

# Introduction to random Tug-of-War games and PDEs

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2009 CIME course

# Inspiration: Games Mathematicians Play

Objective: Understand the methods and give classical proofs of the results in:

- Y. Peres, O. Schramm, S. Sheffield and D. Wilson; *Tug-of-war and the infinity Laplacian*. J. Amer. Math. Soc., 22, (2009), 167-210.
- Y. Peres, S. Sheffield; *Tug-of-war with noise: a game theoretic view of the  $p$ -Laplacian*. Duke Math. J. 145(1), (2008), 91–120.

# Plan of the course

- 1 An example: PDEs on trees ✓
- 2 Probabilistic background: Martingales and Kolmogorov's Consistency Theorem
- 3 Game theory background: Two-person zero-sum games.
- 4 PDE background:  $\Delta_p$  and  $\Delta_\infty$ .
- 5  $p$ -harmonious and  $\infty$ -harmonious functions.

# Probability I: Martingales

- 1 Definition and elementary examples
- 2 Doob's inequality
- 3 Stopping times
- 4 Doob's optional stopping theorem

## Selected References

- *Brownian Motion and Martingales in Analysis* (1984), Out of print.
- *Discrete Martingales and Applications to Analysis* (2002), by José Llorente, Report 87, University of Jyväskylä
- *Probability*, Courant notes by S.S. Varadhan

# Probability II: Kolmogorov's Consistency Theorem

- 1 Review of finite product measures
- 2 Consistent families of finite dimensional distributions
- 3 Kolmogorov's Theorem

## Selected References

- *Probability*, Courant notes by S.S. Varadhan

# Random Zero-Sum Tug-of-War Games

- 1 Tug-of-War games in an undirected graph.
- 2 Strategies and value functions
- 3 DPP

## Selected References

- *Game Theory Alive*, Yuval Peres forthcoming book. Available, FREE FOR THE TIME BEING, at his web page at Berkeley.

# $p$ -Laplacians and $\infty$ -Laplacians

- 1 The divergence form theory
- 2 The non-divergence viscosity theory
- 3 Asymptotic MVP

## Selected References

**JLM** *On the equivalence of viscosity solutions and weak solutions for a quasi-linear elliptic equation*, Juutinen, Lindqvist, M., 2001 SIAM J. Math. Anal.

**SK** *A beginner's Guide to the Theory of Viscosity Solutions* by Sigheaki Koike, 2004. (Original Japanese title: *A secret club on viscosity solutions*)

**MPR1** *An asymptotic mean value property characterization of  $p$ -harmonic functions*, 2009 preprint.

# $p$ -harmonious and $\infty$ -harmonious functions

- 1 Existence and uniqueness
- 2 Strong comparison principle for  $p$ -harmonic functions for  $2 \leq p < \infty$ .
- 3 Approximation of  $p$ -harmonic functions by  $p$ -harmonious functions.

## Selected References

**LG** E. Le Gruyer. *On absolutely minimizing Lipschitz extensions and PDE  $\Delta_\infty(u) = 0$* . 2007 NoDEA .

**MPR2** *On the definition and properties of  $p$ -harmonious functions*, 2009 preprint.