

# Measure Theoretical Methods in Harnack Estimates and Degenerate Parabolic Equations

## Part 1: A Lemma of Measure Theory (Clustering of the Positivity Set of Functions in $W_{loc}^{1,1}$ )

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**Theorem:** Let  $\mu$  be a Radon measure in  $\mathbb{R}^N$  and let  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  be  $\mu$ -locally integrable. Then

$$\lim_{\rho \rightarrow 0} \frac{1}{\mu[B_\rho(x)]} \int_{B_\rho(x)} f d\mu = f(x) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^N.$$

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If  $x$  is one such a point, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

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**An Example:**  $E_\alpha \subset [0, 1]$  Cantor set of measure  $\alpha \in (0, 1)$ . Then

$$\lim_{\rho \rightarrow 0} \frac{|E_\alpha \cap (x - \rho, x + \rho)|}{2\rho} = 1 \quad \text{for a.e. } x \in E_\alpha.$$

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**Issue:** Assumptions on  $f$  to have quantitative statements of  $\varepsilon$  and  $\delta$ .

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**Lemma:** *Let  $u \in W^{1,1}(K_\rho)$  satisfy*

$$\|u\|_{W^{1,1}(K_\rho)} \leq \gamma \rho^{N-1} \quad \text{and} \quad |[u > 1]| \geq \alpha |K_\rho| \quad (1)$$

*for some  $\gamma > 0$  and  $\alpha \in (0, 1)$ . Then, for every  $\delta \in (0, 1)$  and  $0 < \lambda < 1$  there exist  $x_o \in K_\rho$  and  $\eta = \eta(\alpha, \delta, \gamma, \lambda, N) \in (0, 1)$ , such that*

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**Meaning of the Lemma:** If the set  $[u > 1]$  occupies an **appreciable** portion of  $K_\rho$ , then there exists at least one point  $y \in K_\rho$  and a neighborhood  $K_{\eta\rho}(y) \subset K_\rho$  where  $u$  remains **arbitrarily close to 1** in some **arbitrarily large** portion of  $K_{\eta\rho}(x_o)$ . Thus the set where  $u \approx 1$  **clusters** about at least one point of  $K_\rho$ .

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Divide these cubes into two subcollections  $\mathbf{Q}^+$  and  $\mathbf{Q}^-$  by

$$Q_j \in \mathbf{Q}^+ \iff |[u > 1] \cap Q_j| > \frac{\alpha}{2} |Q_j|$$

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Denote by  $\#(\mathbf{Q}^+)$  the number of cubes in  $\mathbf{Q}^+$ . By the assumption

$$\sum_{Q_j \in \mathbf{Q}^+} |[u > 1] \cap Q_j| + \sum_{Q_i \in \mathbf{Q}^-} |[u > 1] \cap Q_i| > \alpha |K_1| = \alpha n^N |Q|$$

where  $|Q|$  is the common measure of the  $Q_\ell$ .

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This would establish the Lemma for  $\eta = 1/n$ .

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**Auxiliary Proposition:** Let  $Q \in \mathcal{Q}^+$  satisfy

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For fixed  $x \in [u \leq \lambda] \cap Q$  and  $y \in [u > \frac{1}{2}(1 + \lambda)] \cap Q$

$$\begin{aligned} \frac{1 - \lambda}{2} < u(y) - u(x) &= \int_0^{|y-x|} Du(x + t\omega) \cdot \omega dt \\ &\leq \int_0^{|y-x|} |Du(x + t\omega)| dt \end{aligned} \quad \omega = \frac{y - x}{|x - y|}.$$

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$$\begin{aligned} \frac{\alpha(1 - \lambda)}{4} |Q| &\leq \int_{|\omega|=1} \int_0^{R(x, \omega)} r^{N-1} \int_0^{|y-x|} |Du(x + t\omega)| dt dr d\omega \\ &\leq N^{N/2} |Q| \int_{|\omega|=1} \int_0^{R(x, \omega)} |Du(x + t\omega)| dt d\omega \\ &= N^{N/2} |Q| \int_Q \frac{|Du(z)|}{|z - x|^{N-1}} dz. \end{aligned}$$

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Here  $x$  is still a variable ranging over the set  $[u \leq \lambda]$ .

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and since  $|Q| = \left(\frac{1}{n}\right)^N$

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If the first of these occurs for some  $Q_j \in \mathbf{Q}^+$ , then we are finished.

**Fact 2:** If the second occurs, for all  $Q_j \in \mathbf{Q}^+$ , then by the Auxiliary Proposition

$$\frac{c}{n^{N-1}} \leq \|u\|_{W^{1,1}(Q_j)} \quad \text{for all cubes } Q_j \in \mathbf{Q}^+.$$

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