

# 1 The Chimera Method for a model problem

Franco Brezzi<sup>1</sup>, Jacques Louis Lions<sup>2</sup>, and Olivier Pironneau<sup>3</sup>

<sup>1</sup> Dipartimento di Matematica e IAN CNR, Pavia *brezzi@ian.pv.cnr.it*

<sup>2</sup> Académie des Sciences

<sup>3</sup> Université Pierre et Marie Curie *pironneau@ann.jussieu.fr*

## 1.1 Introduction

The Chimera method [10] was proposed to bypass the difficulty of generating general unstructured meshes for complex objects like airplanes. It is also quite convenient to improve accuracy of the fictitious domain method as it provides a corrector solved locally on a body-fitted fine mesh around each complex object independently. The method is presented in dimension two on the Laplace equation, but it applies to any elliptic system and also in 3d.

More precisely let  $u_e$  be the (exact) solution of

$$-\Delta u_e = f \text{ in } \Omega, \quad u_e = 0 \text{ on } \Gamma \quad (\Gamma \equiv \partial\Omega), \quad (1.1)$$

where  $\Omega$  is a connected open set. Assume that we are given an overlapping decomposition of  $\Omega = \Omega_1 \cup \Omega_2$  such that both  $\Omega_1$  and  $\Omega_2$  are open .

Let  $\mathcal{T}_h$  be a triangulation of  $\Omega_1$  and  $\mathcal{K}_H$  a triangulation of  $\Omega_2$ . We assume that both decompositions are regular and quasi-uniform, in the sense that, if  $h_M$  and  $h_m$  are the maximum and minimum edges in  $\mathcal{T}_h$ , and  $H_M$  and  $H_m$  are the maximum and minimum edges in  $\mathcal{K}_H$ , then there exists two constants  $C_T$  and  $C_K$  such that

$$h_M \leq C_T h_m \quad H_M \leq C_K H_m, \quad (1.2)$$

see e.g. [2]. Without loss of generality we can also assume, to fix the ideas, that

$$h_M \leq H_M. \quad (1.3)$$

Let  $V_h$  and  $V_H$  be the corresponding spaces of piecewise linear continuous functions. We shall denote by  $V_{0h}$  and  $V_{0H}$  the corresponding subspaces of  $H_0^1(\Omega_1)$  and  $H_0^1(\Omega_2)$ , respectively.

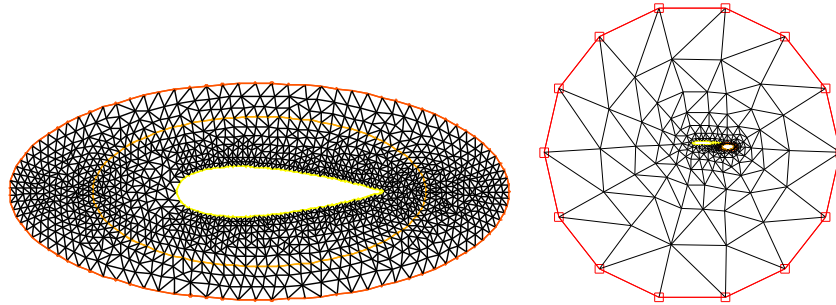
A realistic way of writing the discrete analogue of (1.1) in the finite element subspaces is to proceed *by translation*: we first introduce suitable numerical integration formulae  $(\cdot, \cdot)_h$  and  $(\cdot, \cdot)_H$  in  $\Omega_1$  and  $\Omega_2$  respectively, and then, at each step, we solve the problem: Find  $\{u^{n+1}, v^{n+1}\} \in V_{0h} \times V_{0H}$  solution of

$$\begin{cases} (\nabla(u^{n+1} + v^n), \nabla \hat{u})_h = (f, \hat{u})_h & \forall \hat{u} \in V_{0h}, \\ (\nabla(v^{n+1} + u^n), \nabla \hat{v})_H = (f, \hat{v})_H & \forall \hat{v} \in V_{0H}. \end{cases} \quad (1.4)$$

In [3], [7] it is shown that the method converges if both equations are regularized by adding terms like  $\beta(u^{n+1} - u^n, \hat{u})$  and  $\beta(v^{n+1} - v^n, \hat{v})$  respectively to the

first and second equation in (1.4); we have also proposed to use Gauss quadratures on the gradients, but a proof of convergence in the general case was not given. Here we take up the idea but put the quadrature points at the vertices instead of inside the triangles and show that the method works in a rather general setting; we point out however that the previous integration formula allowed an alternative implementation (by penalty, i.e. putting a large number on the diagonal terms of the lines corresponding to a boundary node in the discrete linear system (see [7])) that is not allowed here.

In the following section, we describe in more details the assumptions on the decompositions  $\mathcal{T}_h$  and  $\mathcal{K}_H$ . In Section 1.3 we present the numerical integration formula. Then, in Section 1.4, we prove a basic ellipticity result for the corresponding bilinear form, and we indicate how this implies the convergence of the iterative methods. In the final section we present the results of some numerical tests. The results contained in this paper were already announced in [1]



**Fig. 1.1.** To compute the stream function around a two-piece airfoil, namely the solution of  $\Delta\psi = 0$  with Dirichlet data by the Chimera method (i.e. Schwarz' algorithm), we build a finer mesh around the smaller airfoil (on the left) and a coarse mesh for the rest of the domain, with an elliptic hole in place of the small airfoil (the scale for both domains is not the same in this picture). The whole domain is the union of the fine and coarse domains.

## 1.2 Assumptions on the decompositions

In what follows  $T$  will denote a generic triangle of the triangulation  $\mathcal{T}_h$  of  $\Omega_1$  and  $K$  a generic triangle of the triangulation  $\mathcal{K}_H$  of  $\Omega_2$ .

Let  $q_T^1, \dots, q_T^R$  be the vertices of  $\mathcal{T}_h$ , and  $q_K^1, \dots, q_K^S$  be the vertices of  $\mathcal{K}_H$ . It will be convenient also to denote the same by  $q_i(T)$  (resp  $q_i(K)$ ),  $i = 1, 2, 3$ , when we refer to the 3 vertices of a triangle.

A crucial assumption that we make is that, in  $\Omega_1 \cap \Omega_2$ , each node  $q_T$  of  $\mathcal{T}_h$  is internal to a triangle  $K$ , and each node  $q_K$  of  $\mathcal{K}_H$  is internal to a triangle  $T$ . This, at first sight, sounds rather restrictive. However, it is clear that one can always reach such a situation by a very small change in the position of the vertices. As we shall

see in the next section, a vertex that is *very close* to an edge of the other decomposition will *not* affect the overall quality of the method; in fact this assumption is necessary only for notational convenience as it makes our quadrature definition unique.

The following lemma will be useful in the sequel.

**Lemma 1.** *If two functions  $u \in V_h$  and  $v \in V_H$  coincide on a connected subset  $\mathcal{S}$  of  $\Omega_1 \cap \Omega_2$ , then both  $u$  and  $v$  are linear (not just piecewise linear) in  $\mathcal{S}$ .*

*Proof.* We notice first that  $\Delta u = \Delta v$  is a distribution with support on the edges of  $\mathcal{T}_h$  and a distribution with support on the edges of  $\mathcal{K}_H$ . But the two sets have in common only isolated points, where an edge of  $\mathcal{T}_h$  crosses an edge of  $\mathcal{K}_H$ . We finally observe that  $\Delta u$  is in  $H^{-1}(\Omega)$  (actually, in  $H^s(\Omega)$  for  $s < -1/2$ ), and hence, as a distribution, its support cannot contain isolated points (see e.g. [9]). Consequently  $u$  is harmonic in  $\mathcal{S}$ , and being piecewise linear is globally linear.

Thanks to the previous result, we can introduce the space

$$V_{hH} := V_{0h} \oplus V_{0H}. \quad (1.5)$$

As we decided to identify functions of  $V_{0h}$  and of  $V_{0H}$  with their extension by zero to the whole  $\Omega$ , every function  $w_{hH}$  in  $V_{hH}$  can be written, in a unique way, as  $w_{hH} = u_h + v_H$  with  $u_h \in V_{0h}$  and  $v_H \in V_{0H}$ .

### 1.3 Quadrature

We are going to introduce now the numerical integration formula to be used in (1.4). Recall that the quadrature formula with integration points at the vertices is exact for polynomials of degree less than or equal to one. In particular, for a given triangle  $\hat{T}$  one has

$$\int_{\hat{T}} g \, dx dy = \frac{|\hat{T}|}{3} \sum_{i=1,2,3} g(q_i) \quad \forall g \in P_1(\hat{T}). \quad (1.6)$$

Hence we introduce the following quadrature rule.

$$\begin{aligned} (\nabla u, \nabla v)_{hH} &:= \sum_{T \in \mathcal{T}_h} \frac{|T|}{3} \sum_{i=1,2,3} \frac{\nabla(u|_T) \cdot \nabla v}{I_{\Omega_1} + I_{\Omega_2}} \Big|_{q_i(T)} \\ &+ \sum_{K \in \mathcal{K}_H} \frac{|K|}{3} \sum_{j=1,2,3} \frac{\nabla(v|_K) \cdot \nabla u}{I_{\Omega_1} + I_{\Omega_2}} \Big|_{q_j(K)}. \end{aligned} \quad (1.7)$$

where  $I_{\Omega_j}(x) = 1$  if  $x \in \Omega_j$  and zero otherwise ( $j = 1, 2$ ).

*Remark 1.* The notation  $\nabla(u|_T)$  is used to indicate that we first restrict the function  $u$  to  $T$ , and then we compute its gradient (which is actually constant in  $T$ ). A similar interpretation holds for  $\nabla(v|_K)$ .

Our main hypothesis, that each vertex in  $\Omega_1 \cap \Omega_2$  is strictly inside a triangle of the other triangulation, allows to write (1.7) with no ambiguity. If it was not the case, for instance if a vertex  $q_i(T)$  were on an edge of  $\mathcal{K}_H$ , then  $\nabla v$  at  $q_i(T)$ , for a function  $v \in V_{0H}$ , would have two possible meanings. Hence, moving slightly the vertex would amount to choosing arbitrarily one of the two meanings, and hence one quadrature formula. Since there is no constant in the proof that follows which depends on the distance of vertices from the edges of the other triangulation, we see that the hypothesis is purely formal.

In the next section we are going to prove that the integration formula (1.7) gives rise to a norm in the space  $V_{hH}$ , that is bounded from below by the usual norm in  $H_0^1(\Omega)$ .

## 1.4 Ellipticity with numerical integration

We start by introducing, for  $w = u + v \in V_{hH}$ , the expression

$$\begin{aligned} |w|_{1,*} \equiv |u + v|_{1,*}^2 &:= h_M^2 \sum_{T \in \mathcal{T}_h} \sum_{i=1,3} |\nabla(u|_T) + \nabla v|^2(q_i(T)) \\ &+ H_M^2 \sum_{K \in \mathcal{K}_H} \sum_{j=1,3} |\nabla u + \nabla(v|_K)|^2(q_j(K)). \end{aligned} \quad (1.8)$$

The notation has to be intended as in Remark 1. It is clear that the quantity  $(\nabla(u+v), \nabla(u+v))_{hH}$  can be bounded (from above and from below) by  $|u+v|_{1,*}^2$ , with constants independent of  $h_M$  and  $H_M$ . We are now going to show that on the space  $V_{hH}$  they are both bounded from below by  $\|\nabla(u+v)\|_{L^2(\Omega)}^2$ . Indeed we have the following theorem.

**Theorem 1.** *For every  $w = u + v \in V_{hH}$  we have:*

$$|u + v|_{1,*} \geq C \|\nabla(u + v)\|_{L^2(\Omega)} \quad (1.9)$$

where  $C$  depends only on  $C_T$  and  $C_K$ .

*Proof.* Before starting the proof, we remember that we extended both  $u$  and  $v$  by zero to the whole  $\Omega$ . We can now consider two bigger subsets of  $\Omega$ , that we call  $\tilde{\Omega}_1$  and  $\tilde{\Omega}_2$ , with  $\Omega_i \subset \tilde{\Omega}_i \subset \Omega$ , ( $i = 1, 2$ ), and correspondingly extend the two decompositions  $\mathcal{T}_h$  and  $\mathcal{K}_H$  to  $\tilde{\mathcal{T}}_h$  and  $\tilde{\mathcal{K}}_H$ , with the same properties of the original ones. We assume that all this is done in such a way that: every triangle  $T \in \mathcal{T}_h$ , having nonempty intersection with  $\Omega_2$ , is completely contained in  $\tilde{\Omega}_2$ , and conversely every triangle  $K \in \mathcal{K}_H$ , having nonempty intersection with  $\Omega_1$ , is completely contained in  $\tilde{\Omega}_1$ . We shall, from now on, denote again the two new decompositions by  $\mathcal{T}_h$  and  $\mathcal{K}_H$ , but we shall keep the distinction between  $\tilde{\Omega}_i$  and  $\Omega_i$ , ( $i = 1, 2$ ). In particular we shall keep the assumption that  $u$  is zero outside  $\Omega_1$  and  $v$  is zero outside  $\Omega_2$ .

We also need some further notation. First, we keep denoting by  $q_T^1, \dots, q_T^R$  the vertices of  $\mathcal{T}_h$ , and by  $q_K^1, \dots, q_K^S$  the vertices of  $\mathcal{K}_H$ . Then, for every  $q_T^r$  (resp.  $q_K^s$ ) we denote by  $T_1^r, \dots, T_{k(r)}^r$  (resp.  $K_1^s, \dots, K_{k(s)}^s$ ) the triangles having  $q_T^r$  (resp.  $q_K^s$ ) as a vertex. Rearranging the terms in the sums of (1.8) we have

$$|u + v|_{1,*}^2 = S_T + S_K, \quad (1.10)$$

where

$$S_T := h_M^2 \sum_{r=1}^R \sum_{\ell=1}^{k(r)} |\nabla(u|_{T_\ell^r}) + \nabla v(q_T^r)|^2 \quad (1.11)$$

and

$$S_K := H_M^2 \sum_{s=1}^S \sum_{\ell=1}^{k(s)} |\nabla u(q_K^s) + \nabla(v|_{K_\ell^s})|^2. \quad (1.12)$$

Then, for every internal node  $q_T^r$  in  $\mathcal{T}_h$  we define  $m_T^r$  as the arithmetic average of the values that  $\nabla(u + v)$  assumes in the triangles  $T$  having  $q_T^r$  as a vertex. We *do not* take into account the measures of the different triangles. A similar meaning holds for  $m_K^s$  when  $q_K^s$  is a vertex in  $\mathcal{K}_H$ . An elementary computation shows that

$$\begin{aligned} \sum_{\ell=1}^{k(r)} |\nabla(u|_{T_\ell^r}) + \nabla v(q_T^r)|^2 = & \quad (1.13) \\ & \sum_{\ell=1}^{k(r)} |\nabla(u|_{T_\ell^r}) + \nabla v(q_T^r) - m_T^r|^2 + k(r) |m_T^r|^2, \end{aligned}$$

and a similar expression holds for the term appearing in  $S_K$ . It will also be convenient to introduce the notation

$$\sigma_T^r := \left( \sum_{\ell=1}^{k(r)} |\nabla(u|_{T_\ell^r}) + \nabla v(q_T^r) - m_T^r|^2 \right)^{1/2}, \quad (1.14)$$

$$\sigma_K^s := \left( \sum_{\ell=1}^{k(s)} |\nabla u(q_K^s) + \nabla(v|_{K_\ell^s}) - m_K^s|^2 \right)^{1/2}. \quad (1.15)$$

Notice that actually  $\sigma_T^r$  does not depend on  $v$  and  $\sigma_K^s$  does not depend on  $u$ . With the above notation we obtain the following new expressions for (1.11) and (1.12):

$$S_T = h_M^2 \sum_{r=1}^R (|\sigma_T^r|^2 + k(r) |m_T^r|^2), \quad (1.16)$$

$$S_K = H_M^2 \sum_{s=1}^S (|\sigma_K^s|^2 + k(s) |m_K^s|^2). \quad (1.17)$$

In particular, from (1.11) and (1.16) we easily have

$$2 S_T \geq h_M^2 \sum_{r=1}^R \left( \sum_{\ell=1}^{k(r)} |\nabla(u|_{T_\ell^r}) + \nabla v(q_T^r)|^2 + |\sigma_T^r|^2 \right) \quad (1.18)$$

and from (1.12) and (1.17)

$$2 S_K \geq H_M^2 \sum_{s=1}^S \left( \sum_{\ell=1}^{k(s)} |\nabla u(q_K^s) + \nabla(v|_{K_\ell^s})|^2 + |\sigma_K^s|^2 \right). \quad (1.19)$$

We notice now that, if  $T_1$  and  $T_2$  are two triangles in  $\mathcal{T}_h$ , having an edge in common, and if  $q_T^r$  is one of the two common vertices, then

$$|\nabla(u|_{T_1}) - \nabla(u|_{T_2})|^2 \leq 2|\sigma_T^r|^2. \quad (1.20)$$

With obvious notation, we have the analogue relation

$$|\nabla(v|_{K_1}) - \nabla(v|_{K_2})|^2 \leq 2|\sigma_K^s|^2. \quad (1.21)$$

In other words, from (1.16), (1.17), (1.20), and (1.21) we see that  $S_T + S_K$  controls both *the values* of  $\nabla(u + v)$  at all the vertices (of  $\mathcal{T}_h$  and  $\mathcal{K}_H$ ) and *the jumps* of  $\nabla(u + v)$  across any edge (of  $\mathcal{T}_h$  and  $\mathcal{K}_H$ ).

Our strategy to prove (1.9) is then the following: we notice that the integral

$$\int_{\Omega} |\nabla u + \nabla v|^2 dx dy$$

is obviously bounded by the sum of the integrals of  $|\nabla u + \nabla v|^2$  over: *i)* the triangles  $T$  that are contained in  $\Omega_1$  but are external to  $\Omega_2$ , *ii)* the triangles  $K$  that are contained in  $\Omega_2$  but are external to  $\Omega_1$ , and *iii)* the pieces  $Q$  that can be written as  $T \cap K$  with  $T \in \mathcal{T}_h$  and  $K \in \mathcal{K}_H$ . Remember that the two decompositions have been extended (keeping the same name) to  $\tilde{\Omega}_1$  and  $\tilde{\Omega}_2$  respectively, so that the union of the sets in *i)*, *ii)*, and *iii)* actually covers  $\Omega$ . Notice also that in each  $Q$  both  $\nabla u$  and  $\nabla v$  are constant.

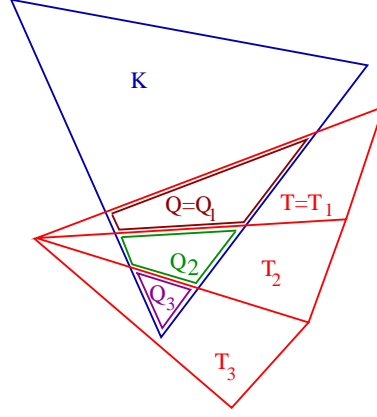
On the triangles  $T$  and  $K$  of *i)* and *ii)* the integration formula is exact, so that we do not have to worry about them.

The integral over the pieces  $Q$  containing at least one vertex of one of the two decompositions can easily be bounded by  $|\nabla u + \nabla v|_{1,*}$ . Hence, in the end, we have to bound only the integrals

$$\int_Q |\nabla u + \nabla v|^2 dx dy \quad (1.22)$$

where  $Q = T \cap K$  *does not contain any vertex*. It is clear that, in any case, the area of  $Q$  is bounded by  $h_M^2$ . The basic idea is then to identify a sequence of pieces  $Q_1, Q_2, \dots, Q_n$  such that *i)*  $Q_1 = Q$ , *ii)* any pair  $Q_i, Q_{i+1}$  has a piece of an edge of  $\mathcal{T}_h$  in common, and *iii)*  $Q_n$  contains a vertex (see the figure, where  $n = 3$ .)

If such a sequence is found, then we can denote by  $T_1, T_2, \dots, T_n$  the elements of  $\mathcal{T}_h$  such that  $Q_i = T_i \cap K$ , and by  $q_{T_i}^r$  one of the two vertices of the edge common to



**Fig. 1.2.** The sequence of polygons  $Q_i$

$T_i$  and  $T_{i+1}$ . As  $Q = T \cap K$  we clearly have that, in  $Q$ ,  $\nabla(u + v) = \nabla u|_T + \nabla v|_K$ . Then we can write, using the triangle inequality ,

$$|\nabla(u|_T) + \nabla(v|_K)| \leq \sum_{i=1}^{n-1} |\nabla(u|_{T_i}) - \nabla(u|_{T_{i+1}})| + |\nabla(u|_{T_n}) + \nabla(v|_K)| \quad (1.23)$$

and using then (1.20) to bound each piece in (1.23):

$$|\nabla(u|_T) + \nabla(v|_K)| \leq \sqrt{2} \sum_{i=1}^{n-1} \sigma_T^{r_i} + |\nabla(u|_{T_n}) + \nabla(v|_K)|. \quad (1.24)$$

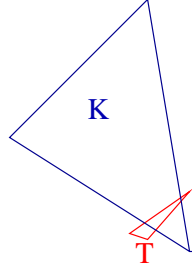
Hence

$$\int_Q |\nabla(u|_T) + \nabla(v|_K)|^2 dx dy \leq h_M^2 C_n \left( \sum_{i=1}^{n-1} |\sigma_T^{r_i}|^2 + |\nabla(u|_{T_n}) + \nabla(v|_K)|^2 \right), \quad (1.25)$$

with  $C_n$  a constant depending only on the number  $n$  of steps in the sequence. Now, each of the pieces of the sum appearing in (1.25) can be bounded by  $S_T$  using (1.16), and the last one can also be bounded by  $S_T$ , since  $T_n \cap K$  contains a vertex of  $\mathcal{T}_h$ .

It is clear that we could also proceed by keeping  $T$  fixed, and changing  $K$  at each step; or keeping one of the two ( $T$  or  $K$ ) fixed, and changing the other ( $K$  or  $T$ ) at each step. It is not difficult to realize that in the above assumptions we have a uniform upper bound for the minimum necessary  $n$  (and hence for  $C_n$ ), depending only on  $C_T$  and  $C_K$ .

Moreover (and this is more delicate,) the *number of times* that we are going to use (1.25) is bounded from above by the “number of vertices in  $\mathcal{K}_H$  that belong to an element with non-empty intersection with  $\Omega_1$ ”. This is pretty easily seen if the



**Fig. 1.3.** Two triangles which intersect but have no vertices inside one another

two triangulations have comparable size. If not, that is if  $h_M$  is much smaller than  $H_M$ , we remark that a small triangle  $T$ , in order to intersect a bigger triangle  $K$  on a piece  $Q$  that does not contain any vertex, has to be sufficiently close (of the order of  $h_M$ !) to a vertex of  $K$  (see the figure.) Hence, only a limited number of such  $T$ 's in this situation can be found for each given vertex in  $\mathcal{K}_H$ .

In conclusion, the sum of all the integrals of the type (1.22) can also be bounded by a fixed constant (depending only on  $C_T$  and  $C_K$ ) times  $|u + v|_{1,*}$ , and (1.9) is proved.

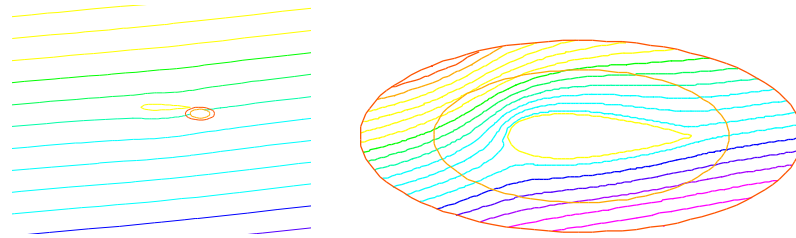
Since  $((\nabla u, \nabla u)_{hH})^{1/2}$  is a norm on  $V_{hH} = V_{0h} \oplus V_{0H}$ , all classical results on the convergence of iterative schemes can be easily applied. For instance, let  $\beta \geq 0$  be some positive scalar, let  $u^0, v^0$  be arbitrary functions of  $V_{0h}$  and  $V_{0H}$  respectively, and consider the loop:

$$\left\{ \begin{array}{l} \text{find } \{u^{n+1}, v^{n+1}\} \in V_{0h} \times V_{0H} \text{ solution of} \\ \beta(u^{n+1} - u^n, \hat{u})_* + (\nabla(u^{n+1} + v^n), \nabla \hat{u})_{hH} = (f, \hat{u})_{**} \quad \forall \hat{u} \in V_{0h}, \\ \beta(v^{n+1} - v^n, \hat{v})_* + (\nabla(v^{n+1} + u^n), \nabla \hat{v})_{hH} = (f, \hat{v})_{**} \quad \forall \hat{v} \in V_{0H}, \end{array} \right. \quad (1.26)$$

where  $(,)_*$  and  $(,)**$  denote suitable integration formulae, possibly based, as (1.7), on vertices. Notice that these choices are less crucial, as they will be used either for right-hand sides, or for products of functions which belong both to the same space (i.e. both in  $V_{0h}$  or both in  $V_{0H}$ ). It is clear that, for  $\beta = 0$ , (1.26) is a particular case of the abstract overlapping Schwarz method analysed in [8]. It is easy to see that the abstract results of [8] imply the geometric convergence of the algorithm for any fixed pair of decompositions, although some additional work would be needed to check whether the contraction constant stays uniformly away from 1 when the meshsizes  $h_M$  and  $H_M$  go to zero. On the other hand, for  $\beta > 0$  the analysis of [7] of the algorithm (1.26) applies unchanged.

## 1.5 Numerical Test

Potential flow around an airfoil involves solving Laplace's equation in a domain outside the airfoil. The finite element method of order one on triangles has been used. The domain is divided in two: a domain near the airfoil which is triangulated with small triangles and the rest of the domain which uses bigger triangles. Here the domain has two airfoils, a large one and a small one. The decomposition must be such that the physical domain is the union of both domains, and the domains must overlap. Then Schwarz algorithm is used with translation and quadratures at the vertices as explained above. Four iterations are sufficient for convergence to machine accuracy.



**Fig. 1.4.** Stream function around a two-pieces airfoil, namely solution of  $\Delta\psi = 0$  with Dirichlet data by the Chimera method (i.e. Schwarz algorithm). The convergence is obtained after 4 iterations.

## References

1. Brezzi, F., Lions, J.L., Pironneau, O. (2001): Analysis of a Chimera Method. C.R.A.S., **332**, 655-660
2. Ciarlet, Ph. G. (1978): The Finite Element Method. North Holland Amsterdam
3. Hecht, F., Lions, J.L., Pironneau, O. (to appear): Domain Decomposition Algorithm for Computed Aided Design. (anniversary book of Necas)
4. Hecht, F., Pironneau, O. (1999): Multiple meshes and the implementation of freefem+. INRIA report. Also on the web at <http://www.ann.jussieu.fr/~pironneau>.
5. Lions, J.L., Pironneau, O. (1998): Algorithmes parallèles pour la solution de problèmes aux limites. C.R.A.S., **327**, 947-352
6. Lions, J.L., Pironneau, O. (1999): Domain decomposition methods for CAD. C.R.A.S., **328** 73-80
7. Lions, J.L., Pironneau, O. (to appear): A Domain Decomposition Algorithm, C.R.A.S. (to appear).
8. Lions, P.L. (1988, 89, 90) On the Schwarz alternating method, I,II,III. Int. Symposium on Domain decomposition Methods for Partial Differential Equations. SIAM Philadelphia
9. Lions, J.L., Magenes, E. (1968): Problèmes aux limites non-homogènes et applications, Vol 1. Dunod Paris
10. Steger J.L. (1991): The Chimera method of flow simulation. Workshop on applied CFD, Univ. of Tennessee Space Institute