Introduction to random Tug-of-War games and PDEs

Juan J. Manfredi

University of Pittsburgh

manfredi@pitt.edu
www.pitt.edu/~manfredi

2009 CIME course

p-harmonious functions

Joint work with
Mikko Parviainen and
Julio D.Rossi.

Definition, $2 \le p < \infty$ ($p = \infty$ Le Gruyer)

Let Ω be a (bounded) domain in $\mathbb{R}^N$ and consider

$$\Gamma_\epsilon = \{x \in \mathbb{R}^N \setminus \Omega \,:\, \mathrm{dist}(x, \partial\Omega) \le \epsilon\}, \qquad \Omega_\varepsilon = \Omega \cup \Gamma_\varepsilon$$

The function u_ε is p-harmonious in Ω with continuous boundary values $F : \Gamma_\varepsilon \to \mathbb{R}$ if $u_\varepsilon(x) = F(x), \;\; x \in \Gamma_\varepsilon$ and

$$u_\varepsilon(x) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon \, dy \quad \text{for every } x \in \Omega,$$

where

$$\alpha = \frac{p-2}{p+N}, \quad \text{and} \quad \beta = \frac{2+N}{p+N}.$$

WARNING! Solutions to the DPP equation may be discontinuous as 1-d examples show.

Tug-of-War Games with Noise $2 \leq p < \infty$

Fix $1 > \alpha \geq 0$, $\beta > 0$ such that $\alpha + \beta = 1$.
Fix $\varepsilon > 0$ and place a token at starting point $x_0 \in \Omega$. Move the token to the next state x_1 as follows:

- With probability α play tug-of war: a fair coin is tossed and the winner of the toss moves the token to any $x_1 \in \overline{B}_\varepsilon(x_0)$.
- With probability β the token moves according to a uniform probability density to a random point in the ball $\overline{B}_\varepsilon(x_0)$.

This procedure yields an infinite sequence of game states $x_0, x_1, \ldots$ where every x_k, except x_0, is a random variable.

Tug-of-War Games with Noise

- A run of the game is $\mathbf{x} = (x_0, x_1, \ldots, x_k, \ldots)$, where $\mathbf{x}(k) = x_k$.
- The game stops the first time it hits Γ_ε. Write

$$\tau(\mathbf{x}) = \min\{k : x_k \in \Gamma_\varepsilon\}.$$

  The random variable τ is a STOPPING TIME. We write

$$\mathbf{x}(\tau(\mathbf{x})) = x_\tau.$$

- $F : \Gamma_\varepsilon \to \mathbb{R}$ is a given (Lipschitz, bounded) *payoff function*. The game payoff is $F(\mathbf{x}) = F(x_\tau)$.
- Player I earns \$ $F(x_\tau)$ while Player II earns \$ $-F(x_\tau)$.

Tug-of-War Games with Noise

- The history of a game up to step k is the vector of the first $k+1$ game states $(x_0, x_1, \ldots, x_k)$. H_k denotes the set of possible sequences of length k. The set of all finite histories is denoted by

$$H = \bigcup_{k=0}^{\infty} H_k.$$

- We endow H_k and H with the natural product topologies and product of Borel σ-algebras.
- A strategy S is a measurable function defined on H that gives the next game position when is the player turn to move

$$S(x_0, x_1, \ldots, x_k) = x_{k+1} \in B_\varepsilon(x_k)$$

Tug-of-War Games with Noise

- Fix strategies S_I and S_{II} for players I and II respectively.
- Start the game at x_0.
- The probability measure $\mathbb{P}^{x_0}_{S_I,S_{II}}$ is defined on H by the transition probabilities

$$\pi_{S_I,S_{II}}(x_0,\ldots,x_k;A) = \frac{\alpha}{2}\left(\delta_{S_I(x_0,\ldots,x_k)}(A) + \delta_{S_{II}(x_0,\ldots,x_k)}(A)\right) + \beta\frac{\left|A\cap\overline{B}_\varepsilon(x_k)\right|}{\left|\overline{B}_\varepsilon(x_k)\right|}$$

and Kolmogorov's extension theorem as follows

Tug-of-War Games with Noise

$\mu^{k,x_0}_{S_I,S_{II}}$ is a probability measure in $H_k \subset \Omega_\varepsilon^{k+1}$.

- $\mu^{0,x_0}_{S_I,S_{II}}(A_0) = \delta_{x_0}(A_0)$
- Conditional probabilities

$$\mu^{k,x_0}_{S_I,S_{II}}(A_0 \times \ldots \times A_{k-1} \times B) = \int_{A_0\times\ldots\times A_{k-1}} \pi_{S_I,S_{II}}(x_0,\ldots,x_{k-1},B)\, d\mu^{k-1,x_0}_{S_I,S_{II}}$$

- Marginals

$$\mu^{k,x_0}_{S_I,S_{II}}(A_0 \times A_1 \times \ldots \times A_k) = \mu^{k+1,x_0}_{S_I,S_{II}}(A_0 \times A_1 \times \ldots \times A_k \times \Omega_\varepsilon).$$

$\mathbb{P}^{x_0}_{S_I,S_{II}} = \lim_{k\to\infty} \mu^{k,x_0}_{S_I,S_{II}}$ probability in $H \subset \Omega_\varepsilon^\infty$

Tug-of-War Games with Noise, $2 \leq p < \infty$

Games end almost surely

$\mathbb{P}^x_{S_I, S_{II}}(H) = 1$ because $\beta > 0$.

Value of the game for player I

$$u^\varepsilon_I(x) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^x_{S_I, S_{II}}[F(x_\tau)]$$

Value of the game for player II

$$u^\varepsilon_{II}(x) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^x_{S_I, S_{II}}[F(x_\tau)]$$

Comparison Principle

$$u^\varepsilon_I(x) \leq u^\varepsilon_{II}(x)$$

Tug-of-War Games with Noise, $2 \leq p < \infty$

Lemma

(recall $\overline{\Omega}$ bounded and F Lipschitz and bounded)

$$u_I^\varepsilon(x) > -\infty \text{ since } \mathbb{P}^x_{S_I, S_{II}}(H) = 1.$$

Existence of quasi-optimal strategies

Let $\delta > 0$ given. Then we can always find strategies S_I^x and S_{II}^x such that $\left| u_I^\varepsilon(x) - \mathbb{E}^x_{S_I^x, S_{II}^x}[F(x_\tau)] \right| < \delta$

DPP $\Longrightarrow$ existence of p-harmonious functions

THEOREM

The value functions u_I^ε and u_{II}^ε are p-harmonious. They satisfy the equation

$$u(x) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy, \qquad x \in \Omega,$$

$$u(x) = F(x), \qquad x \in \Gamma_\varepsilon.$$

(In this case were unable to show directly that the mapping

$$T(u) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy$$

has a fixed point.)

Comparison I

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

- If v_ε is a p-harmonious function with boundary values F_v in Γ_ε such that $F_v(y) \geq u_I^\varepsilon(y)$ for $y \in \Gamma_\varepsilon$, then $v_\varepsilon(x) \geq u_I^\varepsilon(x)$ for $x \in \Omega_\varepsilon$.
- If v_ε is a p-harmonious function with boundary values F_v in Γ_ε such that $F_v(y) \leq u_I^\varepsilon(y)$ for $y \in \Gamma_\varepsilon$, then $v_\varepsilon(x) \leq u_{II}^\varepsilon(x)$ for $x \in \Omega_\varepsilon$.

That is u_I^ε is the smallest p-harmonious function with given boundary values and u_{II}^ε is the largest p-harmonious function with given boundary values.

Comparison I, Proof

Player I arbitrary strategy S_I, player II strategy S_{II}^0 that almost minimizes v_ε, $v_\varepsilon(x_k) \le \inf_{y\in\overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y) + \eta 2^{-k}$

Key Point

$$M_k = v_\varepsilon(x_k) + \eta 2^{-k}$$

is a supermartingale for any $\eta > 0$.

$$\begin{aligned}\mathbb{E}^{x_0}_{S_I,S^0_{II}}[M_k] =& \mathbb{E}^{x_0}_{S_I,S^0_{II}}[v^\varepsilon(x_k) + \eta 2^{-k} \mid x_0,\dots,x_{k-1}] \\ &\le \frac{\alpha}{2}\left\{ \inf_{y\in\overline{B}_\varepsilon(x_{k-1})} v^\varepsilon(y) + \sup_{y\in\overline{B}_\varepsilon(x_{k-1})} v^\varepsilon(y) + \eta 2^{-k} \right\} \\ &+ \beta \fint_{B_\varepsilon(x_{k-1})} v^\varepsilon \, dy \le v^\varepsilon(x_{k-1}) + \eta 2^{-(k-1)} = M_{k-1}\end{aligned}$$

Comparison I, Proof

By optimal stopping

$$\begin{aligned}
u_I^\varepsilon(x_0) &= \sup_{S_I} \inf_{S_{II}} \mathbb{E}^{x}_{S_I,S_{II}}[F(x_\tau)] \\
&\le \sup_{S_I} \mathbb{E}^{x}_{S_I,S_{II}^0}[F(x_\tau)] \\
&\le \sup_{S_I} \mathbb{E}^{x}_{S_I,S_{II}^0}[v_\varepsilon(x_\tau)] \\
&\le \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}^0}[v_\varepsilon(x_\tau) + \eta 2^{-k}] \\
&\le \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}^0}[M_\tau] \\
&\le \sup_{S_I} M_0 = v^\varepsilon(x_0) + \eta
\end{aligned}$$

The game has a value

Theorem ($M_k = u_I^\varepsilon(x_k) + \eta 2^{-k}$ is a supermartingale)

We have $u_I^\varepsilon = u_{II}^\varepsilon$

Need to show that $u_{II}^\varepsilon \leq u_I^\varepsilon$. Player II follows a strategy S_{II}^0 such that at $x_{k-1} \in \Omega_\varepsilon$, he always chooses to step to a point that almost minimizes u_I^ε; that is, to a point x_k such that

$$u_I^\varepsilon(x_k) \leq \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} u_I^\varepsilon(y) + \eta 2^{-k}$$

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_I, S_{II}^0}[u_I^\varepsilon(x_k) + \eta 2^{-k} \mid x_0, \ldots, x_{k-1}] \\
&\leq \frac{\alpha}{2}\left\{ \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} u_I^\varepsilon(y) + \sup_{y \in \overline{B}_\varepsilon(x_{k-1})} u_I^\varepsilon(y) + \eta 2^{-k} \right\} \\
&+ \beta \fint_{B_\varepsilon(x_{k-1})} u_I^\varepsilon \, dy \leq u_I^\varepsilon(x_{k-1}) + \eta 2^{-(k-1)}.
\end{aligned}$$

The game has a value

By optimal stopping

$$\begin{aligned} u_{II}^{\varepsilon}(x_0) &= \inf_{S_{II}} \sup_{S_I} \mathbb{E}^{x}_{S_I,S_{II}}[F(x_\tau)] \\ &\leq \sup_{S_I} \mathbb{E}^{x}_{S_I,S_{II}^0}[F(x_\tau)] \\ &\leq \sup_{S_I} \mathbb{E}^{x}_{S_I,S_{II}^0}[u_I^{\varepsilon}(x_\tau)] \\ &\leq \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}^0}[u_I^{\varepsilon}(x_\tau) + \eta 2^{-k}] \\ &\leq \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}^0}[M_\tau] \\ &\leq \sup_{S_I} M_0 = u_I^{\varepsilon}(x_0) + \eta \end{aligned}$$

Maximum and Comparison Principles

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If u_ε is p-harmonious in Ω with a boundary data F, then $\sup_{\Gamma_\varepsilon} F \geq \sup_\Omega u_\varepsilon$. Moreover, if there is a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = \sup_{\Gamma_\varepsilon} F$, then u_ε is constant in Ω.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. and let u_ε and v_ε be p-harmonious with boundary data $F_u \geq F_v$ in Γ_ε. Then if there exists a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = v_\varepsilon(x_0)$, it follows that $u_\varepsilon = v_\varepsilon \quad \text{in} \quad \Omega$, and, moreover, the boundary values satisfy $F_u = F_v \quad \text{in} \quad \Gamma_\varepsilon$.

Proof of Strong Comparison

The proof uses the fact that $p < \infty$. The strong comparison principle **does not hold** for $p = \infty$.

$$F_u \geq F_v \implies u_\varepsilon \geq v_\varepsilon.$$

We have

$$u_\varepsilon(x_0) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon\right\} + \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy$$

and

$$v_\varepsilon(x_0) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon\right\} + \beta \fint_{B_\varepsilon(x_0)} v_\varepsilon \, dy.$$

Next we compare the right hand sides. Because $u_\varepsilon \geq v_\varepsilon$, it follows that

Proof of Strong Comparison, II

$$\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon - \sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon \geq 0,$$

$$\inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon - \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon \geq 0, \quad \text{and}$$

$$⨍_{B_\varepsilon(x_0)} u_\varepsilon \, dy - ⨍_{B_\varepsilon(x_0)} v_\varepsilon \, dy \geq 0$$

But since

$$u_\varepsilon(x_0) = v_\varepsilon(x_0),$$

and $\beta > 0$ must have $u_\varepsilon = v_\varepsilon$ almost everywhere in $B_\varepsilon(x_0)$. In particular,

$$F_u = F_v \quad \textbf{everywhere in} \quad \Gamma_\varepsilon$$

since F_u and F_v are continuous. By uniqueness $u_\varepsilon = v_\varepsilon$ everywhere in Ω.

Approximation of p-harmonic functions

Boundary Regularity Assumption

Ω bounded domain in $\mathbb{R}^n$ satisfying an exterior sphere condition: For each $y \in \partial\Omega$, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$. $R > 0$ is chosen so that we always have $\Omega \subset B_{R/2}(z)$.

MAIN THEOREM

F is Lipschitz in Γ_ε for small $0 < \varepsilon < \varepsilon_0$. Let u be the unique viscosity solution to

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u)(x) = 0, & x \in \Omega \\ u(x) = F(x), & x \in \partial\Omega, \end{cases}$$

and let u_ε be the unique p-harmonious function with boundary data F in Γ_ε, then $u_\varepsilon \to u$ uniformly in Ω as $\varepsilon \to 0$.

Approximation of p-harmonic functions, Proof I

Ascoli-Arzelá type theorem

Let $\{u_\varepsilon : u_\varepsilon : \overline{\Omega} \to \mathbb{R}, \ \varepsilon > 0\}$ be a set of functions such that

1. there exists $C > 0$ so that $|u_\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,
2. given $\eta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x, x' \in \overline{\Omega}$ with $|x - x'| < r_0$ it holds

$$|u_\varepsilon(x) - u_\varepsilon(x')| < \eta.$$

Then, there exists a sequence $\varepsilon_j \to 0$ and a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ such that

$$u_{\varepsilon_j} \to u$$

uniformly in $\overline{\Omega}$.

Approximation of p-harmonic functions, Proof I

Condition 1 is clear:

$$\min_{y\in\Gamma_\varepsilon} F(y) \le F(x_\tau) \le \max_{y\in\Gamma_\varepsilon} F(y) \implies \min_{y\in\Gamma_\varepsilon} F(y) \le u_\varepsilon(x) \le \max_{y\in\Gamma_\varepsilon} F(y).$$

Condition 2, KEY ESTIMATE

The p-harmonious function u_ε with the boundary data F satisfies

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le \operatorname{Lip}(F)\delta + C(R/\delta)(|x-y| + o(1)),$$

for every small enough $\delta > 0$ and for every two points $x, y \in \Omega \cup \Gamma_\varepsilon$. Here $C(R/\delta) \to \infty$ as $R/\delta \to \infty$. Furthermore the constant in $o(1)$ is uniform in x and y.

Sketch of the Proof of the Key Estimate

We have 3 cases: (i) both points belong to Γ_ε, (ii) one of the two points belongs to Γ_ε and the other to Ω, and (iii) both points belong to Ω

Case (i) is clear since F is Lipschitz

For $x, y \in \Gamma_\varepsilon$ we have

$$|F(x) - F(y)| \leq \mathrm{Lip}(F)|x - y|.$$

Case (ii): The more elaborate case

Suppose $x \in \Omega$ and $y \in \Gamma_\varepsilon$.

By the exterior sphere condition, there exists there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$. Player I chooses a strategy of pulling towards z, denoted by S_I^z, Player II an arbitrary strategy. **We write** $x = x_0$.

Sketch of the Proof of the Key Estimate

$$M_k = |x_k - z| - C\varepsilon^2 k$$

is a supermartingale for a constant C large enough independent of ε.

$$\begin{aligned}
\mathbb{E}^{x_0}_{S_I^z, S_{II}}[M_k] &= \mathbb{E}^{x_0}_{S_I^z, S_{II}}[|x_k - z| \mid x_0, \dots, x_{k-1}] \\
&\le \frac{\alpha}{2}\{|x_{k-1} - z| + \varepsilon + |x_{k-1} - z| - \varepsilon\} + \beta \fint_{B_\varepsilon(x_{k-1})} |x - z|\, dx \\
&\le |x_{k-1} - z| + C\varepsilon^2 = M_{k-1}
\end{aligned}$$

The first inequality follows from the choice of the strategy, and the second from the estimate

$$\fint_{B_\varepsilon(x_{k-1})} |x - z|\, dx \le |x_{k-1} - z| + C\varepsilon^2.$$

Sketch of the Proof of the Key Estimate

By the optional stopping theorem

$$\mathbb{E}^{x_0}_{S_I^z, S_{II}}[|x_\tau - z| - C\varepsilon^2\tau] \leq |x_0 - z|$$

$$\mathbb{E}^{x_0}_{S_I^z, S_{II}}[|x_\tau - z|] \leq |x_0 - z| + C\varepsilon^2\mathbb{E}^{x_0}_{S_I^z, S_{II}}[\tau]$$

We estimate $\mathbb{E}^{x_0}_{S_I^z, S_{II}}[\tau]$ by a stopping time $\tau^\star$ of a random walk in the ring $B_R(z) \setminus \overline{B}_\delta(z)$ starting at x_0. The successor of $x_k \in B_R(z) \setminus \overline{B}_\delta(z)$ is chosen with a uniform probability in $B_\varepsilon(x) \cap B_R(z)$ and

$$\tau^* = \inf\{k \,:\, x_k \in \overline{B}_\delta(z)\}.$$

Sketch of the Proof of the Key Estimate

Random Walk Exit Time Estimates

Consider a random walk on $B_R(y) \setminus \overline{B}_\delta(y)$ such that when at x_{k-1}, the next point x_k is chosen uniformly distributed in $B_\varepsilon(x_{k-1}) \cap B_R(y)$. For $\tau^\star = \inf\{k \,:\, x_k \in \overline{B}_\delta(y)\}$. we have

$$\mathbb{E}^{x_0}(\tau^\star) \leq \frac{C(R/\delta)\,\mathrm{dist}(\partial B_\delta(y), x_0) + o(1)}{\varepsilon^2},$$

for $x_0 \in B_R(y) \setminus \overline{B}_\delta(y)$. Here $C(R/\delta) \to \infty$ as $R/\delta \to \infty$.

This is surely known by experts in probability. We proved it by showing that $g(x) = \mathbb{E}^x(\tau^\star)$ can be estimated by the solution of a mixed Dirichlet-Newman problem in the ring $B_R(y) \setminus \overline{B}_\delta(y)$

Sketch of the Proof of the Key Estimate

$$\varepsilon^2 \mathbb{E}^{x_0}_{S_I^z, S_{II}}[\tau] \leq \varepsilon^2 \mathbb{E}[\tau^\star] \leq C(R/\delta)(\operatorname{dist}(\partial B_\delta(z), x_0) + o(1)).$$

Since $y \in \partial B_\delta(z)$,

$$\operatorname{dist}(\partial B_\delta(z), x_0) = |y - x_0|\,,$$

$$\mathbb{E}^{x_0}_{S_I^z, S_{II}}[|x_\tau - z|] \leq C(R/\delta)(|x_0 - y| + o(1)).$$

We get

$$\begin{aligned} u_\varepsilon(x_0) - u_\varepsilon(y) &= \sup_{S_I} \inf_{S_{II}} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_\tau)] - F(y) \\ &\geq \inf_{S_{II}} \mathbb{E}^{x_0}_{S_I^z, S_{II}}[F(x_\tau)] - F(y) + F(y) - F(z) \end{aligned}$$

Sketch of the Proof of the Key Estimate

For an arbitrary strategy S_{II} we have:

$$\begin{aligned}
\mathbb{E}^{x_0}_{S_I^z,S_{II}}&[F(x_\tau)] - F(y) + F(y) - F(z) \\
&\geq \mathbb{E}^{x_0}_{S_I^z,S_{II}}[F(x_\tau) - F(y)] - \operatorname{Lip}(F)|y-z| \\
&\geq -\mathbb{E}^{x_0}_{S_I^z,S_{II}}[|F(x_\tau) - F(y)|] - \operatorname{Lip}(F)\delta \\
&\geq -\operatorname{Lip}(F)\mathbb{E}^{x_0}_{S_I^z,S_{II}}[|x_\tau - y|] - \operatorname{Lip}(F)\delta \\
&\geq -C(R/\delta)(|x_0 - y| + o(1)) - \operatorname{Lip}(F)\delta
\end{aligned}$$

so that we finally get:

$$u_\varepsilon(x_0) - u_\varepsilon(y) \geq -C(R/\delta)(|x_0 - y| + o(1)) - \operatorname{Lip}(F)\delta$$

Sketch of the Proof of the Key Estimate

Case (iii) x, y both in Ω

Fix strategies, S_I, S_{II} for the game starting at x. Define a virtual game starting at y: use the same coin tosses and random steps as the usual game starting at x. Furthermore, the players adopt virtual strategies, S_I^v, S_{II}^v modeled on S_I and S_{II} from the game starting at x.

The first time $x_k \in \Gamma_\varepsilon$ or $y_k \in \Gamma_\varepsilon$ we have $|x_k - y_k| = |x - y|$, and we may apply the previous steps that work for $x_k \in \Omega$, $y_k \in \Gamma_\varepsilon$ or for $x_k, y_k \in \Gamma_\varepsilon$.

In particular, if we have $x_k \in \Omega$, $y_k \in \Gamma_\varepsilon$, one of the two players starts to point to the point in which the other game ended.

$$\mathbb{E}^x_{S_I,S_{II}}[F(x_\tau)] - \text{Lip}(F)\delta - (C(R/\delta)(|x_\tau - y_\tau| + o(1)) \leq \mathbb{E}^y_{S_I^v,S_{II}^v}[F(y_\tau)]$$

Sketch of the Proof of the Key Estimate

But since $|x_\tau - y_\tau| = |x - y|$

$$\mathbb{E}^x_{S_I, S_{II}}[F(x_\tau)] - \text{Lip}(F)\delta - (C(R/\delta)(|x-y| + o(1)) \leq \mathbb{E}^y_{S^y_I, S^y_{II}}[F(y_\tau)]$$

Since every pair of strategies S_I, S_{II} defines a unique pair of strategies S^y_I, S^y_{II} and vice versa we can take infs and sups to get

Finally!

$$u^\varepsilon(x) - \text{Lip}(F)\delta - (C(R/\delta)(|x-y| + o(1)) \leq u^\varepsilon(y)$$

Proof of DPP

Dynamic Programming Principle

u_I^ε and u_{II}^ε satisfy the equation

$$u(x) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u \right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy, \qquad x \in \Omega$$

Conditional probabilities

$$\begin{aligned}\mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)] &= \int_{H^\infty} F(x_\tau(\omega))\, d\mathbb{P}^{x_0}_{S_I,S_{II}}(\omega) \\ &= \int_{B_\varepsilon(x_0)} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)|x_0, x_1]\, d\pi_{S_I,S_{II}}(x_0, x_1)\end{aligned}$$

Proof of DPP

Density of Conditional Expectations

$$\mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)|x_0,x_1] = \int_{H_{x_1}} F(x_\tau)\, d\mathbb{P}^{x_0,x_1}_{S_I,S_{II}}$$

where

$$H_x = \{h \in H \setminus \{x_0\} \;:\; h(1) = x_0, h(2) = x\}$$

An important point

For $x \neq y$ we have $H_x \cap H_y = \emptyset$. Therefore the probabilities $\mathbb{P}^{x_0,x}_{S_I,S_{II}}$ and $\mathbb{P}^{x_0,y}_{S_I,S_{II}}$ have **DISJOINT** supports.

Proof of DPP

The starting point

$$\mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)] = \frac{\alpha}{2}\Big\{\mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)|x_0, S_I(x_0)] + \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)|x_0, S_{II}(x_0)]\Big\} + \beta \fint_{B_\varepsilon(x_0)} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)|x,y]\,dy$$

Proof of DPP

Step 1: We classify the strategies depending on the first step:

$$S_I(h) = \begin{cases} S_I|_{\{x_0\}}(h), & \text{if} \quad h = \{x_0\}, \\ S_I|_{H\setminus\{x_0\}}(h), & \text{if} \quad h \in H \setminus \{x_0\}, \end{cases}$$

where H denotes a set of all histories, and $\{x_0\}$ is a history consisting only of the starting point. Here $S_I|_{\{x_0\}}(h)$ simply stands for $S_I(x_0)$, but for consistency, we use the first notation.

$$\begin{aligned} u_I^\varepsilon(x_0) &= \sup_{S_I} \inf_{S_{II}} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)] \\ &= \sup_{S_I|\{x_0\}} \sup_{S_I|_{H\setminus\{x_0\}}} \inf_{S_{II}|_{\{x_0\}}} \inf_{S_{II}|_{H\setminus\{x_0\}}} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)]. \end{aligned}$$

Proof of DPP

$S_I|_{\{x_0\}}$ denotes any restricted strategy on $\{x_0\}$ (the first step from x_0) and $S_I|_{H\setminus\{x_0\}}$ any restricted strategy on $H \setminus \{x_0\}$ (the rest). Together these restricted strategies define a complete strategy S_I as suggested by the notation.

Step 2: We further classify strategies with respect to the second step. Set

$$H_x = \{h \in H \setminus \{x_0\} \,:\, h(1) = x_0, h(2) = x\},$$

and write

$$S_I|_{H\setminus\{x_0\}} = \bigcup_{x\in B_\varepsilon(x_0)} S_I|_{H_x}$$

Proof of DPP

$$\sup_{S_I|_{\{x_0\}}} \sup_{S_I|_{H\setminus\{x_0\}}} \inf_{S_{II}|_{\{x_0\}}} \inf_{S_{II}|_{H\setminus\{x_0\}}} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)]$$

$$= \sup_{S_I|_{\{x_0\}}} \inf_{S_{II}|_{\{x_0\}}} \sup_{S_I|_{H\setminus\{x_0\}}} \inf_{S_{II}|_{H\setminus\{x_0\}}} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)],$$

Player I can optimize his strategy on each disjoint set H_x without affecting the strategy on $H_y,\ y \neq x$. Thus we see that Player II gets no extra advantage by choosing his first step.

$$\inf_{S_{II}|_{H\setminus\{x_0\}}} \mathbb{E}_{S_I,S_{II}}[F(x_\tau)]$$

$$= \int_{\Omega_\varepsilon} \inf_{S_{II}|_{H_{x_1}}} \int_{\Omega_\varepsilon\times\ldots} F(x_\tau))\, d\mathbb{P}^{x_0,x_1}_{S_I,S_{II}}(x_2,\ldots)\, d\pi_{S_I,S_{II}}(x_0,x_1)$$

Proof of DPP

Step 3:

$$u_I^\varepsilon(x_0) = \sup_{S_I|_{\{x_0\}}} \inf_{S_{II}|_{\{x_0\}}}$$

$$\Bigg[\frac{\alpha}{2}\Bigg\{ \sup_{S_I|_{H_{S_I(x_0)}}} \inf_{S_{II}|_{H_{S_I(x_0)}}} \int_{\Omega_\varepsilon \times \ldots} F(x_\tau)\, d\mathbb{P}^{x_0, S_I(x_0)}_{S_I, S_{II}}(x_2, \ldots)$$

$$+ \sup_{S_I|_{H_{S_{II}(x_0)}}} \inf_{S_{II}|_{H_{S_{II}(x_0)}}} \int_{\Omega_\varepsilon \times \ldots} F(x_\tau)\, d\mathbb{P}^{x_0, S_{II}(x_0)}_{S_I, S_{II}}(x_2, \ldots)\Bigg\}$$

$$+ \beta \fint_{B_\varepsilon(x_0)} \sup_{S_I|_{H_{x_1}}} \inf_{S_{II}|_{H_{x_1}}} \int_{\Omega_\varepsilon \times \ldots} F(x_\tau)\, d\mathbb{P}^{x_0, x_1}_{S_I, S_{II}}(x_2, \ldots)\, dx_1\Bigg].$$

Proof of DPP

Step 4:

$$\sup_{S_I|_{H_{x_1}}} \inf_{S_{II}|_{H_{x_1}}} \int_{\Omega_\varepsilon \times \ldots} F(x_\tau)\, d\mathbb{P}^{x_0,x_1}_{S_I,S_{II}}(x_2,\ldots)$$

$$= \sup_{S_I|_{\{h\in H \mid h(1)=x_1\}}} \inf_{S_{II}|_{\{h\in H \mid h(1)=x_1\}}} \int_{H^\infty} F(x_\tau)\, d\mathbb{P}^{x_1}_{S_I,S_{II}}(x_1,\ldots)$$

$$= \sup_{S_I|_{\{h\in H \mid h(1)=x_1\}}} \inf_{S_{II}|_{\{h\in H \mid h(1)=x_1\}}} \mathbb{E}^{x_1}_{S_I,S_{II}}[F(x_\tau)]$$

$$= u_I^\varepsilon(x_1).$$

Proof of DPP

Step 5:

$$u_I^\varepsilon(x_0) = \sup_{S_I|_{\{x_0\}}} \inf_{S_{II}|_{\{x_0\}}} \left[\frac{\alpha}{2} \left\{ u_I^\varepsilon(S_I(x_0)) + u_I^\varepsilon(S_{II}(x_0)) \right\} \right.$$
$$\left. + \beta \fint_{B_\varepsilon(x_0)} u_I^\varepsilon(x_1)\, dx_1 \right].$$