Introduction to random Tug-of-War games and PDEs

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2009 CIME course

Asymptotic mean-value properties for p-harmonic functions.

Joint work with
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The case $p = 2$:

Let $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$. Consider the Taylor expansion:

$$u(x+h) = u(x) + \langle \nabla u(x), h \rangle + \frac{1}{2} \langle D^2 u(x) h, h \rangle + o(|h|^2), \text{ as } h \to 0.$$

Averaging on a ball $B_\epsilon(x) \subset \Omega$ we get:

$$\fint_{B_\epsilon(x)} u(x+h)\, dh = u(x) + \frac{1}{(N+2)} \epsilon^2 \Delta(u)(x) + o(\epsilon^2), \text{ as } \epsilon \to 0.$$

Lemma

$u \in C^2(\Omega)$ is harmonic in Ω if and only if for all $x \in \Omega$

$$\fint_{B_\epsilon(x)} u(x+h)\, dh = u(x) + o(\epsilon^2), \text{ as } \epsilon \to 0.$$

The case $p = 2$:

Since viscosity harmonic functions are harmonic in the classical sense, we indeed have:

Lemma

$u \in C(\Omega)$ is harmonic in Ω if and only if for all $x \in \Omega$

$$\fint_{B_\epsilon(x)} u(x+h)\, dh = u(x) + o(\epsilon^2), \text{ as } \epsilon \to 0$$

Similar characterizations hold for equations with variable coefficients $Lu(x) = \sum a_{ij}(x) u_{ij}(x) = 0$. (See Lanconelli's papers and recent book)

The case $p = \infty$, $\nabla u(x) \neq 0$

Let $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$. In the Taylor expansion, use

$$h = \epsilon \frac{\nabla u(x)}{|\nabla u(x)|} \quad \text{and} \quad h = -\epsilon \frac{\nabla u(x)}{|\nabla u(x)|},$$

add, and compute to get:

$$\frac{1}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) = u(x) + \epsilon^2 \Delta_\infty u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0,$$

where

$$\Delta_\infty u(x) = \frac{1}{|\nabla u(x)|^2} \langle D^2 u(x) \nabla u(x), \nabla u(x) \rangle$$

is the *homogeneous* ∞-Laplacian.

The case $p = \infty$, $\nabla u(x) \neq 0$

Lemma

$u \in C^2(\Omega)$, $\nabla u(x) \neq 0$, is ∞-harmonic in Ω if and only if for all $x \in \Omega$

$$\frac{1}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) = u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0.$$

Lemma

Let $u \in C(\Omega)$ be just continuous. Suppose that for all $x \in \Omega$ we have

$$\frac{1}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) = u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0,$$

then ∞-harmonic in Ω.

The case $p = \infty$, $\nabla u(x) \neq 0$

The converse to the previous lemma does not hold.

Example: Aronsson's function near $(x, y) = (1, 0)$

$$u(x, y) = |x|^{4/3} - |y|^{4/3}$$

Aronsson's function is ∞-harmonic in the viscosity sense but it is not of class C^2. A calculation shows that

$$\lim_{\varepsilon \to 0+} \frac{\frac{1}{2}\left\{ \max_{B_\varepsilon(1,0)} u + \min_{B_\varepsilon(1,0)} u \right\} - u(1,0)}{\varepsilon^2} = \frac{1}{18}.$$

But if an asymptotic expansion held in the classical sense, this limit would have to be zero.

The case $1 < p < \infty$, $\nabla u(x) \neq 0$

Let $u \in C^2(\Omega)$ and α, β non-negative such that $\alpha + \beta = 1$.

$$
\begin{aligned}
\frac{\alpha}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) + \beta \fint_{B_\epsilon(x)} u &= u(x) \\
&+ \alpha \Delta_\infty u(x) + \beta \frac{1}{(N+2)} \Delta u(x) \\
&+ o(\epsilon^2), \qquad \text{as } \epsilon \to 0,
\end{aligned}
$$

Let us rewrite the second order operator

$$
\alpha \Delta_\infty u(x) + \beta \frac{1}{(N+2)} \Delta u(x) = \beta \frac{1}{(N+2)} \left(\Delta u(x) + \frac{\alpha}{\beta \frac{1}{(N+2)}} \Delta_\infty u(x) \right).
$$

The case $1 < p < \infty$, $\nabla u(x) \neq 0$

Next, choose $2 < p < \infty$ such that

$$p - 2 = \frac{\alpha}{\beta \frac{1}{(N+2)}}.$$

We then have

$$\Delta u(x) + \frac{\alpha}{\beta \frac{1}{(N+2)}} \Delta_\infty u(x) = |\nabla u(x)|^{2-p} \mathrm{div} \left(|\nabla u(x)|^{p-2} \nabla u(x) \right).$$

Lemma

$u \in C^2(\Omega)$, $\nabla u(x) \neq 0$, is p-harmonic in Ω if and only if for all $x \in \Omega$

$$\frac{\alpha}{2} \left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) + \beta \fint_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \qquad \text{as } \epsilon \to 0,$$

The case $1 < p < \infty$

Lemma

Let be $u \in C(\Omega)$. Suppose that for all $x \in \Omega$ we have

$$\frac{\alpha}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) + \beta \fint_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \qquad \text{as } \epsilon \to 0,$$

where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta = 1$ and

$$\frac{p-2}{N+2} = \frac{\alpha}{\beta},$$

then u is p-harmonic in Ω

Question: Can we modify these lemmas so that they characterize p-harmonic functions?

The case $1 < p \leq \infty$

Theorem

$u \in C(\Omega)$ is p-harmonic in Ω if and only if for all $x \in \Omega$ we have that the asymptotic expansion

$$\frac{\alpha}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) + \beta \fint_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \qquad \text{as } \epsilon \to 0,$$

holds in the **VISCOSITY SENSE**, where $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$ and

$$\frac{p-2}{N+2} = \frac{\alpha}{\beta}.$$

Similar results hold for p-subharmonic and p-superharmonic functions.

Asymptotic Mean Value Expansions

Definition

A continuous function u verifies

$$u(x) = \frac{\alpha}{2}\left\{\max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u\right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2)$$

as $\varepsilon \to 0$ *in the viscosity sense* if
(i) for every $\phi \in C^2$ that touches u from below at x ($u - \phi$ has a strict minimum at the point $x \in \overline{\Omega}$ and $u(x) = \phi(x)$) we have

$$\phi(x) \geq \frac{\alpha}{2}\left\{\max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi\right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\,dy + o(\varepsilon^2).$$

Asymptotic Mean Value Expansions

Definition (continued)

(ii) for every $\phi \in C^2$ that touches u from above at x ($u - \phi$ has a strict maximum at the point $x \in \overline{\Omega}$ and $u(x) = \phi(x)$) we have

$$\phi(x) \leq \frac{\alpha}{2} \left\{ \max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi \right\} + \beta \fint_{B_\varepsilon(x)} \psi(y)\, dy + o(\varepsilon^2).$$

Proof

u p-harmonic $\iff$ u p-harmonic in the viscosity sense $\iff$
Use Taylor theorem applied to the test function ϕ.
(We can safely avoid points x for which $\nabla u(x) = 0$)