Introduction to random Tug-of-War games and PDEs

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2009 CIME course

Inspiration: Games Mathematicians Play

Objective: Understand the methods and give classical proofs of the results in:

- Y. Peres, O. Schramm, S. Sheffield and D. Wilson; *Tug-of-war and the infinity Laplacian.* J. Amer. Math. Soc., 22, (2009), 167-210.
- Y. Peres, S. Sheffield; *Tug-of-war with noise: a game theoretic view of the p-Laplacian.* Duke Math. J. 145(1), (2008), 91–120.

Trees

A directed tree with regular 3-branching T consists of

- the empty set $\emptyset$,
- 3 sequences of length 1 with terms chosen from the set $\{0, 1, 2\}$,
- 9 sequences of length 2 with terms chosen from the set $\{0, 1, 2\}$,
- $\dots$
- 3^r sequences of length r with terms chosen from the et $\{0, 1, 2\}$

and so on. The elements of T are called *vertices*.

Calculus on Trees

Each vertex v al level r has three children (successors)

$$v_0, v_1, v_2.$$

Gradient

Let $u\colon T \mapsto \mathbb{R}$ be a real valued function. The gradient of u at the vertex v is the vector in $\mathbb{R}^3$

$$\nabla u(v) = \{u(v_0) - u(v), u(v_1) - u(v), u(v_2) - u(v)\}.$$

Divergence

The averaging operator or *divergence* of a vector $X = (x, y, z) \in \mathbb{R}^3$ as

$$\mathrm{div}(X) = x + y + z.$$

Harmonic Functions on Trees

Harmonic functions

A function u is harmonic if satisfies the Laplace equation

$$\operatorname{div}(\nabla u) = 0.$$

The Mean Value Property

A function u is harmonic if and only if it satisfies the mean value property

$$u(v) = \frac{u(v_0) + u(v_1) + u(v_2)}{3}.$$

Thus the values of harmonic function at level r determine its values at all levels smaller than r.

The boundary of the tree

In probability theory u is called a **martingale.** Think of a random walk on the tree.

Branches and boundary

A **branch** of T is an infinite sequence of vertices each followed by its immediate successor (this corresponds to the level $r = \infty$.) The collection of all branches is the boundary of the tree T is denoted by ∂T.

The mapping $g \colon \partial T \mapsto [0, 1]$ given by

$$g(b) = \sum_{r=1}^{\infty} \frac{b_r}{3^r} \quad \text{(also denoted by } b\text{)}$$

is a bijection (think of an expansion in base 3 of the numbers in [0,1]).

The Dirichlet problem

We have a natural metric and natural measure in ∂T inherited from the interval $[0, 1]$.

The **classical Cantor set** C is the subset of ∂T formed by branches that don't go through any vertex labeled 1.

The Dirichlet problem

Given a (continuous) function $f\colon \partial T \mapsto \mathbb{R}$ find a harmonic function $u\colon\ T \mapsto \mathbb{R}$ such that

$$\lim_{r\to\infty} u(b_r) = f(b)$$

for every branch $b = (b_r) \in \partial T$.

Dirichlet problem, II

Given a vertex $v \in T$ consider the subset of ∂T consisting of all branches that start at v. This is always an interval that we denote by I_v.

Solution to the Dirichlet problem

The we have

$$u(v) = \frac{1}{|I_v|} \int_{I_v} f(t)\, dt.$$

Note that u is a *martingale*.

We see that we can in fact solve the Dirichlet problem for $f \in L^1([0, 1])$.

Harmonic measure

Definition of Harmonic Measure

Given a Borel set $E \subset \partial T$ we can consider the function u_E that is harmonic in T has has boundary values 1 in E and zero in $\partial T \setminus E$. The value of $u_E(\emptyset)$ is called the harmonic measure of the set E relative to the top vertex.

Proposition

We have

$$u_E(\emptyset) = |E|$$

the length (or Lebesgue measure) of the set E.

Game interpretation

Random Walk

Start at $\emptyset$. Move downward by choosing successors at random with uniform probability. When you get at ∂T at the point t you get paid $f(t)$ dollars.

Two player tug-of-War game

Instead of moving at random, each player chooses the successor vertex. The game ends when we reach ∂T at a point t in which case player II pays $f(t)$ dollars to player I.

Two player random Tug-of-War game

A coin is tossed. The player who wins the coin toss chooses the successor vertex (heads for player I, tails for player II.) The game ends when we reach ∂T at a point t in which case player II pays $f(t)$ dollars to player I.

More on Random Walk Game interpretation

Every time we run the game we get a sequence of vertices

$$v_1, v_2, \ldots, v_k, \ldots..$$

that determines a point on t the boundary ∂T.
If we are at vertex v_1 and run the game, player II pays $f(t)$ dollars to player I. Let us average out over all possible games that start at v_1.

The value function is harmonic

$$\text{Expected pay off} = \mathbb{E}^{v_1}[f(t)] = u(v_1) = \frac{1}{|I_{v_1}|}\int_{I_{v_1}} f(t)\,dt.$$

Say that f is large near 1 and small near 0. When player I moves he tries to move to the right. When player II moves he moves to the left. These are examples of *strategies*.

DPP for Random Tug-of-War

Definition of Value functions

$$u^I(v) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^v[f(t)] \quad \textit{and} \quad u^{II}(v) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^v[f(t)]$$

DPP (Dynamic Programming Principle)

We have

$$u^I = u^{II}$$

and if we denote the common function by u, it is the only function on the tree such that:

$$u = f \text{ on } \partial T, \quad u(v) = \frac{1}{2}\left[\max_i \{u(v_i)\} + \min_i \{u(v_i)\}\right].$$

Random Walk + Tug-of-War

Let us combine random choice of successor plus tug of war. Choose $\alpha \geq 0$, $\beta \geq 0$ such that $\alpha + \beta = 1$. Start at $\emptyset$. With probability α the players play Tug-of-War. With probability β move downward by choosing successors at random. When you get at ∂T at the point t you get paid $f(t)$ dollars.

DPP for Tug-of-War with noise, DPP = MVP

The value function u verifies the equation

$$u(v) = \frac{\alpha}{2}\left(\max_i\{u(v_i)\} + \min_i\{u(v_i)\}\right) + \beta\left(\frac{u(v_0) + u(v_1) + u(v_2)}{3}\right).$$

Where are the PDEs?

Setting

$$\mathrm{div}_\infty(X) = \max\{x, y, z\} + \min\{x, y, z\}$$

the value function u of the tug-of-war game satisfies

$$\mathrm{div}_\infty\,(\nabla u) = 0$$

Setting

$$\mathrm{div}_p(X) = \frac{\alpha}{2}\,(\max\{x, y, z\} + \min\{x, y, z\}) + \beta\left(\frac{x + y + z}{3}\right)$$

the value function u of the tug-of-war game with noise satisfies

$$\mathrm{div}_p\,(\nabla u) = 0.$$

This operator is non linear, not in divergence form , elliptic and degenerate.

The p-Laplacian on trees

The equations

$$\mathrm{div}_2(\nabla u) = 0, \quad \mathrm{div}_p(\nabla u) = 0, \quad \mathrm{div}_\infty(\nabla u) = 0$$

DPP for Tug-of-War with noise

$$u(v) = \frac{\alpha}{2}\left(\max_i\{u(v_i)\} + \min_i\{u(v_i)\}\right) + \beta\left(\frac{u(v_0) + u(v_1) + u(v_2)}{3}\right).$$

1. The case $p = 2$ corresponds to $\alpha = 0$, $\beta = 1$.
2. The case $p = \infty$ corresponds to $\alpha = 1$, $\beta = 0$.
3. In general, there is no explicit solution formula for $p \neq 2$