APPROXIMATION

$$\frac{\partial v}{\partial t} = \Delta_p v$$

$$0 \le v(x,t) \le L, \qquad (x,t) \in \Omega$$

$$v_\varepsilon(x,t) = \inf_{(y,t)\in\Omega} \left\{ \frac{|x-y|^2 + (t-\tau)^2}{2\varepsilon} + v(y,\tau) \right\}$$

- $v_\varepsilon(x,t) \nearrow v(x,t)$
- $v_\varepsilon(x,t) - \dfrac{|x|^2 + t^2}{2\varepsilon}$ loc. concave
- $\dfrac{\partial v_\varepsilon}{\partial t}$ and ∇v_ε exist and belong to $L^\infty_{loc}(\Omega)$

PROPOSITION The approximant v_ε is a viscosity supersolution in
$$\Omega_\varepsilon = \{(x,t) \mid \operatorname{dist}(x,t) > \sqrt{2L\varepsilon}\}.$$

LEMMA The approximant v_ε is a weak supersolution in Ω_ε. That is

$$\iint\left(-v_\varepsilon\frac{\partial\varphi}{\partial t}+\langle|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon,\nabla\varphi\rangle\right)dx\,dt$$

$$\geq 0,\quad \varphi\in C_0^\infty(\Omega_\varepsilon),\quad \varphi\geq 0.$$

(OBSTACLE PROBLEM NEEDED)

LEMMA (CACCIOPPOLI)

$$\iint_\Omega \zeta^p|\nabla v_\varepsilon|^p dx\,dt \leq C L^p\iint_\Omega|\nabla\zeta|^p dx\,dt + C L^2\iint_\Omega\left|\frac{\partial\zeta^p}{\partial t}\right|dx\,dt$$

$\zeta\geq 0$, $\zeta\in C_0^\infty(\Omega_\varepsilon)$

<u>Proof:</u>

$$\varphi(x,t)=\left(L-v_\varepsilon(x,t)\right)\zeta^p(x,t)\qquad\square$$

CONCLUSION

$\nabla v \in L^p_{loc}(\Omega)$ exists $\quad (0 \le v \le L)$

$\nabla v_\varepsilon \longrightarrow \nabla v$ weakly in $L^p_{loc}(\Omega)$

No good bound on $\frac{\partial v_\varepsilon}{\partial t}$

$\underline{Ex}$: $\quad v(x,t) = \begin{cases} 1, & \text{when } t > 0, \\ 0, & \text{when } t \le 0. \end{cases}$

$$\iint \left(-v_\varepsilon \frac{\partial \varphi}{\partial t} + \langle |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon, \nabla \varphi \rangle \right) \ge 0$$

¿ $\varepsilon \longrightarrow 0$? Passage to the limit ¿

$$\left| |b|^{p-2} b - |a|^{p-2} a \right|$$

$$\le (p-1) |b-a| \left(|b| + |a| \right)^{p-2}$$

+ Hölder

ENOUGH WITH

NOT p

$$\nabla v_\varepsilon \longrightarrow \nabla v \text{ in } L^{p-1}_{loc}(\Omega)$$

strongly.

THM Suppose that $v_1, v_2, v_3, \dots$ is a sequence of Lipschitz cont. weak supersolutions such that

$$0 \le v_k \le L \quad \text{in } Q_T = Q \times (0,T),$$

$$v_k \longrightarrow v \quad \text{in } L^p(Q_T).$$

Then $\nabla v_1, \nabla v_2, \nabla v_3, \dots$ is a Cauchy sequence in $L^{p-1}_{loc}(Q_T)$.

<u>Proof</u> $\delta > 0$

TCHEBYCHEV

$$\text{mes}\{|v_j - v_k| > \delta\} \le \delta^{-p} \|v_j - v_k\|_p^p$$

$$\theta \in C_0^\infty(Q_T), \quad 0 \le \theta \le 1$$

$$\iint_{\theta = 1} |\nabla v_k|^p \, dx\,dt \le A^p \quad (k = 1, 2, \dots)$$

Caccioppoli

$v_k = v_{\varepsilon_k}$

Fix k, j.

$$\iint \left(\langle |\nabla v_j|^{p-2} \nabla v_j, \nabla \varphi \rangle - v_j \frac{\partial \varphi}{\partial t} \right) dx\,dt \geq 0$$

$$\varphi = (\delta - w_{jk})\,\Theta$$

$$w_{jk} = \begin{cases} \delta, & v_j - v_k > \delta \\ v_j - v_k, & |v_j - v_k| \leq \delta \\ -\delta, & v_j - v_k < -\delta \end{cases}$$

$$|w_{jk}| \leq \delta, \quad \varphi \geq 0$$

In the eqn for v_k, use

$$(\delta + w_{jk})\,\Theta$$

Add
~~Subtract~~ the equations and arrange the terms

$$\iint\limits_{|v_j - v_k| \le \delta} \Theta \left\langle |\nabla v_j|^{p-2}\nabla v_j - |\nabla v_k|^{p-2}\nabla v_k \,,\, \nabla v_j - \nabla v_k \right\rangle$$

$$\le \delta \int_0^T\!\!\int_Q \left\langle |\nabla v_j|^{p-2}\nabla v_j + |\nabla v_k|^{p-2}\nabla v_k \,,\, \nabla\Theta \right\rangle$$

$$- \int_0^T\!\!\int_Q w_{jk} \left\langle |\nabla v_j|^{p-2}\nabla v_j - |\nabla v_k|^{p-2}\nabla v_k \,,\, \nabla\Theta \right\rangle$$

$$+ \int_0^T\!\!\int_Q (v_j - v_k)\frac{\partial}{\partial t}(\Theta w_{jk}) \quad \longleftarrow \text{TERM } \mathrm{III}$$

$$- \delta \int_0^T\!\!\int_Q (v_j + v_k)\frac{\partial \Theta}{\partial t} \qquad = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$$

$$\mathrm{III} = \underbrace{\iint \Theta \frac{\partial}{\partial t}\left(\frac{w_{jk}^2}{2}\right)}_{-\frac{1}{2}\iint w_{jk}^2 \frac{\partial \Theta}{\partial t}} + \iint (v_j - v_k) w_{jk} \frac{\partial \Theta}{\partial t}$$

$$\overline{\underline{\mathrm{III}}} \leq \frac{1}{2}\delta^2 \|\theta_t\|_1 + 2L\delta\|\theta_t\|_1$$

$$\leq \delta C_3$$

$$\overline{\underline{\mathrm{IV}}} \leq 2\delta L\|\theta_t\|_1 = \delta C_4$$

The terms I, II are easy

$$\mathrm{I} \leq \delta C_1 \quad , \quad \mathrm{II} \leq \delta C_2 \, .$$

$$\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} \leq C\delta$$

$$\iint_{|v_j - v_k| \leq \delta} \theta |\nabla v_j - \nabla v_k|^p \leq 2^{p-2}\delta C$$

$$\iint_{|v_j - v_k| \leq \delta} \theta |\nabla v_j - \nabla v_k|^{p-1} = \mathcal{O}(\delta^{1-\frac{1}{p}})$$

$$\iint_{|v_j - v_k| > \delta} \Theta \, |\nabla v_j - \nabla v_k|^{p-1}$$

$$\leq \delta^{-1} \|v_j - v_k\|_p \left(\|\nabla v_j\|_p + \|\nabla v_k\|_p \right)^{p-1}$$

$$\leq (2A)^{p-1} \delta^{-1} \|v_j - v_k\|_p$$

FINALLY,

$$\int_0^T \int_Q \Theta \, |\nabla v_j - \nabla v_k|^{p-1}$$ Independent of δ!

$$\leq O(\delta^{1-\frac{1}{r}}) + C_5 \, \delta^{-1} \underbrace{\|v_j - v_k\|_p}_{\longrightarrow 0}$$

THM Let v be a bounded viscosity supersolution of $v_t = \Delta_p v$ in Ω, $p \geq 2$. Then

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n}\right)$$

exists in Sobolev's sence, $\nabla v \in L^p_{loc}(\Omega)$, and

$$\iint \left(-v\varphi_t + \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle\right) dx\,dt$$

$$\geq 0.$$

SOBOLEV'S INEQ

$u \in L^p(0,T; W_0^{1,p}(Q))$

$$\int_0^T \int_Q |u|^{p(1+\frac{2}{n})} \, dx\,dt$$

$$\le C \int_0^T \int_Q |\nabla u|^p \, dx\,dt \left\{ \operatorname*{ess\,sup}_{0<t<T} \int_Q |u(x,t)|^2 dx \right\}^{\frac{p}{n}}$$

$$Q \times (t_1, t_2)$$

$$\varphi = 0 \quad \text{on} \quad \partial Q \times [t_1, t_2]$$

$$\int_{t_1}^{t_2} \int_Q \left(\langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle - v \frac{\partial \varphi}{\partial t} \right) dx\, dt$$

$$+ \int_Q v(x, t_2)\varphi(x, t_2)\, dx \geq \int_Q v(x, t_1)\varphi(x, t_1)\, dx$$

If v is zero on the lateral boundary, take $\varphi = v$ (!)

$$\boxed{\frac{1}{2} \underset{\text{PAST}}{\int_Q v(x, t_1)^2 dx} \leq \int_{t_1}^{t_2} \int_Q |\nabla v|^p dx\, dt + \frac{1}{2} \underset{\text{FUTURE}}{\int_Q v(x, t_2)^2 dx}}$$

START FROM

$$\int_0^{\cancel{\tau}\, t_1}\!\!\int_Q |\nabla v_j|^p \, dx\, dt \le j \int_0^T\!\!\int_Q |\nabla v_1|^p \, dx\, dt$$

$$+\; j \int_Q v_j(x,\tau)\, dx$$

THIS WAS PROVED

$0 \le \underline{t_1} < \tau < T$ Integrate with respect to τ over $[t_1, T]$

$$(T-t_1)\int_0^{t_1}\!\!\int_Q |\nabla v_j|^p \le j\,(T-t_1)\,K$$

$$+\; \underbrace{j \int_{t_1}^T\!\!\int_Q v_j(x,t)\, dx\, dt}_{\le\, j^{2-\varepsilon} \int_{t_1}^T\!\int_Q v^{\varepsilon}}$$

$$\int_0^{t_1}\!\!\int_Q |\nabla v_j|^p \le j^{2-\varepsilon} K \qquad (t_1 < T)$$

Estimating again the measures $|E_j|$, but starting with the bound

$$j^{\gamma} K \qquad (\gamma = 2 - \varepsilon)$$

in place of $j^2 K$, yields

$$|E_j| \lesssim j^{\gamma - p}$$

$$= j^{2-p-\varepsilon} \qquad 0 < \varepsilon \leq 1$$

RESULT

$$\int_0^{t_2} \int_Q v^q \, dx\, dt < \infty, \text{ when } \quad 0 < q < p - \gamma$$

$$\begin{array}{ll} q_1 = \varepsilon & T \\ q_2 = p - 2 + \varepsilon & t_1 \\ q_3 = 2(p-2) + \varepsilon & t_2 \end{array}$$

CONTINUE TILL

$$k(p-2)+\varepsilon > p-1$$

NOW

$$v \in L^{p-1}(Q_{T'}).$$

IN PARTICULAR

$v \in L^1$

$$\frac{1}{2}\int_Q v_j^2(x,t)\,dx \le \int_0^\tau\int_Q |\nabla v_j|^p\,dx\,dt + \frac{1}{2}\int_Q v_j^2(x,\tau)\,dx$$

PAST

FUTURE

$$\operatorname*{ess\,sup}_{0<t<t_1}\int_Q v_j^2(x,t)\,dx \le 2\int_0^\tau\int_Q |\nabla v_j|^p\,dx\,dt + \int_Q v_j(x,\tau)\,dx$$

$$\le 2j\int_0^\tau\int_Q |\nabla v_1|^p + 2j\int_Q v_j(x,\tau)\,dx + \int_Q v_j(x,\tau)\,dx$$

$$\operatorname*{ess\,sup}_{0<t<t_1} \int_Q v_j^2(x,t)\,dx \qquad (t_1 < \tau < T)$$

$$\leq 2j \int_0^T \int |\nabla v_1|^p\,dx\,dt + \underbrace{3j \int_Q v_j(x,\tau)\,dx}_{*}$$

Integrate τ over $[t_1, T]$.

Then $(*)$ becomes : $\dfrac{3j}{T-t_1} \displaystyle\int_{t_1}^{T} \int_Q v_j\,dx\,dt$

We had

$$\int_0^{t_1} \int |\nabla v_j|^p \leq j \int\int |\nabla v_1|^p \qquad \underline{\varepsilon = 1}$$

$$+ \frac{j^{2-\varepsilon}}{T-t_1} \int_{t_1}^{T} \int_Q v^\varepsilon\,dx\,dt .$$

$$\boxed{\int_0^{t_1} \int_Q |\nabla v_j|^p + \operatorname*{ess\,sup}_{0<t<t_1} \int_Q v_j^2(x,t)\,dx \leq 3j \int_0^T \int |\nabla v_1|^p + \frac{4j}{T-t_1} \int_{t_1}^{T} \int_Q v} \approx K_j$$

LEMMA If

$$\int_0^T\int_Q |\nabla v_j|^p + \operatorname*{ess\,sup}_{0<t<T}\int v_j^2\,dx \le jK$$

$$j = 1, 2, 3, \dots,$$

then

$$v \in L^q(Q_T) \quad \text{whenever} \quad 0 < q < p - 1 + \frac{p}{n}$$

$$\nabla v \in L^q(Q_T) \quad \text{— " —} \quad 0 < q < p - 1 + \frac{1}{n+1}$$

<u>Proof:</u> $E_j = \{(x,t) \mid j \le v(x,t) < 2j\}$

$$\varkappa = 1 + \frac{2}{n}$$

$$j^{\varkappa p}|E_j| \le \iint_{E_j} v_{2j}^{\varkappa p} \le \iint_{Q_T} v_{2j}^{\varkappa p}$$

$$\le C \int_0^T\int_Q |\nabla v_{2j}|^p \left(\operatorname*{ess\,sup}_{0<t<T}\int_Q v_{2j}^2\right)^{p/n}$$

$$\le C K^{1+\frac{p}{n}} j^{1+\frac{p}{n}}$$

$$|E_j| \lesssim j^{1-p-\frac{p}{n}}$$

$$\int_0^T \int_Q v^q \, dx\, dt \le T|Q| + \sum_{j=1}^{\infty} \iint_{E_{2^{j-1}}} v^q \, dx\, dt$$

$$\le T|Q| + \sum_{j=1}^{\infty} 2^{jq} |E_{2^{j-1}}|$$

$$\sim \sum_{j=1}^{\infty} 2^{j(q+1-p-\frac{p}{n})}$$

converges in the designed range

$$\iint |\nabla v_k|^q = \iint_{E_0} |\nabla v|^q + \sum_{j=1}^{\infty} \iint_{E_{2^{j-1}}} |\nabla v_k|^q$$

$$\lesssim \sum \left(\iint_{E_{2^{j-1}}} |\nabla v_k|^p \right)^{\frac{q}{p}} |E_{2^{j-1}}|^{1-q/p}$$

$$\le \sum 2^{(j-1)(1-\frac{q}{p})(1-p-\frac{p}{n})} \Bigg(\overbrace{\iint_{E_{2^{j-1}}} |\nabla v_{2^j}|^p}^{2^j K} \Bigg)^{\frac{q}{p}}$$