

**Measure Theoretical Methods in Harnack
Estimates and Degenerate Parabolic Equations**

**Part 3: Mean Value-Type Properties for
Non-Negative Solutions of Degenerate Parabolic
Equations**

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Issue: Is there a Harnack estimate for such solutions ?

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$$\mathcal{B}(x, t) = \frac{1}{t^{N/\lambda}} \left\{ 1 - b \left(\frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}} \quad \begin{aligned} b &= \frac{1}{\lambda^{\frac{1}{p-1}}} \frac{p-2}{p} \\ \lambda &= N(p-2) + p. \end{aligned}$$

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Therefore the Harnack inequality in its classical form fails.

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Theorem: Let u be a non-negative harmonic function in E . Then for all $x \in B_\rho(x_o) \subset B_R(x_o) \subset E$

$$\left(\frac{R}{R+\rho}\right)^{N-2} \frac{R-\rho}{R+\rho} u(x_o) \leq u(x) \leq \left(\frac{R}{R-\rho}\right)^{N-2} \frac{R+\rho}{R-\rho} u(x_o).$$

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Therefore if $\rho = \sigma R$ for some $\sigma \in (0, 1)$

$$c_1(\sigma) \sup_{B_\rho(x_o)} u \leq u(x_o) \leq c_2(\sigma) \inf_{B_\rho(x_o)} u.$$

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$$\begin{aligned} Q_\rho^+ &= B_\rho(x_o) \times (t_o, t_o + \theta] \\ Q_\rho^- &= B_\rho(x_o) \times (t_o - \theta, t_o]. \end{aligned} \quad \theta = [c u(x_o, t_o)]^{2-p} \rho^p$$

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Bonus: Harnack implies Hölder (interior).

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