, \tag{2.3}$$

where $Q(t) := \{x \in \mathbb{R}^n : (x,t) \in Q\}$.

We start with a simple lemma, which holds in both cases (D) and (ND). Its proof is straightforward.

LEMMA 2.1. *Let Q be an open set in $\mathbb{R}^{n+1}$. Then for arbitrary functions $u \in C^{2,1}(Q)$ and $G \in C^2(\mathbb{R}^1)$, the function $v := G(u)$ belongs to $C^{2,1}(Q)$ and satisfies*

$$Dv = G'(u)\, Du, \quad Lv = G'(u)\, Lu - G''(u)\,(aDu, Du) \quad \text{in} \quad Q.$$

In particular, if $u \geq 0, Lu \leq 0$ in Q, and $G, G', G'' \geq 0$, then $v \geq 0, Lv \leq 0$ in Q.

The following theorem is a modification of well known *energy estimates* (see [**LSU**], Sec. 3.2; [**L**], Sec. 6.1). It holds for equations in the divergence form (D).

THEOREM 2.2 (Energy Estimate). *Let u be a function in $C^{2,1}(C_2)$, where $C_2 = C_2(Y) = B_2(y) \times (s-4, s)$, $Y = (y,s) \in \mathbb{R}^{n+1}$, and let*

$$Lu := a_0 u_t - (D, aDu) = f_0 + (D, f) = f_0 + \sum_{i=1}^n D_i f_i \tag{2.4}$$

in C_2, with functions $f_0 \in C(C_2)$, $f = (f_1, \cdots, f_n)^t$, $Df_i \in C(C_2)$ for $i = 1, \cdots, n$. Take a standard [1] *smooth cut-off function η, such that*

$$0 \leq \eta \leq 1 \quad \text{on } \overline{C_2}, \quad \eta \equiv 1 \quad \text{on} \quad \overline{C_1(Y)}, \quad \text{and} \quad \eta = 0 \quad \text{near} \quad \partial_p C_2. \tag{2.5}$$

Then

$$||\eta u||^2_{V(C_2)} \leq NF^2, \quad \text{where} \quad F^2 := \int\limits_{C_2} \left(u^2 + f_0^2 + |f|^2\right) dX, \tag{2.6}$$

and the constant $N = N(n, \nu)$. This estimate remains true if instead of the equality (2.4) in C_2, we impose two inequalities

$$u \geq 0, \quad Lu \leq f_0 + (D, f). \tag{2.7}$$

PROOF. Consider the function

$$I(t) := \int a_0 \eta^2 u^2(x,t)\, dx, \quad s - 4 \leq t \leq s,$$

where the integral is taken over $B_2(y)$. From (2.4) or (2.7) it follows

$$\begin{aligned} I'(t) &= \int \left(a_0 \eta^2\right)_t \cdot u^2 dx + \int 2\eta^2 u \cdot a_0 u_t \, dx \\ &\leq N \int u^2 dx + 2 \int \eta^2 u \left[(D, aDu) + f_0 + (D, f)\right] dx. \end{aligned}$$

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[1] By standard functions we mean smooth functions with derivatives bounded by constants depending only on the order of derivative and the dimension n.

Then we integrate by parts and use the estimates $2uf_0 \le u^2 + f_0^2$, and for $0<\varepsilon<1$,

$$\begin{aligned} -2\left(D\left(\eta^2 u\right), f\right) &= -2\eta\left(D\left(\eta u\right), f\right) - 2\eta u\left(D\eta, f\right) \\ &\le \varepsilon\left|D\left(\eta u\right)\right|^2 + \varepsilon^{-1}\left|f\right|^2 + N\left(u^2 + |f|^2\right), \\ -2\left(D\left(\eta^2 u\right), aDu\right) &= -2\left(D\left(\eta u\right) + uD\eta, a\eta Du\right) \\ &= -2\left(D\left(\eta u\right) + uD\eta, aD\left(\eta u\right) - uaD\eta\right) \\ &\le -2\nu\left|D\left(\eta u\right)\right|^2 + N\left|u\right| \cdot \left|D\left(\eta u\right)\right| + Nu^2 \\ &\le \left(\varepsilon - 2\nu\right)\left|D\left(\eta u\right)\right|^2 + N\varepsilon^{-1}u^2. \end{aligned}$$

This gives us

$$I'\left(t\right) \le 2\left(\varepsilon-\nu\right)\int\left|D\left(\eta u\right)\right|^2 dx + N\varepsilon^{-1}\int\left(u^2 + f_0^2 + |f|^2\right)dx, \quad 0<\varepsilon<1.$$

For $\varepsilon = \frac{\nu}{2}$, this implies

$$\nu\int\left|D\left(\eta u\right)\right|^2 dx + I'\left(t\right) \le N\int\left(u^2 + f_0^2 + |f|^2\right)dx, \qquad s-4\le t\le s.$$

Integrating with respect to t and using Fubini's theorem, we conclude

$$\nu\int_{C_2}\left|D\left(\eta u\right)\right|^2 dX \le NF^2, \qquad \sup_{s-4<t<s} I\left(t\right) \le NF^2.$$

Since $a_0 \ge \nu > 0$, we also have

$$\nu\int_{B_2(y)}\left(\eta u\right)^2\left(x,t\right)\,dx \le I\left(t\right).$$

From these estimates it follows (2.6) with a constant $N = N\left(n,\nu\right)$. □

Corollary 2.3. *Let a function $u \in C^{2,1}\left(\overline{Q}\right)$, where Q is an open set in $\mathbb{R}^{n+1}$, and let*

$$\begin{aligned} u &\ge 0, \quad Lu := a_0 u_t - \left(D, aDu\right) \le 0 \quad \text{in} \quad Q; \\ u &= 0 \quad \text{on} \quad \left(\partial Q\right)\cap C_2, \end{aligned}$$

where $C_2 = C_2\left(Y\right) = B_2\left(y\right)\times\left(s-4,s\right)$, $Y = \left(y,s\right)\in\mathbb{R}^{n+1}$. Let η be a standard smooth cut-off function η satisfying the conditions (2.5) in the cylinder $C_2 = C_2\left(Y\right)$. Then

$$(2.8) \qquad \left\|\eta u\right\|^2_{V(Q\cap C_2)} \le NF^2, \quad \text{where} \quad F^2 := \int_{Q\cap C_2} u^2 dX,$$

and the constant $N = N\left(n,\nu\right)$.

Proof. We approximate u by smooth functions $u_\varepsilon = G_\varepsilon\left(u\right)$ for small $\varepsilon>0$, where G_ε are smooth functions such that $G_\varepsilon, G'_\varepsilon, G''_\varepsilon \ge 0$ on $\mathbb{R}^1$, $G_\varepsilon\left(u\right)\equiv 0$ for $u\le\varepsilon$, and $G'_\varepsilon\left(u\right)\equiv 1$ for $u\ge 2\varepsilon$. Then

$$(2.9) \qquad u - 2\varepsilon \le u_\varepsilon \le \max\left(u-\varepsilon, 0\right) \quad \text{on} \quad \overline{Q}, \quad Du_\varepsilon = Du \quad \text{on} \quad \left\{u\ge 2\varepsilon\right\}.$$

Moreover, since $u_\varepsilon \equiv 0$ near $(\partial Q) \cap C_2(Y)$, we can extend $u_\varepsilon \equiv 0$ on $C_2 \setminus Q$, the extended functions $u_\varepsilon \in C^{2,1}(C_2)$, and by Lemma 2.1, $u_\varepsilon \geq 0, Lu_\varepsilon \leq 0$ in C_2. From Theorem 2.2 it follows that the functions u_ε satisfy the estimate

$$||\eta u_\varepsilon||_{V(C_2)} \leq N(n,\nu) \int\limits_{C_2} u_\varepsilon^2 dX, \quad \varepsilon > 0.$$

Taking the limit as $\varepsilon \to 0^+$ and using (2.9), we get the estimate (2.8). □

For solutions of second order elliptic and parabolic equations $Lu = f$ with measurable coefficients in the non-divergence form (ND), there are pointwise estimates through the norm $||f||_p$, where $p = n$ in the elliptic case, and $p = n+1$ in the parabolic case. In the elliptic case, such estimates were established by Aleksandrov [**A**] (see also [**GT**], Cf. 9, for further discussion and references to related works by Bakel'man, Pucci and other mathematicians). A similar estimate for solutions of parabolic equations was obtained by Krylov [**K76**], who used a completely different method in his proof. Later Tso [**TK**] found a simpler proof of Krylov's result by further adjustment of the original Aleksandrov's method. In the next theorem, we present the Aleksandrov-Krylov estimate, in its simplified form. In the proof, we basically follow Tso [**TK**].

THEOREM 2.4. *Let a function* $u \in C^{2,1}\left(\overline{Q}\right)$*, where* Q *is an open subset of* $C_1 = C_1(Y_0) \subset \mathbb{R}^{n+1}$*, and let*

$$Lu = u_t - (aD, Du) \leq f \quad \text{in} \quad Q, \qquad u \leq 0 \quad \text{on} \quad \partial_p Q, \tag{2.10}$$

where $f \in L^{n+1}(Q)$*. Then*

$$M := \sup_Q u \leq NF, \quad \text{where} \quad F := ||f||_{n+1,Q}, \tag{2.11}$$

and the constant $N = N(n,\nu)$*.*

PROOF. Obviously, it suffices to consider the case $M > 0$. Then we have $M = u(X_0)$ for some $X_0 = (x_0, t_0) \in \overline{Q} \setminus \partial_p Q$. We define the *upper contact set* $\Gamma = \Gamma_u$ of u as the set of all $X = (x,t) \in Q$ such that

$$u(X) + (y - x, Du(X)) \geq u(Y)$$

for all $Y = (y,s) \in Q$ with $s \leq t$. Introduce the mapping $\Phi : \Gamma \to \mathbb{R}^{n+1}$ by formulas

$$\Phi = (p,h), \qquad p = Du, \qquad h = u - (x - x_0, Du).$$

Since $D_j h = -(x - x_0, DD_j u)$ and $h_t = u_t - (x - x_0, Du_t)$, the Jacobian matrix [2]

$$J_\Phi = \frac{\partial \Phi}{\partial X} = \frac{\partial(p,h)}{\partial(x,t)} = \begin{bmatrix} D_{ij}u & D_i u_t \\ (x_0 - x, DD_j u) & u_t + (x_0 - x, Du_t) \end{bmatrix}$$

has determinant $\det J_\Phi = u_t \det[D_{ij}u] = u_t \det D^2 u$.

We claim that $\Phi(\Gamma)$ contains the cylinder

$$C = \left\{ (p,h) \in \mathbb{R}^{n+1} : \ |p| < \frac{1}{4}M, \ \frac{1}{2}M < h < M \right\} \subset \Phi(\Gamma).$$

[2]Actually the calculation of Du_t in this matrix requires higher smoothness of u. However, by a simple approximation argument (see Remark 6.1) the estimate (2.11) is extended from functions $u \in C^\infty(\overline{Q})$ to more general class $W^{2,1}_{n+1,loc}(Q) \cap C(\overline{Q})$.

Indeed, fix $(p,h) \in C$ and consider the function $l(x) = (x - x_0, p) + h$. Since $Q \subset C_1$, we have

$$|x - x_0| < 2,\ l(x) > 0 \geq u(x,t) \quad \text{for} \quad (x,t) \in \partial_p Q, \quad l(x_0) = h < M = u(x_0, t_0).$$

Therefore, the set $Q \cap \{u = l\}$ is not empty. Choose a point $X = (x,t)$ in this set, which has the minimal possible coordinate t. Then

$$\begin{aligned} u(x,t) - l(x) &= \max[u(\cdot,t) - l(\cdot)] = 0, \\ u(X) &= l(x), \quad Du(X) = Dl(x) = p, \\ u(Y) &= u(y,s) \leq l(y) = l(x) + (y - x, p) = u(X) + (y - x, Du(X)) \end{aligned}$$

for all $Y = (y,s) \in Q,\ s \leq t$. This argument shows that $X = (x,t) \in \Gamma$ and

$$\Phi(X) = (Du(X), u(X) - (x - x_0, Du(X))) = (p,h).$$

Since (p,h) is an arbitrary point in C, we have proved $C \subset \Phi(\Gamma)$. This implies the inequality for the Lebesgue measures:

$$\frac{1}{N} M^{n+1} = |C| \leq |\Phi(\Gamma)| = \int_\Gamma |\det J_\Phi| \, dX = \int_\Gamma \left| u_t \det D^2 u \right| dX. \tag{2.12}$$

From geometrical properties of Γ, we have $u_t \geq 0$ and $D_{ii} u \leq 0$ for all i on Γ. Further, for any fixed point $X \in \Gamma$, we can rotate the coordinate axes in such a way that the matrix $D^2 u = [D_{ij} u]$ becomes diagonal at X. In the new coordinates,

$$\det D^2 u = \prod_{i=1}^n D_{ii} u, \quad Lu = u_t - \sum_{i=1}^n a_{ii} D_{ii} u \geq \nu (u_t - \Delta u) \geq 0 \quad \text{at } X.$$

Using the inequality between the geometrical mean and the arithmetical mean of $n+1$ nonnegative numbers $u_t, -D_{11}u, \ldots, -D_{nn}u$, we get

$$(n+1) \left| u_t \det D^2 u \right|^{\frac{1}{n+1}} \leq u_t - \Delta u \leq \nu^{-1} Lu \leq \nu^{-1} f \quad \text{on} \quad \Gamma.$$

Now the desired estimate (2.11) follows from (2.12). □

3. First growth theorem

In this section, we formulate and prove the first *growth theorem*, Theorem 3.3. This theorem, as well as the second and third growth theorems, Theorems 4.2 and 5.3, have exactly same formulation in both divergence (D) and non-divergence (ND) cases. Roughly speaking, they claim that from the inequality $Lu \leq 0$ in a cylinder $C_r(Y)$, and some additional information on the set $\{u \leq 0\}$, it follows

$$M_{r/2}(Y) \leq \beta \cdot M_r(Y), \quad \text{where} \quad M_r(Y) := \sup_{C_r(Y)} u_+, \tag{3.1}$$

with a constant $\beta \in (0,1)$. It is convenient to have a special notation for a class of functions satisfying this estimate.

Definition 3.1. Let $Y = (y,s) \in \mathbb{R}^{n+1},\ r > 0$, and $\beta = \mathit{const} \in (0,1)$ be fixed. Denote $\mathcal{M}(\beta, Y, r)$ the class of all functions u which are defined on $C_r(Y)$ and satisfy the estimate (3.1).

Obviously, this estimate imposes some restriction only on the positive part $u_+ = \max(u, 0)$ of u; the class $\mathcal{M}(\beta, Y, r)$ automatically contains all functions u satisfying $u \leq 0$ on $C_r(Y)$.

In the divergence case (D), in addition to the energy estimate in Theorem 2.2, we need the following technical lemma.

LEMMA 3.2. *Let Ω be a bounded open set in $\mathbb{R}^n$, and let u be a continuously differentiable function on $\overline{\Omega}$ which vanishes on the boundary $\partial\Omega$ of Ω. Then*

$$||u||_1 := \int_\Omega |u|\ dx \le N\,|\Omega|^{1/n} \int_\Omega |Du|\ dx, \tag{3.2}$$

where the constant $N = N(n) > 0$.

PROOF. This inequality with $N = 1$ follows immediately from Hölder's inequality and the Gagliardo-Nirenberg-Sobolev imbedding theorem (see [**GT**], Sec. 7.7, or [**LSU**], Sec. 2.2). Indeed, since (3.2) is invariant under rescaling $x \to const \cdot x$, we may assume $|\Omega| = 1$, and then

$$||u||_1 \le ||u||_{\frac{n}{n-1}} \le \int_\Omega |Du|\ dx.$$

For completeness, we give another elementary proof of (3.2). We assume

$$u \equiv 0, \quad Du \equiv 0 \quad \text{on} \quad \mathbb{R}^n \setminus \Omega.$$

Then for arbitrary $y \in \mathbb{R}^n$ and θ in the unit sphere $S := \{|\theta| = 1\} \subset \mathbb{R}^n$,

$$\begin{aligned} u\,(y) &= -\int_0^\infty \frac{d}{dr} u\,(y - r\theta)\ dr = \int_0^\infty (\theta, Du\,(y - r\theta))\ dr, \\ |u\,(y)| &\le \int_0^\infty |Du\,(y - r\theta)|\ dr. \end{aligned}$$

Integrating over S with respect to θ, considering r and θ as radial and angular coordinates of $x = r\theta \in \mathbb{R}^n$, and applying Fubini's theorem, we obtain

$$\begin{aligned} |u\,(y)| &\le \frac{1}{|S|} \int_S \int_0^\infty |Du\,(y - r\theta)|\ dr\, d\theta = \frac{1}{|S|} \int_{\mathbb{R}^n} |Du\,(y - x)| \cdot |x|^{1-n}\, dx \\ &= \frac{1}{|S|} \int_\Omega |Du\,(x)| \cdot |y - x|^{1-n}\, dx, \end{aligned}$$

$$\int_\Omega |u|\ dy = \frac{1}{|S|} \int_\Omega \int_\Omega |Du\,(x)| \cdot |y - x|^{1-n}\, dx\, dy = \int_\Omega |Du\,(x)| \cdot I\,(x)\ dx, \tag{3.3}$$

where

$$I\,(x) := \frac{1}{|S|} \int_\Omega |y - x|^{1-n}\, dy.$$

Now it remains to show that $I\,(x) \le N\,|\Omega|^{1/n}$ for all x. For the proof of this estimate, choose $\rho > 0$ such that $|B_\rho| = |\Omega|$. Then obviously $|\Omega \setminus B_\rho\,(x)| = |B_\rho\,(x) \setminus \Omega|$. Moreover,

$$\sup_{\Omega \setminus B_\rho(x)} |y - x|^{1-n} \le \rho^{1-n} \le \inf_{B_\rho(x) \setminus \Omega} |y - x|^{1-n}.$$

Comparing the integrals, we get

$$\begin{aligned}\int_{\Omega} |y-x|^{1-n}\,dy &\le \int_{B_\rho(x)} |y-x|^{1-n}\,dy = \int_{B_\rho(0)} |y|^{1-n}\,dy \\ &= \int_S \int_0^\rho r^{1-n}\cdot r^{n-1} dr\,d\theta = |S|\cdot\rho,\end{aligned}$$

hence

$$I(x) \le \rho = |B_1|^{-1/n}\cdot|B_\rho|^{1/n} = N\,|\Omega|^{1/n},$$

where $N = N(n) = |B_1|^{-1/n}$. From the last estimate and (3.3), the estimate (3.2) follows. $\square$

Theorem 3.3 (First Growth Theorem). *Let a function $u \in C^{2,1}\left(\overline{C_r}\right)$, where $C_r := C_r(Y)$, $Y = (y,s) \in \mathbb{R}^{n+1}$, $r > 0$, and let $Lu \le 0$ in C_r. In addition, suppose*

$$|\{u>0\}\cap C_r| \le \mu_1\cdot|C_r|, \quad \textit{where} \quad \mu_1 = const \in (0,1]. \tag{3.4}$$

Then $u \in \mathcal{M}(\beta_1, Y, r)$ with a constant $\beta_1 = \beta_1(n,\nu,\mu_1) \in (0,1]$, such that $\beta_1 \to 0^+$ as $\mu_1 \to 0^+$.

Proof. By Definition 3.1 of classes $\mathcal{M}$, we have to prove the estimate (3.1) with $\beta = \beta_1 \to 0^+$ as $\mu_1 \to 0^+$. Note that instead of (3.1), it suffices to prove a weaker estimate

$$u(Y) \le \beta_1\cdot M_r(Y), \quad \text{where} \quad M_r(Y) := \sup_{C_r(Y)} u_+. \tag{3.5}$$

Indeed, for arbitrary $Z \in C_{r/2}(Y)$ we have $C_{r/2}(Z) \subset C_r(Y) =: C_r$, and (3.4) implies

$$\left|\{u>0\}\cap C_{r/2}(Z)\right| \le |\{u>0\}\cap C_r| \le \mu_1\cdot|C_r| = 2^{n+2}\mu_1\cdot\left|C_{r/2}\right|.$$

Hence we can use (3.5) with Z, $r/2$, $2^{n+2}\mu_1$, instead of Y, r, μ_1, correspondingly. Then $u(Z) \le \beta_1\cdot M_{r/2}(Z) \le \beta_1\cdot M_r(Y)$, and since Z is an arbitrary point in $C_{r/2}(Y)$, the estimate (3.1) follows. For the proof of (3.5), we consider separately cases (D) and (ND).

Case (D). Using rescaling $x \to cx, t \to c^2 t$ with an appropriate constant $c > 0$, and replacing u by $const\cdot u$, we reduce the proof to the case $r = 2$ and $M_2(Y) = 1$. We set $Q := \{u>0\}\cap C_2$, and for $t \in I := (s-4, s)$,

$$f(t) := |Q(t)|, \quad g(t) := \int_{Q(t)} |Dw(x,t)|^2\,dx, \quad W(t) := \int_{Q(t)} |w(x,t)|\,dx,$$

where $Q(t) := \{x \in \mathbb{R}^n : (x,t) \in Q\}$, $w = \eta u$, η is a standard cut-of function satisfying (2.5) in $C_2(Y)$. By Fubini's theorem and (3.4),

$$||f||_{1,I} = \int_I |Q(t)|\,dt = |Q| \le N\mu_1, \quad \text{where} \quad N = N(n) = |C_2|. \tag{3.6}$$

Moreover, since $u \le 1$ in Q, by Corollary 2.3 the function $w := \eta u$ satisfies

$$||w||^2_{V(Q)} \le N\int_Q u^2 dX \le N\cdot|Q| \le N\mu_1,$$

with different constants $N = N(n, \nu)$. This estimate includes

$$(3.7)\qquad ||g||_{1,I} = \int\limits_I \int\limits_{Q(t)} |Dw(x,t)|^2 \, dx \, dt = \int\limits_Q |Dw|^2 \, dX \le N\mu_1$$

and

$$\int\limits_{Q(t)} w^2(x,t) \, dx \le N\mu_1 \quad \text{for} \quad t \in I.$$

Using Hölder's inequality, we get

$$W^2(t) = ||w(\cdot,t)||^2_{1,Q(t)} \le |Q(t)| \cdot \int\limits_{Q(t)} w^2(x,t) \, dx \le N\mu_1 f(t).$$

On the other hand, by Lemma 3.2 and Hölder's inequality,

$$W^2(t) \le N |Q(t)|^{\frac{2}{n}} ||Dw(\cdot,t)||^2_{1,Q(t)} \le N f^{\frac{2}{n}+1}(t) \cdot g(t).$$

These two estimates allow us to write

$$W(t) = W^{2\alpha}(t) \cdot W^{1-2\alpha}(t) \le N\mu_1^{\alpha} f^{\alpha_1}(t) \cdot g^{\alpha_2}(t), \quad \alpha_1 := \frac{1}{2} + \frac{1-2\alpha}{n}, \ \alpha_2 := \frac{1}{2} - \alpha,$$

with an arbitrary $\alpha \in \left(0, \frac{1}{2}\right)$. We take $\alpha := \frac{1}{n+2}$ in order to guarantee the equality $\alpha_1 + \alpha_2 = 1$. Using Fubini's theorem, Hölder's inequality, and estimates (3.6), (3.7), we obtain

$$\int\limits_Q w(X) \, dX = \int\limits_I W(t) \, dt \le N\mu_1^{\alpha} \int\limits_I f^{\alpha_1} g^{\alpha_2} \, dt \le N\mu_1^{\alpha} \cdot ||f||^{\alpha_1}_{1,I} \cdot ||g||^{\alpha_2}_{1,I} \le N\mu_1^{1+\alpha}.$$

Replacing μ_1 in (3.4) by a smaller constant if necessary, we may assume $|Q| = |\{u > 0\} \cap C_2| = \mu_1 \cdot |C_2|$. Since $w = \eta u \equiv u$ on $Q \cap C_1$, the previous estimate implies

$$\int\limits_{C_1} u_+ \, dX \le \int\limits_Q w \, dX \le N \cdot |\{u > 0\} \cap C_2|^{1+\alpha}, \quad \alpha = \frac{1}{n+2} > 0.$$

The previous arguments remain valid for the functions $u - h$, $h = const \ge 0$, with Q being replaced by $\{u > h\} \cap C_2$. Then the last estimate has the form

$$\int_{C_1} (u-h)_+ \, dX \le N \cdot |\{u > h\} \cap C_2|^{1+\alpha}, \qquad h \ge 0.$$

Using rescaling $x \to \rho x$, $t \to \rho^2 t$, we can rewrite this estimate for arbitrary cylinder $C_{2\rho}$ instead of C_2. Since $|C_\rho| = |B_1| \cdot \rho^{n+2}$ and $\alpha = \frac{1}{n+2}$, after cancellations we arrive at

$$(3.8)\qquad \int_{C_\rho} (u-h)_+ \, dX \le \frac{N_0}{\rho} \cdot |\{u > h\} \cap C_{2\rho}|^{1+\alpha}, \quad 0 < \rho \le 1, \ h \ge 0,$$

where $N_0 = N_0(n, \nu) > 0$.

By our assumption $M_2(Y) = 1$, we can rewrite (3.5) as $u(Y) \le \beta_1$. In order to prove this estimate (with a small $\beta_1 > 0$ for small $\mu_1 > 0$) it suffices to show that the estimate

$$(3.9)\qquad m(\rho) := |\{u > (1-\rho)\beta_1\} \cap C_{2\rho}(Y)| \le \mu_1 \rho^{\gamma} |C_{2\rho}|$$

holds for a sequence $\rho = \rho_k \to 0^+$ with a constant $\gamma > 0$. Indeed, if (3.5) fails, i.e. $u(Y) > \beta_1$, then also $u > \beta_1 > (1-\rho)\beta_1$ on $C_{2\rho}(Y)$ for small $\rho > 0$, and (3.9) also fails.

For $\rho = 1$, (3.9) coincides with the given estimate (3.4). If we show that from (3.9) with some $\rho \in (0,1]$ it follows

$$m\left(\frac{\rho}{2}\right) \le \mu_1 \left(\frac{\rho}{2}\right)^{\gamma} |C_\rho|, \tag{3.10}$$

then by induction (3.9) holds for $\rho = \rho_k = 2^{-k},\ k = 0,1,2,\cdots$. According to this plan, suppose (3.9) is true for some $\rho \in (0,1]$ and set $h := (1-\rho)\beta_1$. Note that

$$\frac{2}{\rho\beta_1}(u-h) \ge 1 \quad \text{on the set} \quad \left\{u > \left(1 - \frac{\rho}{2}\right)\beta_1\right\}.$$

Using (3.8) and (3.9), we derive

$$\begin{aligned} m\left(\frac{\rho}{2}\right) &= \left|\left\{u > \left(1-\frac{\rho}{2}\right)\beta_1\right\} \cap C_\rho\right| \le \frac{2}{\rho\beta_1}\int_{C_\rho}(u-h)_+\,dX \\ &\le \frac{2N_0}{\rho^2\beta_1}\left|\{u>h\}\cap C_{2\rho}\right|^{1+\alpha} \le \frac{2N_0}{\rho^2\beta_1}\left(\mu_1\rho^{\gamma}|C_{2\rho}|\right)^{1+\alpha}. \end{aligned}$$

Since $\alpha = \frac{1}{n+2}$ and $|C_\rho| = N\rho^{n+2}$, the last expression will coincide with the right-hand side of (3.10) if we take $\gamma = n+2$ and $\beta_1 = N_1\mu_1^{\alpha}$ with an appropriate constant $N_1 = N_1(n,\nu) > 0$. By the above arguments

$$u(Y) \le \beta_1 = \beta_1(n,\nu,\mu_1) = \min(1, N_1\mu_1^{\alpha}) \to 0 \quad \text{as} \quad \mu_1 \to 0^+.$$

This completes the proof of theorem in the divergence case.

Case (ND). Using rescaling and replacing u by $const \cdot u$, we may assume without loss of generality that $r = 1$, $M_1(Y) := \sup_{C_1(Y)} u = 1$, and $u(Y) > 0$. Consider the functions

$$v(X) = v(x,t) = u(x,t) + t - s - |x-y|^2 \qquad \text{in} \quad Q := \{v > 0\} \cap C_1(Y).$$

The set Q is nonempty, because $v(Y) = u(Y) > 0$ and $Y \in \partial Q$. From the inequality $v \le u$ in Q and (3.4) it follows

$$|Q| \le |\{u > 0\} \cap C_1(Y)| \le N\mu_1.$$

Since $v \le 0$ on $\partial_p C_1(Y)$, we must have $v = 0$ on $\partial_p Q$. Furthermore, the inequality $Lu \le 0$ yields

$$Lv \le 1 + 2\,\mathrm{tr}\,a \le 1 + 2n\nu^{-1} \quad \text{in} \quad Q.$$

Applying Theorem 2.4 to the function v in Q, we obtain

$$u(Y) = v(Y) \le N\left|\left|(Lv)_+\right|\right|_{n+1,Q} \le N|Q|^{\frac{1}{n+1}} \le N_1\mu_1^{\frac{1}{n+1}},$$

with different constants N and N_1, depending only on n and ν. Now the estimate (3.5) follows with

$$\beta_1 = \beta_1(n,\nu,\mu_1) = \min(1, N_1\mu_1^{\alpha}), \quad \alpha = \frac{1}{n+1}.$$

Theorem is proved. □

The following theorem follows from Theorem 3.3 by general arguments, which work for both cases (D) and (ND).

THEOREM 3.4. *Let a function* $u \in C^{2,1}(C_r)$, *where* $C_r := C_r(Y)$, $Y = (y,s) \in \mathbb{R}^{n+1}$, $r > 0$, *and let* $Lu \le 0$ *in* C_r. *Then for arbitrary* $p > 0$,

$$u_+^p(Y) \le \frac{N}{|C_r|} \int_{C_r} u_+^p \, dX, \tag{3.11}$$

where $N = N(n, \nu, p) > 0$.

PROOF. Since (3.11) is invariant under rescaling $x \to rx$, $t \to r^2 t$, we may assume $r = 1$. For $X \in C_1 = C_1(Y)$, define

$$d(X) := \sup\{\rho > 0 : \ C_\rho(X) \subset C_1(Y)\}.$$

Obviously, $d(Y) = 1$, so that

$$u(Y) = ud^\gamma(Y) \le M := \sup_{C_1} d^\gamma u, \tag{3.12}$$

where $\gamma := \frac{n+2}{p} > 0$. This choice of γ will be useful below. Note that $d^\gamma u(X)$ is a continuous function on $\overline{C_1}$, which vanishes on $\partial_p C_1$. Therefore,

$$M = d^\gamma u(X_0) \quad \text{for some} \quad X_0 \in \overline{C_1} \setminus \partial_p C_1.$$

It is easy to see that

$$d(X) \ge r_0 := \frac{1}{2} d(X_0) \quad \text{on} \quad C_0 := C_{r_0}(X_0).$$

Hence

$$\sup_{C_0} u \le r_0^{-\gamma} \sup_{C_0} d^\gamma u \le r_0^{-\gamma} M = 2^\gamma u(X_0),$$

and the function $v := u - \frac{1}{2} u(X_0)$ satisfies

$$v(X_0) = 2^{-1} u(X_0) \ge 2^{-\gamma-1} \sup_{C_0} u > 2^{-\gamma-1} \sup_{C_0} v. \tag{3.13}$$

Let $\mu_1 = \mu_1(n, \nu, p) \in (0,1]$ be the constant in Theorem 3.3, which corresponds to $\beta_1 = \beta_1(n, \nu, p) = 2^{-\gamma-1} \in (0,1]$. The previous inequality (3.13) says that v does not belong to $\mathcal{M}(\beta_1, X_0, r_0)$, and therefore by this theorem, the corresponding estimate (3.4) fails, i.e. the set

$$Q_0 := \{v > 0\} \cap C_0 = \left\{ u > \frac{1}{2} u(X_0) \right\} \cap C_0$$

has measure $|Q_0| > \mu_1 \cdot |C_0|$. Combining the above estimates, we obtain

$$\begin{aligned} u_+^p(Y) &\le M^p = (d^\gamma u)^p(X_0) = (2r_0)^{\gamma p} u^p(X_0) \\ &\le \frac{(2r_0)^{\gamma p}}{|Q_0|} \int_{Q_0} (2u)^p \, dX \le \frac{2^{\gamma p + p} r_0^{\gamma p}}{\mu_1 |C_0|} \int_{C_1} u_+^p \, dX. \end{aligned}$$

By the choice of γ, we have $r_0^{\gamma p} = r_0^{n+2} = N \cdot |C_0|$, and the estimate (3.11) follows. $\square$

REMARK 3.5. In the above proof, we derived Theorem 3.4 from Theorem 3.3 by general arguments, which do not use further properties of solutions. In turn, Theorem 3.3 follows easily from Theorem 3.4. Indeed if u satisfies (3.4) and (3.11), then

$$u_+^p(Y) \le \frac{N}{|C_r|}\int\limits_{C_r} u_+^p\,dX \le N\frac{|\{u>0\}\cap C_r|}{|C_r|}\sup_{C_r} u_+^p \le N\mu_1 \sup_{C_r} u_+^p,$$

and $u \in \mathcal{M}(\beta_1, Y, r)$ with $\beta_1 = N(n,\nu,p)\cdot \mu_1^{1/p}$.

4. Second growth theorem

For a fixed point $Y=(y,s)\in\mathbb{R}^{n+1}$ with $s>0$, and $r>0$, introduce the *slant cylinder*

$$V_r = V_r(Y) := \left\{X=(x,t)\in\mathbb{R}^{n+1} : \left|x-\frac{t}{s}y\right|<r,\ 0<t<s\right\}. \tag{4.1}$$

The following lemma is the main technical tool in this section.

Lemma 4.1 (Slant Cylinder Lemma). *Let a function $u \in C^{2,1}\left(\overline{V_r}\right)$ satisfy $Lu\le 0$ in a slant cylinder V_r, which is defined in (4.1) with $Y=(y,s)\in\mathbb{R}^{n+1}, s>0, r>0$, such that*

$$K^{-1}r\,|y| \le s \le Kr^2, \quad \text{where} \quad K = const \ge 1. \tag{4.2}$$

In addition, suppose

$$u\le 0 \quad \text{on} \quad D_r := B_r(0)\times\{0\}. \tag{4.3}$$

Then

$$u(Y) \le \beta_2 \sup_{V_r(Y)} u_+, \tag{4.4}$$

with a constant $\beta_2=\beta_2(n,\nu,K)<1$.

Proof. Using rescaling and replacing u by $const\cdot u$, we reduce the proof to the case $r=1$, $\sup\limits_{V_1(Y)} u = 1$. Moreover, approximating u by $u^\varepsilon = G^\varepsilon(u)$ as in the proof of Corollary 2.3, we may assume $u\in C^\infty\left(\overline{V_1}\right)$ and

$$0\le u<1,\ Lu\le 0 \quad \text{on } \overline{V_1}; \quad u\equiv 0 \quad \text{on} \quad D_1 := B_1(0)\times\{0\}. \tag{4.5}$$

Now consider separately cases (D) and (ND).

Case (D). Introduce the function $v=G(u):=-\ln(1-u)$. Obviously, $v\ge 0$ in V, $v\equiv 0$ on D_1. Since

$$G'(u) = (1-u)^{-1}>0, \quad G''(u) = (1-u)^{-2}>0 \quad \text{for} \quad u<1,$$

by Lemma 2.1 we have

$$\begin{aligned} Lv &= G'(u)\,Lu - G''(u)\,(aDu,Du) \\ &\le -(1-u)^{-2}(aDu,Du) = -(aDv,Dv) \le -\nu\,|Dv|^2. \end{aligned}$$

This gives us the estimate

$$a_0 v_t = (D,aDv)+Lv \le (D,aDv) - \nu\,|Dv|^2 \quad \text{in} \quad V_1. \tag{4.6}$$

Further, take a smooth cut-off function $\eta=\eta(x)$ in $B_1=B_1(0)$, such that

$$0\le\eta\le 1 \quad \text{on} \quad B_1=B_1(0), \quad \eta\equiv 1 \quad \text{on} \quad B_{1/2}, \quad \eta\equiv 0 \quad \text{near} \quad \partial B_1,$$

and define

$$I(t) := \int_{B_1(tl)} a_0(x) v(x,t) \eta^2(x - tl)\, dx, \quad 0 \le t \le s, \tag{4.7}$$

where $l := s^{-1}y$. By (4.2), we have $K^{-1}|y| \le s \le K$, hence $|l| \le K$. Since the integral function vanishes near $\partial B_1(tl)$, we may assume that the integral is taken over $\mathbb{R}^n$. Differentiating with respect to t, we obtain

$$\begin{aligned} I'(t) &= \int \left[a_0 v_t \eta^2 - a_0 v \left(l, D\left(\eta^2\right)\right)\right] dx \\ &\le \int \left[(D, aDv)\eta^2 - \nu |Dv|^2 \eta^2 + \nu^{-1} K v \left| D\left(\eta^2\right)\right| \right] dx. \end{aligned}$$

Integrating by parts and using Schwartz's inequality, we have

$$\begin{aligned} \int (D, aDv)\eta^2(x)\, dx &= -2\int (D\eta, aDv)\eta(x)\, dx \\ &\le N \int |Dv| \eta\, dx \le \frac{\nu}{2} \int |Dv|^2 \eta^2\, dx + N_1. \end{aligned}$$

This gives us

$$I'(t) \le -\frac{\nu}{2}\int |Dv|^2 \eta^2\, dx + N_1 + \nu^{-1} K \int v \left| D\left(\eta^2\right)\right| dx,$$

where $v = v(x,t)$, $\eta = \eta(x - tl)$.

We may assume that $\eta = \eta(x)$ depends only on $r = |x|$ and η is nonincreasing with respect to r on $[0,1]$. Then $D\left(\eta^2(x)\right)$ is directed along the unit vector $-|x|^{-1}x$ for $\frac{1}{2} \le |x| \le 1$. Therefore,

$$\left| D\left(\eta^2(x)\right)\right| = -|x|^{-1}\left(x, D\left(\eta^2(x)\right)\right) \quad \text{for} \quad \frac{1}{2} \le |x| \le 1,$$

and $D\left(\eta^2(x)\right) \equiv 0$ for $|x| \le \frac{1}{2}$. We can write

$$\left| D\left(\eta^2(x)\right)\right| = -\left(\phi(x), D\left(\eta^2(x)\right)\right) \quad \text{for} \quad |x| < 1,$$

where $\phi = (\phi_1, \cdots, \phi_n)$ is an arbitrary smooth vector field on $B_1(0)$, such that $\phi(x) \equiv |x|^{-1}x$ for $|x| \ge \frac{1}{2}$. Integrating by parts, we estimate

$$\begin{aligned} \int v(x,t) \left| D\left(\eta^2(x - tl)\right)\right| dx &= -\int v(x,t)\left(\phi(x - tl), D\left(\eta^2(x - tl)\right)\right) dx \\ &= \int (D, v\phi)\eta^2\, dx \le N_2 \int (|v| + |Dv|)\eta^2\, dx. \end{aligned}$$

Here

$$\begin{aligned} \int v\eta^2 dx &\le \nu^{-1}\int a_0(x) v(x,t)\eta^2(x - tl)\, dx = \nu^{-1} I(t), \\ \int |Dv|\eta^2 dx &\le \int |Dv|\eta\, dx \le \varepsilon \int |Dv|^2 \eta^2 dx + N\varepsilon^{-1}, \quad 0 < \varepsilon < 1. \end{aligned}$$

Taking $\varepsilon := \frac{1}{2}\nu^2 K^{-1} N_2^{-1}$, we derive

$$\nu^{-1} K \int v \left| D\left(\eta^2\right)\right| dx \le \frac{\nu}{2}\int |Dv|^2 \eta^2 dx + N + N I(t).$$

Combining together these estimates, we conclude

$$I'(t) \le N_3 + N_3 I(t), \quad 0 < t \le s \le K$$

with a constant $N_3 = N_3(n, \nu, K)$. In addition, since $v = u = 0$ on $B_1(0) \times \{0\}$, we also have $I(0) = 0$. Integrating the inequality

$$\frac{d}{dt} \ln (I(t) + 1) \le N_3,$$

we find

$$I(t) \le e^{N_3 t} - 1 < e^{N_3 K}, \quad 0 \le t \le s,$$

and

$$\int_V a_0(x) v(x,t) \eta^2(x - tl)\, dX = \int_0^s I(t)\, dt \le N = N(n, \nu, K).$$

Notice that $a_0 \ge \nu > 0$ and $\eta(x - tl, t) \equiv 1$ on $C_\rho(Y)$ for some small $\rho = \rho(K) > 1$. Hence

$$||v||_{1, C_\rho(Y)} = \int_{C_\rho(Y)} v(X)\, dX \le N = N(n, \nu, K).$$

By Theorem 3.4 applied to the function v with $p = 1, r = \rho$, we finally obtain $v(Y) \le N_4 = N_4(n, \nu, K)$, and

$$u(Y) = 1 - e^{-v(Y)} \le \beta_2 := 1 - e^{-N_4} < 1.$$

This completes the proof of lemma in the divergence case.

Case (ND). We set

$$v(x,t) := e^{-\gamma t} w^2(x,t), \quad w(x,t) := 1 - |x - tl|^2 > 0 \quad \text{in} \quad V_1,$$

where $\gamma = \mathit{const} > 0$. Then

$$\begin{aligned} Lv &= v_t - (aD, Dv) = e^{-\gamma t} \left[-\gamma w^2 + 2wLw - 2(aDw, Dw) \right] \\ &\le e^{-\gamma t} \left(-\gamma w^2 + Nw - 2\nu |Dw|^2 \right) \quad \text{in} \quad V_1. \end{aligned}$$

Notice that $|Dw|^2 = 4|x - tl|^2 = 4(1 - w)$. Therefore,

$$Lv \le e^{-\gamma t} \left(-\gamma w^2 + Nw - 8\nu \right) \le 0 \quad \text{in} \quad V_1,$$

provided $\gamma = \gamma(n, \nu, K) > 0$ is chosen large enough. This inequality together with (4.5) imply

$$L(u + v - 1) \le 0 \quad \text{in} \quad V_1.$$

Now compare u and v on the "bottom" $D_1 := B_1(0) \times \{0\}$ and the lateral side $S := \{(x,t) : \ |x - tl| = 1, \ 0 \le t \le s\}$. From our assumptions it follows

$$u \equiv 0, \quad 0 \le v \le 1 \quad \text{on} \quad D_1; \qquad 0 \le u \le 1, \quad v \equiv 0 \quad \text{on} \quad S.$$

Hence $u + v - 1 \le 0$ on the parabolic boundary $\partial_p V = D_1 \cup S$, and by the maximum principle, $u + v - 1 \le 0$ on $\overline{V}_1$. In particular,

$$u(Y) \le 1 - v(Y) = 1 - e^{-\gamma s} \le \beta_2 := 1 - \exp(-\gamma K).$$

Lemma is completely proved. □

THEOREM 4.2 (Second Growth Theorem). *Let a function* $u \in C^{2,1}\left(\overline{C_r}\right)$, *where* $C_r := C_r(Y)$, $Y = (y,s) \in \mathbb{R}^{n+1}$, $r > 0$, *and let* $Lu \le 0$ *in* C_r. *In addition, suppose*

$$u \le 0 \quad on \quad D_\rho := B_\rho(z) \times \{\tau\}, \tag{4.8}$$

where

$$B_\rho(z) \subset B_r(y) \quad and \quad s - r^2 \le \tau \le s - \frac{1}{4}r^2 - \rho^2. \tag{4.9}$$

Then $u \in \mathcal{M}(\beta_2, Y, r)$ *with a constant* $\beta_2 = \beta_1(n,\nu,\rho/r) < 1$.

PROOF. Using rescaling and parallel translation in $\mathbb{R}^{n+1}$, we reduce the proof to the case $r = 1$ and $(z,\tau) = 0 \in \mathbb{R}^{n+1}$. For an arbitrary $Y' \in C_{1/2}(Y)$, we can apply Lemma 4.1 to the slant cylinder $V_\rho(Y') \subset C_1(Y)$, having in mind that the corresponding constant K in this lemma depends only on ρ. This gives us

$$u(Y') \le \beta_2 \sup_{V_\rho(Y')} u_+ \le \beta_2 \sup_{C_1(Y)} u_+.$$

Since this is true for all $Y' \in C_{1/2}(Y)$, we have

$$\sup_{C_{1/2}(Y)} u_+ \le \beta_2 \sup_{C_1(Y)} u_+,$$

i.e. $u \in \mathcal{M}(\beta_2, Y, 1)$. □

We need one more estimate of same kind, with more explicit dependence of constants on the ratio ρ/r.

LEMMA 4.3. *Let a function* $v \in C^{2,1}\left(\overline{C_r}\right)$ *satisfy* $v \ge 0$, $Lv \ge 0$ *in* $C_r := C_r(Y)$, $Y = (y,s) \in \mathbb{R}^{n+1}, r > 0$. *Then for arbitrary disks* $D_\rho := B_\rho(z) \times \{\tau\}$ *and* $D^0 := B_{r/2}(y) \times \{\sigma\}$, *such that* $B_\rho(z) \subset B_r(y)$ *and*

$$s - r^2 \le \tau < \tau + h^2r^2 \le \sigma \le s,\ h = const \in (0,1),$$

we have

$$\inf_{D_\rho} v \le \left(\frac{2r}{\rho}\right)^\gamma \inf_{D^0} v, \quad where \quad \gamma = \gamma(n,\nu,h) > 0. \tag{4.10}$$

PROOF. Without loss of generality, we assume

$$m := \inf_{D_\rho} v > 0, \quad r = 1, \quad z = 0, \quad \tau = 0, \quad \sigma = s = h^2,$$

so that $D_\rho = B_\rho(0) \times \{0\}$. Applying the additional linear transformation along the t-axis, we can also reduce the proof to the case $h = 1$. After these preparations, fix the integer $k \ge 0$ in such a way that $2^{-k-1} < \rho \le 2^{-k}$, and for $j = 0, 1, \ldots,$ introduce

$$y^j \;:\; = y^* + 2^{-j}(y - y^*), \quad B^j := B_{2^{-j}}\left(y^j\right), \quad \text{where} \quad y^* := \frac{\rho}{\rho - 1}y;$$
$$Y^j \;:\; = \left(y^j, 4^{-j}\right), \quad C^j := C_{2^{-j}}\left(Y^j\right), \quad D^j := B_{2^{-j-1}}\left(y^j\right) \times \left\{4^{-j}\right\}.$$

By the choice of y^*, we have $0 = y^* + \rho\cdot(y - y^*)$, so that

$$B_\rho(0), \quad B^j \quad \in \quad \{B_\theta(y^* + \theta\cdot(y - y^*));\quad 0 \le \theta \le 1\}.$$

From our assumption $B_\rho(0) \subset B_1(y)$ it follows $|y| \le 1 - \rho$, $|y - y^*| \le 1$, and

$$B^{k+1} \subset B_\rho(0) \subset B^k \subset B^{k-1} \subset \ldots \subset B^1 \subset B^0 = B_1(y).$$

We can apply Theorem 4.2 to the function $u := 1 - m^{-1}v$ in C^k with

$$r = 2^{-k}, \quad \rho = 2^{-k-1}, \quad Y = Y^k, \quad z = 0, \quad \tau = 0.$$

This gives us

$$\sup_{D_k} u \le \sup_{C_{2^{-k-1}}(Y^k)} u \le \beta_2 \sup_{C^k} u \le \beta_2 = \beta_1 \left(n, \nu, \frac{1}{2}\right) < 1,$$

or, equivalently,

$$\inf_{D_\rho} v = m \le (1 - \beta_2)^{-1} \inf_{D^k} v = 2^\gamma \inf_{D^k} v,$$

where $\gamma = \gamma(n, \nu) := -\log_2 (1 - \beta_2) > 0$. Similarly, if $k \ge 1$, we also have

$$\inf_{D^j} v \le 2^\gamma \inf_{D^{j-1}} v \quad \text{for} \quad j = 1, 2, \cdots, k.$$

Combining these inequalities, we obtain

$$\inf_{D_\rho} v \le 2^\gamma \inf_{D^k} v \le 2^{2\gamma} \inf_{D^{k-1}} v \le \cdots \le 2^{(k+1)\gamma} \inf_{D^0} v \le \left(\frac{2}{\rho}\right)^\gamma \inf_{D^0} v.$$

□

5. Third growth theorem

The first growth theorem, Theorem 3.1, describes the relation between the constants β_1 and μ_1 in (3.1) and (3.4) in such a way that $\beta_1 = \beta_1(n, \nu, \mu_1) \to 0^+$ as $\mu_1 \to 0^+$. In this section we show that, roughly speaking, $\beta_1 < 1$ if $\mu_1 < 1$. The proof of this fact, which we call the *third growth theorem*, is based on the first and second growth theorems. We also need two covering lemmas, which are similar to those in [**KS**] and [**S80**].

Lemma 5.1. *Let a constant $\mu_0 \in (0,1)$ be fixed. For an arbitrary measurable set $\Gamma \subset \mathbb{R}^{n+1}$ with finite measure $|\Gamma|$, introduce the family of cylinders*

$$\mathcal{A} := \{C = C_r(Y) : \ |C \cap \Gamma| \ge (1 - \mu_0) \cdot |C|\}. \tag{5.1}$$

Then the open set $E := \cup \{C : \ C \in \mathcal{A}\}$ satisfies

$$|\Gamma \setminus E| = 0 \quad \text{and} \quad |E| \ge q_0 |\Gamma|, \quad \text{where} \quad q_0 := 1 + 3^{-n-1}\mu_0 > 1. \tag{5.2}$$

Proof. The first statement $|\Gamma \setminus E| = 0$ follows from the fact that almost every point of Γ is a point of density. Indeed, suppose $|\Gamma \setminus E| > 0$. Then one can choose a cylinder $C^* := B_{1/m}(y) \times \left(s - \frac{1}{m}, s\right)$, with natural m, such that

$$|C^* \cap (\Gamma \setminus E)| > (1 - \mu_0) \cdot |C^*|. \tag{5.3}$$

Since C^* is a union of m disjoint *parabolic* cylinders

$$C_k^* := B_{1/m}(y) \times \left(s - \frac{k+1}{m^2}, s - \frac{k}{m^2}\right), \quad k = 0, 1, \cdots, m-1,$$

the inequality (5.3) must be true for some C_k^* instead of C^*. But then $C_k^* \in \mathcal{A}$, $C_k^* \subset E$, $C_k^* \cap (\Gamma \setminus E)$ is empty and (5.3) cannot be true for C_k^*. This contradiction proves $|\Gamma \setminus E| = 0$.

Further, for each $C = C_r(Y) \in \mathcal{A}$ with $|C \cap \Gamma| > (1 - \mu_0) \cdot |C|$, we can continuously increase r until we get the equality $|C \cap \Gamma| = (1 - \mu_0) \cdot |C|$ (note that $|\Gamma| < \infty$). Therefore, we can write

$$E = \cup \{C : \ C \in \mathcal{A}_0\}, \quad \mathcal{A}_0 := \{C = C_r(Y) : \ |C \cap \Gamma| = (1 - \mu_0) \cdot |C|\}. \tag{5.4}$$

Next, we reproduce the well known argument in the proof of the classical Vitali covering theorem (with parabolic cylinders instead of usual balls or cubes). We construct a (countable or finite) sequence of cylinders $C^k, k = 1, 2, \cdots$, as follows. For this, we denote $R_1 := \sup\{r : C_r(Y) \in \mathcal{A}_0\}$. Easy compactness argument shows that this supreme is attained for some cylinder $C_{R_1}(Y_1) \in \mathcal{A}_0$. We pick an arbitrary one of (possibly many) such cylinders: $C^1 := C_{R_1}(Y_1)$. Assuming that $C^i := C_{R_i}(Y_i)$ for $i = 1, 2, \cdots, k$ have been already selected for some $k \geq 1$, we set

$$\mathcal{A}_{k+1} := \left\{C = C_r(Y) \in \mathcal{A}_0 : C \cap C^i = \emptyset, \ i = 1, 2, \cdots, k\right\}.$$

If $\mathcal{A}_{k+1}$ is nonempty, then we denote

$$R_{k+1} := \sup\{r > 0 : C_r(Y) \in \mathcal{A}_{k+1} \text{ for some } Y\}$$

this supreme is attained for $C^{k+1} := C_{R_{k+1}}(Y_{k+1}) \in \mathcal{A}_{k+1}$.

In the case when all the sets $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots$ are nonempty, we obtain a countable sequence of cylinders $C^i = C_{R_i}(Y_i), i = 1, 2, \cdots$. If however, $\mathcal{A}_i \neq \emptyset$ for $i = 1, 2, \cdots, k$ and $\mathcal{A}_{k+1} = \emptyset$, then we have a finite sequence of cylinders $C^i, i = 1, 2, \cdots, k$. In the latter case we set, by definition, $R_{k+1} = R_{k+2} = \cdots = 0$. We note that by construction the cylinders C^i are pairwise disjoint, $R_1 \geq R_2 \geq \cdots$, and $R_i \to 0$ as $i \to \infty$.

Take an arbitrary cylinder $C_r(Y) \in \mathcal{A}_0$. We have $R_1 \geq R_2 \geq \cdots \geq R_k \geq r > R_{k+1}$ for some integer $k \geq 1$. Since $r > R^{k+1}$, the cylinder $C_r(Y)$ does not belong to $\mathcal{A}_{k+1}$, and therefore $C_r(Y) \cap C^i \neq \emptyset$ for some $i \leq k$. Since $r < R_i$, this implies $C_r(Y) \subset \tilde{C}^i$, where

$$\tilde{C} := B_{3\rho}(z) \times \left(s - 2\rho^2, s + \rho^2\right) \quad \text{for} \quad C = C_\rho(z, s) = B_\rho(z) \times \left(s - \rho^2, s\right).$$

Thus arbitrary $C_r(Y) \in \mathcal{A}_0$ is a subset of $\tilde{C}^i$ for some $i \geq 1$. From (5.4) we conclude

$$E \subset \bigcup_i \tilde{C}^i, \quad |E| \leq \sum_i \left|\tilde{C}^i\right| = 3^{n+1} \sum_i \left|C^i\right|.$$

On the other hand, since $C^i \in \mathcal{A}_0$ are pairwise disjoint,

$$|E \setminus \Gamma| \geq \sum_i \left|(E \setminus \Gamma) \cap C^i\right| = \mu_0 \sum_i \left|C^i\right|.$$

From the previous relations we obtain

$$\frac{|E|}{|\Gamma|} = 1 + \frac{|E \setminus \Gamma|}{|\Gamma|} \geq 1 + \frac{|E \setminus \Gamma|}{|E|} \geq 1 + \frac{\mu_0}{3^{n+1}} = q_0 > 1.$$

The lemma is proved. □

Lemma 5.2. *For a fixed constant $K_1 > 1$ and any standard cylinder $C = C_r(Y) = B_r(y) \times \left(s - r^2, s\right)$ denote $\hat{C} := B_r(y) \times \left(s + r^2, s + K_1 r^2\right)$. Then for arbitrary family $\mathcal{A}$ of standard cylinders $C = C_r(Y)$, the $n+1$-dimensional measures of the sets $E := \cup\{C : C \in \mathcal{A}\}$ and $\hat{E} := \cup\left\{\hat{C} : C \in \mathcal{A}\right\}$ satisfy*

$$\left|\hat{E}\right| \geq q_1 |E|, \quad \text{where} \quad q_1 := \frac{K_1 - 1}{K_1 + 1}. \tag{5.5}$$

Proof. By Fubini's theorem,

$$|E| = \int |E_x| \, dx, \quad \left|\hat{E}\right| = \int \left|\hat{E}_x\right| dx,$$

where

$$E_x := \{t \in \mathbb{R}^1 : (x,t) \in E\}, \quad \hat{E}_x := \left\{t \in \mathbb{R}^1 : (x,t) \in \hat{E}\right\}, \quad x \in \mathbb{R}^n.$$

Therefore, for the proof of lemma it suffices to get the estimate $\left|\hat{E}_x\right| \geq q_1 |E_x|$ for all $x \in \mathbb{R}^n$.

Fix an arbitrary x for which E_x is nonempty. For this x, the open set $\hat{E}_x$ is a union of disjoint open intervals $\hat{I}_k,\ k = 1, 2, \cdots$. If $t \in E_x$, then (x,t) belongs to a cylinder $C = B_r(y) \times (s - r^2, s)$ in $\mathcal{A}$, and $(s + r^2, s + K_1 r^2) \subset \hat{I}_k$ for some k. Obviously, $r^2 \leq r_k^2 := (K_1 - 1)^{-1} \left|\hat{I}_k\right|$. We also choose s_k such that $\hat{I}_k = (s_k + r_k^2, s_k + K_1 r_k^2)$. Then $s_k + r_k^2 \leq s + r^2,\ s_k - r_k^2 \leq s - r^2$, and

$$t \in (s - r^2, s) \subset J_k := (s_k - r_k^2, s_k + K_1 r_k^2).$$

Our argument shows that $E_x \subset \bigcup_k J_k$, hence

$$q_1 |E_x| \leq q_1 \left|\bigcup_k J_k\right| \leq q_1 \sum_k |J_k| = \sum_k \left|\hat{I}_k\right| = \left|\hat{E}_x\right|.$$

This completes the proof of lemma. □

THEOREM 5.3 (Third Growth Theorem). *Let a function* $u \in C^{2,1}\left(\overline{C_r}\right)$, *where* $C_r := C_r(Y),\ Y = (y,s) \in \mathbb{R}^{n+1},\ r > 0,$ *and let* $Lu \leq 0$ *in* C_r. *In addition, suppose*

$$\left|\{u > 0\} \cap C^0\right| \leq \mu \cdot \left|C^0\right| \tag{5.6}$$

where

$$C^0 := C_{r/2}\left(Y^0\right), \quad Y^0 := \left(y, s - \frac{3}{4}r^2\right), \quad \mu = const < 1.$$

Then $u \in \mathcal{M}(\beta, Y, r)$ *with a constant* $\beta = \beta(n, \nu, \mu) < 1$.

PROOF. For certainty, we assume $Y = (y,s) = 0$ and $r = 2$. As before, the general case is reduced to this one by a linear transformation. Note that $C^0 := B_1(0) \times (-4, -3)$, hence $\left|C^0\right| = |B_1|$ depends only on n.

Consider the set $\Gamma := \{u \leq 0\} \cap C^0$. From (5.6) it follows

$$|\Gamma| \geq (1 - \mu)\left|C^0\right| = (1 - \mu)|B_1| =: c_0 = c_0(n, \mu) > 0. \tag{5.7}$$

In Theorem 3.3, the constant $\beta_1 = \beta_1(n, \nu, \mu) \to 0^+$ as $\mu \to 0^+$. Fix a constant $\mu_0 = \mu_0(n, v) \in (0,1)$ such that the corresponding value $\beta_1(n, \nu, \mu_0) \leq \frac{1}{2}$. For this constant μ_0 and the given set Γ, consider the family of cylinders $\mathcal{A}$ defined by the formula (5.1). By Lemma 5.1, the set $E := \cup\{C : C \in A\}$ has measure

$$|E| \geq q_0 |\Gamma|, \quad \text{where} \quad q_0 := 1 + 3^{-n-1}\mu_0 > 1.$$

Denote $\varepsilon_0 := 3^{-n-2}\mu_0$, so that $q_0 = 1 + 3\varepsilon_0$. Next, choose the constant $K_1 > 0$ satisfying the equality

$$q_1 := \frac{K_1 - 1}{K_1 + 1} = \frac{1 + 2\varepsilon_0}{1 + 3\varepsilon_0}, \quad \text{i.e.} \quad K_1 = K_1(n, \nu) := 5 + 2\varepsilon_0^{-1}.$$

Combining Lemmas 5.1 and 5.2, we find that the corresponding set $\hat{E}$ has measure

$$\left|\hat{E}\right| \geq q_1 |E| \geq q_0 q_1 |\Gamma| = (1 + 2\varepsilon_0)|\Gamma|. \tag{5.8}$$

Further, choose another open cylinder C^1 such that

$$(5.9)\quad C^0 \subset C^1, \quad \left|C^1 \setminus C^0\right| \le \varepsilon_0 c_0, \quad \text{and} \quad dist\left(C^0, \partial C^1\right) \ge c_1 = c_1(n,\nu,\mu) > 0.$$

Consider separately cases (a) $\hat{E} \setminus C^1 \ne \emptyset$ and (b) $\hat{E} \setminus C^1 = \emptyset$.

(a) $\hat{E} \setminus C^1 \ne \emptyset$. This can be true only if some cylinders $C \in \mathcal{A}$ are large enough: there exists $C = C_r(Y) \in \mathcal{A}$ with $r \ge r_0 = r_0(n,\nu,\mu) > 0$. Note that

$$|\{u > 0\} \cap C| \le |C \setminus \Gamma| \le \mu_0 |C| \quad \text{for} \quad C \in \mathcal{A}.$$

By Theorem 3.3 and the choice of μ_0,

$$(5.10)\qquad \sup_{C_{\frac{r}{2}}(Y)} u \le \frac{1}{2} \sup_{C_r(Y)} u_+ \le \frac{1}{2} M \quad \text{for} \quad C = C_r(Y) \in \mathcal{A},$$

where $M := \sup_{C_2} u_+$, hence

$$u - \frac{1}{2} M \le 0 \quad \text{on} \quad D := B_{r/2}(y) \times \{s\} \quad \text{for} \quad C_r(Y) \in \mathcal{A},\ Y = (y,s).$$

In our case, we can apply Theorem 4.2 to the function $u - \frac{1}{2}M$ with $\rho = \frac{1}{2} r_0$. By this theorem, $u - \frac{1}{2}M \in \mathcal{M}(\beta_2, 0, 2)$ with $\beta_2 < 1$ depending only on n, ν, μ. Then

$$\begin{aligned} \sup_{C_1} u &= \frac{1}{2} M + \sup_{C_1}(u - \frac{1}{2} M) \\ &\le \frac{1}{2} M + \beta_2 \sup_{C_2}(u - \frac{1}{2} M)_+ = \frac{1}{2}(1+\beta_2) M, \end{aligned}$$

and $u \in \mathcal{M}(\beta_0, 0, 2)$ with $\beta_0 := \frac{1}{2}(1+\beta_2) < 1$.

(b) $\hat{E} \setminus C^1 = \emptyset$. In this case $\hat{E} \subset C^1$, and by the estimates (5.7)–(5.9), the set $\Gamma_1 := \hat{E} \cap C^0$ has measure

$$(5.11)\quad |\Gamma_1| = \left|\hat{E} \cap C^0\right| \ge \left|\hat{E}\right| - \left|C^1 \setminus C^0\right| \ge (1 + 2\varepsilon_0) |\Gamma| - \varepsilon_0 c_0 \ge (1+\varepsilon_0) |\Gamma|.$$

The previous argument in (a) shows that for arbitrary $C \in \mathcal{A}$, from (5.10) it follows $u \le \beta_0 M$ on $\hat{C}$ with $\beta_0 = \beta_0(n,\nu,\mu) < 1$. Since such sets $\hat{C}$ cover Γ_1, we have $u \le \beta_0 M$ on Γ_1. Combining this estimate with (5.11), we obtain

$$(5.12)\qquad \left|\{u \le \beta_0 M\} \cap C^0\right| \ge (1+\varepsilon_0) \left|\{u \le 0\} \cap C^0\right|.$$

We have proved that either (a) $u \le \beta_0 M$ on C_1, or (b) u satisfies (5.12). Starting from $u_0 := u, M_0 := M$, define

$$u_{k+1} := u_k - \beta_0 M_k, \quad M_{k+1} := \sup_{C_2} u_{k+1} \quad \text{for} \quad k = 0,1,2,\cdots.$$

It is easy to see that

$$u_k = u - \left[1 - (1-\beta_0)^k\right] M, \quad M_k = (1-\beta_0)^k M \qquad \text{for all} \quad k.$$

If the alternative (b) holds for all u_k with $k = 0,1,2,\cdots, m-1$, then for such k

$$\begin{aligned} \left|C^0\right| &\ge \left|\{u_m \le 0\} \cap C^0\right| = \left|\{u_{m-1} \le \beta_0 M_{m-1}\}\right| \ge (1+\varepsilon_0) \left|\{u_{m-1} \le 0\} \cap C^0\right| \\ &\ge \cdots \ge (1+\varepsilon_0)^m \left|\{u_0 \le 0\} \cap C^0\right| \ge (1+\varepsilon_0)^m c_0. \end{aligned}$$

If we take m such that $(1+\varepsilon_0)^m c_0 > \left|C^0\right|$, then (b) fails for some u_k with $k \le m-1$. Therefore, $u_m \le u_{k+1} \le 0$, and

$$u \le \left[1 - (1-\beta_0)^m\right] M \quad \text{on} \quad C_1.$$

The last inequality exactly means

$$u \in \mathcal{A}(\beta, 0, 2) \quad \text{with} \quad \beta := 1 - (1 - \beta_0)^m < 1.$$

□

Corollary 5.4. *Let a function* $v \in C^{2,1}\left(\overline{C_r}\right)$ *satisfy* $v \geq 0,\ Lv \geq 0$ *in* C_r, *and in addition,* $\left|\{v \geq 1\} \cap C^0\right| \geq (1-\mu) \cdot \left|C^0\right|$. *Then* $v \geq 1 - \beta > 0$ *on* $C_{r/2}$. *Here* $\beta = \beta(n, \nu, \mu) < 1$ *for* $\mu < 1$.

Proof. We assume $r = 2$. The function $u := 1 - v$ satisfies $Lv \leq 0$ in C_2, and

$$\left|\{u > 0\} \cap C^0\right| = \left|\{v < 1\} \cap C^0\right| = |C_0| - \left|\{v \geq 1\} \cap C^0\right| \\ \leq |C_0| - (1-\mu) \cdot \left|C^0\right| = \mu \cdot \left|C^0\right|.$$

By the previous theorem,

$$\sup_{C_1} u \leq \beta \sup_{C_2} u_+ \leq \beta, \quad \text{and} \quad \inf_{C_1} v = 1 - \sup_{C_1} u \geq 1 - \beta.$$

□

6. Proof of Theorem 1.5. Some generalizations and applications

Proof. We may assume $Y = 0, r = 1$, so that $C_1 = B_1(0) \times (-1, 0)$, $C^0 = B_1(0) \times (-3, -2)$. Obviously,

$$d(X) := \sup\{\rho > 0 :\ C_\rho(X) \subset C_2(0)\} \geq 1 \quad \text{on} \quad C^0.$$

Therefore,

$$\sup_{C^0} u \leq M := \sup_{C^1} d^\gamma u, \quad \text{where} \quad C^1 := B_2(0) \times (-3, -2),$$

and $\gamma > 0$ is the constant in Lemma 4.3 corresponding to $h = \frac{1}{2}$. Following the proof of Theorem 3.4, we have

$$M = d^\gamma u(X_0) \quad \text{for some} \quad X_0 = (x_0, t_0) \in \overline{C^1} \setminus \partial_p C_1,$$

and then for

$$\rho := \frac{1}{4} d(X_0) \in \left(0, \frac{1}{2}\right], \quad C_0 := C_\rho(X_0), \quad Q_0 := \left\{u > \frac{1}{2} u(X_0)\right\} \cap C_0,$$

we have $|Q_0| > \mu_1 |C_0|$ with a constant $\mu_1 = \mu_1(n, \nu) > 0$. Now we can apply Corollary 5.4 with

$$v = \frac{2}{u(X_0)} u, \quad C_r = C_{2\rho}(Y_0), \quad Y_0 := \left(x_0, t_0 + 3\rho^2\right), \quad C^0 = C_0, \quad 1 - \mu = \mu_1.$$

As a result, we get the estimate

$$u \geq \beta u(X_0) \quad \text{on} \quad C_\rho(Y_0) \quad \text{with} \quad \beta = \beta(n, \nu) > 0.$$

Further, we use Lemma 4.3 with

$$v = u, \quad r = 2, \quad D_\rho = B_\rho(x_0) \times \left\{t_0 + 2\rho^2\right\} \subset \overline{C_\rho(Y_0)},$$

and $D^0 = B_1(0) \times \{\tau\}$ with an arbitrary $\tau \in (-1, 0)$. This yields

$$\beta u(X_0) \leq \inf_{D_\rho} u \leq \left(\frac{4}{\rho}\right)^\gamma \inf_{C_1(0)} u.$$

Finally,

$$\sup_{C^0} u \leq M = d^\gamma u(X_0) = (4\rho)^\gamma u(X_0) \leq \beta^{-1} 4^{2\gamma} \inf_{C_1(0)} u.$$

This proves the theorem with $N = N(n, \nu) = \beta^{-1} 4^{2\gamma}$. $\square$

Remark 6.1. The main statements, such as the Harnack inequality and growth theorems, are naturally extended to more general classes, which are different the cases (D) and (ND). These are classes obtained by closure of the set $C^{2,1}$ (or C^∞) with respect to the integral norms discussed in Section 2. We demonstrate this approach in the case (ND) for the proof of the Harnack inequality. We assume

$$u \geq 0, \quad Lu := u_t - \sum_{i,j} a_{ij} D_{ij} u = 0 \quad \text{in} \quad C_{2r},$$

where the coefficients a_{ij} are measurable functions, and u belongs to the Sobolev space $W^{2,1}_{n+1}(C_{2r}) \cap C\left(\overline{C_{2r}}\right)$, i.e. $u_t, D_{ij} u \in L^{n+1}$. One can approximate a_{ij} and u by smooth functions $a^\varepsilon_{ij} \to a_{ij}$ (a. e.) and $u^\varepsilon \to u$ (in $W^{2,1}_{n+1}$) as $\varepsilon \to 0^+$. Then

$$f^\varepsilon := L^\varepsilon u^\varepsilon := u^\varepsilon_t - \sum_{i,j} a^\varepsilon_{ij} D_{ij} u^\varepsilon \to 0 \quad (\text{in } L^{n+1}) \text{ as } \varepsilon \to 0^+.$$

Relying on the existence results for equations with smooth coefficients, we can write $u^\varepsilon = v^\varepsilon + w^\varepsilon$, where

$$\begin{aligned} L^\varepsilon v^\varepsilon &= 0 \quad \text{in} \quad C_{2r}, \qquad v^\varepsilon = u^\varepsilon \quad \text{on} \quad \partial_p C_{2r}; \\ L^\varepsilon w^\varepsilon &= f^\varepsilon \quad \text{in} \quad C_{2r}, \qquad w^\varepsilon = 0 \quad \text{on} \quad \partial_p C_{2r}. \end{aligned}$$

By Theorem 2.4, $v^\varepsilon \to 0$ in L^∞, while v^ε satisfies the Harnack inequality (1.2). By easy limit passage, (1.2) also holds for the original function u.

Remark 6.2. In the divergence case, we do not assume that $a_{ij} = a_{ji}$. This makes it easy to extend the results to more general equations

$$L'u = a_0 u_t - \sum_{i,j=1}^n D_i(a_{ij} D_j u) - \sum_{j=1}^n b_j D_j u - \sum_{i=1}^n D_i(c_i u) = 0, \tag{6.1}$$

which include the lower order terms with bounded coefficients b_j, c_i. We set

$$a^0_{00} = A = \mathit{const} > 0, \quad a^0_{i0} = c_i, \quad a^0_{0j} = b_j, \quad a^0_{ij} = a_{ij}, \quad \text{for} \quad i, j = 1, \cdots, n.$$

Then the $(n+1) \times (n+1)$-matrix $\left[a^0_{ij}\right]^n_{i,j=0}$ satisfies the condition the uniform parabolicity condition (U) with a smaller constant $\nu > 0$, provided $A > 0$ is large enough. Introduce a new variable $x_0 \in \mathbb{R}^1$. Then the function

$$u^0(x_0, x, t) = e^{x_0 + At} u(x, t),$$

satisfies the parabolic equation without lower order terms:

$$L^0 u^0 = a_0 u^0_t - \sum_{i,j=0}^n D_i(a^0_{ij} D_j u^0) = Au^0 + e^{x_0 + At} L'u - Au^0 = 0.$$

Thus the Harnack inequality and some other results are easily extended to more general equations (6.1).

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