COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **8**, Number **1**, January **2009**

10.3934/cpaa.2009.8.1

pp. –

REDUCED SYMMETRY ELEMENTS IN LINEAR ELASTICITY

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Dedicated to professor Philippe G. Ciarlet's for his 70th birthday

Abstract. In continuum mechanics problems, we have to work in most cases with symmetric tensors, symmetry expressing the conservation of angular momentum. Discretization of symmetric tensors is however difficult and a classical solution is to employ some form of reduced symmetry. We present two ways of introducing elements with reduced symmetry. The first one is based on Stokes problems, and in the two-dimensional case allows to recover practically all interesting elements on the market. This however is (definitely) not true in three dimensions. On the other hand the second approach (based on a very nice property of several interpolation operators) works for three-dimensional problems as well, and allows, in particular, to prove the convergence of the Arnold-Falk-Winther element with simple and standard arguments, without the use of the Berstein-Gelfand-Gelfand resolution.

1. **Introduction.** Mixed methods are an appealing technique for the numerical solution of elasticity problems. They ensure the equilibrium condition, a basic property in solid mechanics, and they make the constitutive law more explicit. The stress tensor becomes the main variable but the symmetry of this tensor, however, makes the construction of suitable elements much more complicated than what can be done in the thermal problems where families of elements such as the RT_k and BDM_k are now classical.

The idea of using stress tensors having only a *reduced* symmetry goes back to Fraeijs de Veubeke [11], but the introduction and the analysis of specific elements having symmetry only *in average* was done first in [1], while an even weaker form of symmetry (namely, orthogonality to piecewise linear continuous functions) was proposed and studied in [2]. Since then their use underwent alternate periods of popularity and oblivion. See e.g. [7], [14], [16, 17], [13] and many others. See also [8], [5], [9], and the references therein.

Recently, a general construction of elements with reduced symmetry was presented in [4], [3], and [9]. Their construction relies on a very elegant but quite abstract procedure, requiring rather sophisticated instruments. We present here

2000 *Mathematics Subject Classification.* Primary: 58F15, 58F17; Secondary: 53C35.
Key words and phrases. Linear elasticity, mixed formulations, reduced symmetry.
The first author is supported by NSF grant ????

a new proof of some of their result and related ones, using more elementary and classical techniques. It is clear to us that the construction in [4] still has the merit of having inspired the choice of these elements (and having provided the first proof of their convergence). Nevertheless we believe that our much simpler construction might be interesting for many readers.

The plan of the paper is as follows. In Section 2 we recall the mixed formulation of linear elasticity problems and the general setting for their approximation. In Section 3 we recall first the *non-symmetric* formulation, in which the variational problem is posed in a space of non-symmetric tensors, and the symmetry is then imposed by means of a suitable Lagrange multiplier. The well-posedness of such formulation is well known, but it is proved here with a different approach, based on the solution of suitable auxiliary Stokes problems. The discrete counterpart of this approach is then used to construct families of reduced-symmetry elements, first in two dimensions and then in three dimensions, that include in particular most known two-dimensional elements in the literature. An alternative approach for proving stability and error bounds for elements with reduced symmetry is introduced Section 4. This approach is not based on Stokes auxiliary problems, but rather the possibility of having interpolation operators that respect the reduced symmetry. This new and nice feature is then used to prove the convergence of the Arnold-Falk-Winther family in the three-dimensional case.

Throughout the paper we shall use the standard notation for Sobolev spaces and Sobolev norms (see e.g. [8]). For the sake of simplicity we shall assume that the computational domain Ω is a convex polygon (or a convex polyhedron in 3 dimensions), although (as it could be easily seen looking at the proofs) much more general assumptions would be sufficient.

2. **Linear elasticity problems; stress methods.**

2.1. **Continuous formulation of stress methods.** We consider in this paper a *mixed* approach to linear elasticity problems, that is we use as main variable a symmetric stress tensor, chosen in a suitable space. We therefore define

$$\underline{\underline{H}}(\underline{\mathrm{div}};\Omega) := \{\underline{\underline{\tau}} \in (L^2(\Omega))^{n\times n} |\ \text{such that } \underline{\mathrm{div}}\underline{\underline{\tau}} \in (L^2(\Omega))^n\}, \tag{2.1}$$

$$\underline{\underline{H}}(\underline{\mathrm{div}};\Omega)_S := \{\underline{\underline{\tau}} \in \underline{\underline{H}}(\underline{\mathrm{div}};\Omega) |\ \text{such that } \tau_{i,j} = \tau_{j,i} \quad \forall\, i\, j = 1,..,n\}, \tag{2.2}$$

$$\Sigma = \underline{\underline{H}}(\underline{\mathrm{div}};\Omega),\ \Sigma_S = \underline{\underline{H}}(\underline{\mathrm{div}};\Omega)_S,\ U = (L^2(\Omega))^n. \tag{2.3}$$

We recall the definition of the *trace* of a tensor

$$tr(\underline{\underline{\tau}}) := \sum_{i=1}^{n} \underline{\underline{\tau}}_{ii} \tag{2.4}$$

and of the *deviatoric*

$$\underline{\underline{\tau}}^D := \underline{\underline{\tau}} - \frac{1}{n} tr(\underline{\underline{\tau}})\mathbb{I}, \tag{2.5}$$

where $\mathbb{I}$ is the *identity* tensor. Note that $tr(\mathbb{I}) = n$, thus giving $tr(\underline{\underline{\tau}}^D) = 0$ in (2.5). Note as well that (2.5) can equally be written as

$$tr(\underline{\underline{\tau}})\mathbb{I} = n\,(\underline{\underline{\tau}} - \underline{\underline{\tau}}^D), \tag{2.6}$$

which, applied to the case of a tensor $\underline{\underline{\tau}} = \underline{\underline{\nabla v}}$ (for some $\underline{v}$), gives

$$(\mathrm{div}\,\underline{v})\,\mathbb{I} \equiv\ tr(\underline{\underline{\nabla v}})\,\mathbb{I} = n\,(\underline{\underline{\nabla v}} - \underline{\underline{\nabla v}}^D). \tag{2.7}$$

At this point we can set

$$a(\underline{\underline{\sigma}},\underline{\underline{\tau}}) := \int_\Omega \Big[\frac{1}{2\mu}\underline{\underline{\sigma}}^D : \underline{\underline{\tau}}^D + \frac{1}{n(n\lambda+2\mu)} tr(\underline{\underline{\sigma}}) tr(\underline{\underline{\tau}})\Big]\,dx, \tag{2.8}$$

$$b(\underline{\underline{\tau}},\underline{v}) := \int_\Omega \underline{\operatorname{div}}(\underline{\underline{\tau}})\cdot\underline{v}\,dx, \tag{2.9}$$

and we can write our simple linear elasticity problem as: find $(\underline{\underline{\sigma}},\underline{u}) \in \Sigma_S \times U$ such that

$$\begin{cases} a(\underline{\underline{\sigma}},\underline{\underline{\tau}}) + b(\underline{\underline{\tau}},\underline{u}) = 0, & \forall \underline{\underline{\tau}} \in \Sigma_S, \\ b(\underline{\underline{\sigma}},\underline{v}) + (\underline{f},\underline{v}) = 0, & \forall \underline{v} \in U. \end{cases} \tag{2.10}$$

Remark 1. The first equation represents the constitutive law and the second one the equilibrium condition. It must be clear that although we consider a linear model, the results can be transposed to more realistic non linear models.

We thus have to consider the standard conditions for existence and uniqueness of the solution to this problem. It is very easy to check that

$$\inf_{\underline{v}\in U} \sup_{\underline{\underline{\tau}}\in\Sigma_S} \frac{b(\underline{\underline{\tau}},\underline{v})}{\|\underline{\underline{\tau}}\|_1 \|\underline{v}\|_0} \geq c > 0, \tag{2.11}$$

and that

$$a(\underline{\underline{\tau}},\underline{\underline{\tau}}) \geq \frac{1}{n(n\lambda+2\mu)} \|\underline{\underline{\tau}}\|_0^2, \quad \forall \underline{\underline{\tau}} \in \Sigma. \tag{2.12}$$

We thus have an inf-sup condition and coercivity, so that our problem is well posed. However, trouble arises when we have to deal with a very large λ (nearly incompressible materials). In fact it is clear that the coercivity constant which appears in (2.12) goes to zero like $1/\lambda$ when $\lambda \to +\infty$ so that the stability properties of problem (2.10) seem to deteriorate for large values of λ. Actually, the situation is not as bad as it seems, because, as is well known, we do not need coercivity to hold for every $\underline{\underline{\tau}} \in \Sigma$ (or Σ_h) but only for $\underline{\underline{\tau}} \in \ker B$ (respectively, $\ker B_h$ for discrete problems). In particular, the continuous formulation (2.10) does not break down when $\lambda \to \infty$, because of the following proposition, whose proof can be found in [2] or in [8] [5].

Proposition 1. *There exists a constant $C > 0$ such that, for every $\underline{\underline{\tau}} \in \Sigma$ satisfying*

$$\int_\Omega tr(\underline{\underline{\tau}})\,dx = 0, \tag{2.13}$$

we have

$$\|\underline{\underline{\tau}}\|_0 \leq C(\|\underline{\underline{\tau}}^D\|_0 + \|\underline{\operatorname{div}}\underline{\underline{\tau}}\|_0). \tag{2.14}$$

If we work in the subspace

$$\tilde{\Sigma} = \Big\{\underline{\underline{\tau}} \mid \underline{\underline{\tau}} \in \Sigma, \int_\Omega tr(\underline{\underline{\tau}})\,dx = 0\Big\}, \tag{2.15}$$

we know that the set

$$\ker B = \big\{\underline{\underline{\tau}} \mid \underline{\underline{\tau}} \in \tilde{\Sigma} \text{ such that } b(\underline{\underline{\tau}},\underline{v}) = 0,\ \forall \underline{v} \in U\big\} \tag{2.16}$$

is precisely made of tensors satisfying (2.13) and

$$\underline{\operatorname{div}}\underline{\underline{\tau}} = 0. \tag{2.17}$$

Hence, from Proposition 1 we have

$$a(\underline{\underline{\tau}},\underline{\underline{\tau}}) \geq \frac{1}{2\mu}\|\underline{\underline{\tau}}^D\|_0^2 \geq C(\mu)\,\|\underline{\underline{\tau}}\|_0^2 = C(\mu)\|\underline{\underline{\tau}}\|^2_{\underline{\underline{H}}(\underline{\mathrm{div}};\Omega)_s}, \quad \forall\underline{\underline{\tau}} \in \ker B. \tag{2.18}$$

The stability constant of our problem is therefore independent of λ.

Remark 2. It must be noted that condition (2.13) refers to the fact that with Dirichlet boundary conditions, in incompressible problems, pressure is defined only up to an additive constant. The condition can then be applied *a posteriori*. It disappears whenever Neumann boundary conditions are imposed on a part of the boundary. From the mathematical point of view we can also remark that taking $\underline{\underline{\tau}} = \underline{\underline{I}}$ in the first equation of (2.10) we immediately have that the solution $\underline{\underline{\sigma}}$ belongs to $\tilde{\Sigma}$.

To avoid unnecessary complications, we shall often use, in what follows, the spaces Σ and Σ_S instead of $\tilde{\Sigma}$ and $\tilde{\Sigma}_S$.

2.2. **Numerical approximations of stress formulations.** If we now choose some finite-dimensional subspaces Σ_{Sh} of Σ_S and U_h of U, we must be careful to have the discrete analogues of (2.11) and (2.18) verified. However we have to face a delicate point. In order to prove an inequality of type (2.18) we *needed*, in Proposition 1, to have $\underline{\mathrm{div}}\underline{\underline{\tau}} = 0$. Hence our life would be a lot easier if we had the "inclusion of the kernels property": $\ker B_h \subset \ker B$. In other words, we would like our spaces Σ_{Sh} and U_h to satisfy the following property:

$$\begin{aligned} \ker B_h = \{\underline{\underline{\tau}}_h \in \Sigma_{Sh}\,,\ b(\underline{\underline{\tau}}_h, \underline{v}_h) = 0,\ \forall \underline{v}_h \in U_h\} \\ \subset \ker B = \{\underline{\underline{\tau}} \in \Sigma|_S, \underline{\mathrm{div}}\underline{\underline{\tau}} = 0\}. \end{aligned} \tag{2.19}$$

At the same time, the inf-sup condition (2.11) is related to the existence of an operator $\Pi_h : \Sigma_S \to \Sigma_{Sh}$ such that

$$b(\underline{\underline{\tau}} - \Pi_h\underline{\underline{\tau}}, \underline{v}_h) = 0,\ \forall \underline{v}_h \in U_h, \tag{2.20}$$

$$\|\Pi_h\underline{\underline{\tau}}\|_\Sigma \leq c\,\|\underline{\underline{\tau}}\|_\Sigma, \quad \forall\underline{\underline{\tau}} \in \Sigma. \tag{2.21}$$

There are in the literature many examples of discrete spaces satisfying (2.19), (2.20), and (2.21) for the approximation of spaces of type $H(\mathrm{div};\Omega)$ and $L^2(\Omega)$ when dealing with mixed formulations. The most popular are surely the Raviart Thomas elements [15] and the Brezzi-Douglas-Marini [6], but many others are available: see e.g. [8], [5]. It seems, at the first sight, that we could just use a *pair* of vectors in $H(\mathrm{div};\Omega)$ (or a triplet in 3 dimensions) to approximate Σ, but we should not forget the symmetry of the tensors in Σ_S. The problem of finding subspaces of Σ_S and U satisfying (2.19), (2.20), and (2.21) is actually very difficult. One of the (nowadays) classical remedies is to give up the symmetry of $\underline{\underline{\tau}}$ and enforce it back in a weaker form by some Lagrange multiplier. This is what we are going to do in the next section.

3. **Relaxed symmetry.**

3.1. **Continuous formulation of the relaxed symmetry approach.** The idea of relaxing symmetry was, to our knowledge, first used by [12] and his school; it was then used by [1] and then by [2]. Other results can be found in [7], [14], in [16, 17], and in several other papers. See [9] for a more detailed list of references.

Remark 3. It is worth recalling that the symmetry of the stress tensor is in fact a simplified way of expressing a conservation law, namely the conservation of angular momentum. This should make it easier to understand why symmetry is difficult to enforce. Conservation laws are not easily imposed exactly.

In fact the point of using spaces like $H(\underline{\mathrm{div}};\Omega)$ and its discrete counterparts is to get a strong form for conservation of momentum. For the moment, let us try to define a variational formulation suitable for our purpose. Given a tensor $\underline{\underline{\tau}} \in \Sigma$, we define its skew-symmetric part as

$$\underline{\underline{as}}(\underline{\underline{\tau}}) := \frac{1}{2}\Big\{\underline{\underline{\tau}} - \underline{\underline{\tau}}^t\Big\}. \tag{3.1}$$

We now define a space of skew-symmetric tensors

$$X = \{\underline{\underline{\gamma}} \in L^2(\Omega)^{n\times n} \text{ such that } \underline{\underline{as}}\,(\underline{\underline{\gamma}}) = \underline{\underline{\gamma}}\}, \tag{3.2}$$

and we introduce a new bilinear form:

$$c(\underline{\underline{\tau}},\underline{\underline{\gamma}}) := \int_\Omega \underline{\underline{as}}(\underline{\underline{\tau}}) : \underline{\underline{\gamma}}\, dx \equiv \int_\Omega \underline{\underline{as}}(\underline{\underline{\gamma}}) : \underline{\underline{\tau}}\, dx. \tag{3.3}$$

We can see that, in general, an approximation of (2.10) with relaxed symmetry requirements corresponds to a conforming approximation of the following continuous problem: find $(\underline{\underline{\sigma}},\underline{u},\underline{\underline{\omega}}) \in \tilde{\Sigma}\times U\times X$ such that

$$\begin{cases} a(\underline{\underline{\sigma}},\underline{\underline{\tau}}) + b(\underline{\underline{\tau}},\underline{u}) + c(\underline{\underline{\tau}},\underline{\underline{\omega}}) = 0, \quad \forall \underline{\underline{\tau}} \in \tilde{\Sigma}, \\ b(\underline{\underline{\sigma}},\underline{v}) = (\underline{f},\underline{v}), \quad \forall \underline{v} \in U, \\ c(\underline{\underline{\sigma}},\underline{\underline{\gamma}}) = 0, \quad \forall \underline{\underline{\gamma}} \in X. \end{cases} \tag{3.4}$$

We now have to prove an existence result for this problem. With respect to the general theory, what we need is an inf-sup condition of the form

$$\inf_{\underline{v}\in U,\,\underline{\underline{\gamma}}\in X}\ \sup_{\underline{\underline{\tau}}\in\Sigma} \frac{b(\underline{\underline{\tau}},\underline{v}) + c(\underline{\underline{\tau}},\underline{\underline{\gamma}})}{\|\underline{\underline{\tau}}\|_\Sigma\left(\|\underline{v}\|_U\ +\ \|\underline{\underline{\gamma}}\|_X\right)} \geq C > 0 \tag{3.5}$$

where, here and in all the sequel, C denotes a generic constant that is independent of h.

The above condition is indeed satisfied, as we can see in the following proposition.

Proposition 2. *There exists a constant C such that for any $\underline{v} \in U$ and $\underline{\underline{\gamma}} \in X$, there exists $\underline{\underline{\tau}} \in \Sigma$ such that*

$$b(\underline{\underline{\tau}},\underline{v}) + c(\underline{\underline{\tau}},\underline{\underline{\gamma}}) = \|\underline{v}\|_U^2 + \|\underline{\underline{\gamma}}\|_X^2 \tag{3.6}$$

and

$$\|\underline{\underline{\tau}}\|_\Sigma \leq C\,(\|\underline{v}\|_U + \|\underline{\underline{\gamma}}\|_X). \tag{3.7}$$

Proof. We first give the proof for the two-dimensional case, and then for the three-dimensional one. We give it in detail because the technique will be relevant for the construction of the discrete approximations. The construction of $\underline{\underline{\tau}}$ will be done in two steps. The first one is to build a tensor $\underline{\underline{\tau}}^1 \in \Sigma$ such that

$$\begin{cases} b(\underline{\underline{\tau}}^1,\underline{w}) = (\underline{v},\underline{w}), \quad \forall \underline{w} \in U, \\ \|\underline{\underline{\tau}}^1\|_\Sigma \leq C\,\|\underline{v}\|_U. \end{cases} \tag{3.8}$$

This is easily done, even with a symmetric $\underline{\underline{\tau}}^1$. One could, for instance, solve a classical elasticity problem and take the associated stress field. The second step is to correct this tensor by a divergence-free tensor $\underline{\underline{\tau}}^2$ such that $\underline{\underline{as}}(\underline{\underline{\tau}}^2) = \underline{\underline{\gamma}} - \underline{\underline{as}}(\underline{\underline{\tau}}^1)$. In the two-dimensional case this divergence free tensor is obtained by taking the $i-th$ row $(i = 1, 2)$ made by the curl of the $i - th$ component of a vector $\underline{\Psi} \equiv (\psi_1, \psi_2)$, that is

$$\underline{\underline{\tau}}^2 = \begin{pmatrix} -\partial_2\psi_1 & \partial_1\psi_1 \\ -\partial_2\psi_2 & \partial_1\psi_2 \end{pmatrix}. \tag{3.9}$$

One sees immediately that the condition $\underline{\underline{as}}(\underline{\underline{\tau}}^2) = \underline{\underline{\gamma}} - \underline{\underline{as}}(\underline{\underline{\tau}}^1)$ is equivalent to

$$\frac{1}{2}\mathcal{S}^2(\partial_1\psi_1 + \partial_1\psi_2) = \frac{1}{2}\mathcal{S}^2(\operatorname{div}\underline{\Psi}) = \underline{\underline{as}}(\underline{\underline{\tau}}^2) = \underline{\underline{\gamma}} - \underline{\underline{as}}(\underline{\underline{\tau}}^1), \tag{3.10}$$

where for every scalar q the skew-symmetric tensor $\mathcal{S}^2(q)$ is defined by

$$\mathcal{S}^2(q) = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}. \tag{3.11}$$

To satisfy equation (3.10) with the required continuity condition, it is then sufficient to solve a Stokes problem for $\underline{\Psi}$.

In the three-dimensional case, the situation is slightly more complex. The first step (3.8) holds unchanged, but the divergence-free tensor $\underline{\underline{\tau}}^2$ will now be the curl of another tensor of the form

$$\underline{\underline{\Psi}} = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{pmatrix}. \tag{3.12}$$

This means that we will look for a $\underline{\underline{\tau}}^2$ of the form:

$$\underline{\underline{\tau}}^2 = \begin{pmatrix} \partial_2\psi_{13} - \partial_3\psi_{12} & \partial_3\psi_{11} - \partial_1\psi_{13} & \partial_1\psi_{12} - \partial_2\psi_{11} \\ \partial_2\psi_{23} - \partial_3\psi_{22} & \partial_3\psi_{21} - \partial_1\psi_{23} & \partial_1\psi_{22} - \partial_2\psi_{21} \\ \partial_2\psi_{33} - \partial_3\psi_{32} & \partial_3\psi_{31} - \partial_1\psi_{33} & \partial_1\psi_{32} - \partial_2\psi_{31} \end{pmatrix}, \tag{3.13}$$

whose skew-symmetric part $\underline{\underline{as}}(\underline{\underline{\tau}}^2)$ is individuated by the vector

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \tau^2_{32} - \tau^2_{23} \\ \tau^2_{13} - \tau^2_{31} \\ \tau^2_{21} - \tau^2_{12} \end{pmatrix} = \begin{pmatrix} \partial_1(-\psi_{22} - \psi_{33)} + \partial_2\psi_{21} + \partial_3\psi_{31} \\ \partial_1\psi_{12} + \partial_2(-\psi_{33} - \psi_{11}) + \partial_3\psi_{32} \\ \partial_1\psi_{13} + \partial_2\psi_{23} + \partial_3(-\psi_{11} - \psi_{22}) \end{pmatrix}. \tag{3.14}$$

The construction of $\underline{\underline{\tau}}^2$ such that $\underline{\underline{as}}(\underline{\underline{\tau}}^2) = \underline{\underline{\gamma}} - \underline{\underline{as}}(\underline{\underline{\tau}}^1)$ is thus equivalent to solving three Stokes problems,

$$\begin{cases} -\Delta\underline{\phi}^{(k)} + \underline{\operatorname{grad}}\, p^{(k)} = 0 \\ \operatorname{div}\underline{\phi}^{(k)} = s_k \end{cases}, \tag{3.15}$$

for $k = 1, 2, 3$. Once $\phi^{(1)}$, $\phi^{(2)}$ and $\phi^{(3)}$ have been found, we can construct $\underline{\underline{\Psi}}$ such that

$$\psi_{22} + \psi_{33} = -\phi^{(1)}_1, \psi_{21} = \phi^{(1)}_2, \psi_{31} = \phi^{(1)}_3. \tag{3.16}$$

$$\psi_{12} = \phi^{(2)}_1, \psi_{33} + \psi_{11} = -\phi^{(2)}_2, \psi_{32} = \phi^{(2)}_3, \tag{3.17}$$

$$\psi_{13} = \phi^{(3)}_1, \psi_{23} = \phi^{(3)}_2, \psi_{11} + \psi_{22} = -\phi^{(3)}_3, \tag{3.18}$$

Denoting by $\underline{\underline{\Phi}}$ the tensor having $\underline{\phi}^{(1)}$, $\underline{\phi}^{(2)}$, and $\underline{\phi}^{(3)}$ as columns, it is immediate to see that

$$\underline{\underline{\Phi}} = \underline{\underline{\Psi}} - tr(\underline{\underline{\Psi}})\underline{\underline{I}},$$

giving easily

$$\underline{\underline{\Psi}} = \underline{\underline{\Phi}} - \frac{tr(\underline{\underline{\Phi}})}{2}\mathbb{I}. \tag{3.19}$$

We may write the algebraic relation (3.19) as

$$\underline{\underline{\mathbf{\Psi}}} = \mathcal{A}\underline{\underline{\mathbf{\Phi}}}. \tag{3.20}$$

It is also immediate to verify that the above construction, which relies on the solution of well posed Stokes problems, satisfies the required continuity conditions as well. □

Remark 4. The above construction is not the most general in the three-dimensional case. It is not necessary to get $\underline{\underline{\mathbf{\Psi}}}$ with all the regularity implied by our procedure. This will have as a consequence that some approximations cannot be generated with the discrete equivalent procedure.

3.2. **Numerical approximation of relaxed-symmetry formulations.** We can now start considering the approximation of the variational formulation (3.4). We want to choose subspaces Σ_h, U_h, X_h of Σ, U, X and to solve the problem: find $(\underline{\underline{\sigma}}_h, \underline{u}_h, \underline{\underline{\omega}}_h) \in \tilde{\Sigma}_h \times U_h \times X_h$ such that

$$\begin{cases} a(\underline{\underline{\sigma}}_h, \underline{\underline{\tau}}_h) + b(\underline{\underline{\tau}}_h, \underline{u}_h) + c(\underline{\underline{\tau}}_h, \underline{\underline{\omega}}_h) = 0, \quad \forall \underline{\underline{\tau}}_h \in \tilde{\Sigma}_h, \\ b(\underline{\underline{\sigma}}_h, \underline{v}_h) = (\underline{f}, \underline{v}_h), \quad \forall \underline{v}_h \in U_h, \\ c(\underline{\underline{\sigma}}_h, \underline{\underline{\gamma}}_h) = 0, \quad \forall \underline{\underline{\gamma}}_h \in X_h. \end{cases} \tag{3.21}$$

Here again, we shall rely on the general theory. We shall try to build approximations satisfying the "inclusion of kernels property"(2.19). To do so, we can use some of the finite element discretizations of $H(\mathrm{div}, \Omega)$ available in the literature. Assuming that we made a choice that takes care of that, we still must check the discrete inf-sup condition:

$$\inf_{\underline{v}_h \in U_h, \underline{\underline{\gamma}}_h \in X_h} \sup_{\underline{\underline{\tau}}_h \in \Sigma_h} \frac{b(\underline{\underline{\tau}}_h, \underline{v}_h) + c(\underline{\underline{\tau}}_h, \underline{\underline{\gamma}}_h)}{\|\underline{\underline{\tau}}_h\|_\Sigma \left(\|\underline{v}_h\|_U + \|\underline{\underline{\gamma}}_h\|_X\right)} \geq C > 0. \tag{3.22}$$

It is well known (see e.g. [10] or [8]) that this can be done by building an interpolation operator $\Pi_h : \Sigma \to \Sigma_h$ satisfying,

$$\begin{aligned} & b(\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}, \underline{v}_h) + c(\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}, \underline{\underline{\gamma}}_h) = 0, \forall \underline{v}_h \in U_h, \forall \underline{\underline{\gamma}}_h \in X_h, \\ & \|\Pi_h \underline{\underline{\tau}}\|_\Sigma \leq C \|\underline{\underline{\tau}}\|_\Sigma. \end{aligned} \tag{3.23}$$

To do this we shall try to proceed in the same way that we used to prove the continuous inf-sup condition: we shall first build $\underline{\underline{\tau}}_h^1$ so that its divergence satisfies the first requirement

$$\begin{aligned} & b(\underline{\underline{\tau}} - \underline{\underline{\tau}}_h^1, \underline{v}_h) = 0, \forall \underline{v}_h \in U_h, \\ & \|\underline{\underline{\tau}}_h^1\|_\Sigma \leq C \|\underline{\underline{\tau}}\|_\Sigma, \end{aligned} \tag{3.24}$$

and then we correct this tensor by a divergence-free tensor $\underline{\underline{\tau}}_h^2$ to obtain the required asymmetry,

$$\begin{aligned} & c(\underline{\underline{\tau}} - \underline{\underline{\tau}}_h^2, \underline{\underline{\gamma}}_h) = c(\underline{\underline{\tau}}_h^1, \underline{\underline{\gamma}}_h), \forall \underline{\underline{\gamma}}_h, \in X_h \\ & \|\underline{\underline{\tau}}_h^2\|_\Sigma \leq C \|\underline{\underline{\tau}} - \underline{\underline{\tau}}_h^1\|_\Sigma. \end{aligned} \tag{3.25}$$

Referring to the continuous case, we can try to build $\underline{\underline{\tau}}_h^2$ by solving (discrete) Stokes problems (one in two dimensions and three in three dimensions) returning a tensor $\underline{\underline{\mathbf{\Psi}}}_h$ and then by taking $\underline{\underline{\tau}}_h^2 = \mathrm{curl}(\underline{\underline{\mathbf{\Psi}}}_h)$. It follows that one possible key to our constructions will be stable elements for the Stokes problems, together with the *inclusion of the kernels* property, that we now have to require in Σ_h (rather than Σ_{Sh} as in (2.19)):

$$\ker B_h \equiv \{\underline{\underline{\tau}}_h \in \Sigma_h\,,\ b(\underline{\underline{\tau}}_h, \underline{v}_h) = 0,\ \forall \underline{v}_h \in U_h\} \\ \subset \ker B \equiv \{\underline{\underline{\tau}} \in \Sigma, \underline{\mathrm{div}}\underline{\underline{\tau}} = 0\}. \tag{3.26}$$

But the approach through Stokes problems is less effective for the three-dimensional case, and we shall present an alternative one in Section 4.

Before considering specific constructions, let us state the error estimate that one can expect for the discrete problem (3.21).

Theorem 3.1. *Let us suppose that the spaces $\Sigma_h \times U_h \times X_h$ are such that*

- *$\Sigma_h \times U_h$ satisfies* (3.26)
- *$\Sigma_h \times U_h \times X_h$ satisfies* (3.22)*.*

Then (3.21) *has a unique solution. Moreover, if $(\underline{\underline{\sigma}}, \underline{u}, \underline{\underline{\omega}}) \in \Sigma \times U \times X$ is the solution of* (3.4) *and $(\underline{\underline{\sigma}}_h, \underline{u}_h, \underline{\underline{\omega}}_h) \in \Sigma_h \times U_h \times X_h$ is the solution of* (3.21)*, then we have*

$$\|\underline{\underline{\sigma}}_h - \underline{\underline{\sigma}}\|_0 + \|\underline{u}_h - \underline{u}\|_0 + \|\underline{\underline{\omega}}_h - \underline{\underline{\omega}}\|_0 \\ \leq C\Big(\inf_{\underline{\underline{\tau}} \in \Sigma_h} \|\underline{\underline{\tau}}_h - \underline{\underline{\sigma}}\|_0 + \inf_{\underline{v}_h \in U_h} \|\underline{v}_h - \underline{u}\|_0 + \inf_{\underline{\underline{\phi}}_h \in X_h} \|\underline{\underline{\phi}}_h - \underline{\underline{\omega}}\|_0 \Big) \tag{3.27}$$

The proof is an easy consequence of the general theory on mixed formulations (see e.g.[5]).

One can see from (3.27) that it is important to balance the quality of the approximation for the three components of the solution. In particular, symmetry must be imposed at least to the same precision as the approximation properties for the other variables

Remark 5. It is clear that the inclusion of kernels (3.26) is not necessary, but it makes the theory easier.

Remark 6. It is easy to see that if the space Σ_h contains (as we implicitly assume) the constant identity tensor $\underline{\underline{I}}$, then solving the problem in $\tilde{\Sigma}_h \times U_h \times X_h$ or in $\Sigma_h \times U_h \times X_h$ gives exactly the same result.

Remark 7. In a few cases, we shall be able to build explicitly a basis for the space Σ_{Sh} of discrete symmetric tensors. We shall then be able to consider the problem

$$\begin{cases} a(\underline{\underline{\sigma}}_h, \underline{\underline{\tau}}_h) + b(\underline{\underline{\tau}}_h, \underline{u}_h) = 0, & \forall \underline{\underline{\tau}}_h \in \Sigma_{Sh}, \\ b(\underline{\underline{\sigma}}_h, \underline{v}_h) = (\underline{f}, \underline{v}_h), & \forall \underline{v}_h \in U_h. \end{cases} \tag{3.28}$$

3.3. **A first family of relaxed symmetry elements in two dimensions.** We start by considering the *two-dimensional* case. Let us suppose that we can choose a pair $\Sigma_h \times U_h$ such that (3.26) is satisfied, and moreover the "first part"of the inf-sup condition holds that is

$$\inf_{\underline{v}_h \in U_h} \sup_{\underline{\underline{\tau}}_h^1 \in \Sigma_h} \frac{b(\underline{\underline{\tau}}_h^1, \underline{v}_h)}{\|\underline{\underline{\tau}}_h^1\|_\Sigma \|\underline{v}_h\|_U} \geq C > 0. \tag{3.29}$$

This is equivalent to be able to build, for any $\underline{\underline{\tau}}$, a tensor $\underline{\underline{\tau}}_h^1$ such that,

$$\begin{aligned} & b(\underline{\underline{\tau}} - \underline{\underline{\tau}}_h^1, \underline{v}_h) = 0\ \forall \underline{v}_h \in U_h, \\ & \|\underline{\underline{\tau}}_h^1\|_\Sigma \leq C \|\underline{\underline{\tau}}\|_\Sigma. \end{aligned} \tag{3.30}$$

To get the full inf-sup condition, we must control symmetry. Let us suppose that we have a stable approximation $V_h \times Q_h$ for the Stokes problem such that

$$\mathrm{curl}(V_h) \subset \Sigma_h. \tag{3.31}$$

We then solve for $(\underline{\psi}_h, p_h) \in V_h \times Q_h$:

$$\begin{cases} \int_\Omega 2\mu\, \underline{\underline{\varepsilon}}(\underline{\psi}_h) : \underline{\underline{\varepsilon}}(\underline{\phi}_h)\, dx + \int_\Omega p_h \operatorname{div} \underline{\phi}_h \, dx = (\underline{f}, \underline{\phi}_h),\ \forall \underline{\phi}_h \in V_h, \\ \int_\Omega q_h \operatorname{div} \underline{\psi}_h \, dx = \int_\Omega (\underline{\underline{\tau}} - \underline{\underline{\tau}}_{1h}) : \mathcal{S}^2(q_h)\, dx \quad \forall q_h \in Q_h. \end{cases} \tag{3.32}$$

If we now compute $\underline{\underline{\tau}}_h^2 = \underline{\underline{\tau}}_h^1 - \mathrm{curl}\, \underline{\psi}_h$, we do not invalidate (3.29) and we have:

$$\begin{aligned} &c(\underline{\underline{\tau}} - \underline{\underline{\tau}}_h^2, \underline{\underline{\gamma}}_h) = c(\underline{\underline{\tau}}_h^1, \underline{\underline{\gamma}}_h)\, \forall \underline{\underline{\gamma}}_h \in X_h, \\ &\|\underline{\underline{\tau}}_h^2\|_\Sigma \le C \|\underline{\underline{\tau}} - \underline{\underline{\tau}}_h^1\|_\Sigma. \end{aligned} \tag{3.33}$$

Considering (3.29) and (3.33), it is then natural to take $X_h := \mathcal{S}^2(Q_h)$ where $\mathcal{S}^2(q)$ is always defined in (3.11), and we easily obtain the required inf-sup condition (3.22) for the triplet (Σ_h, U_h, X_h) . With respect to our elasticity problem, we work then with a reduced symmetry property, "symmetry weighted by q_h". This can be summarized in the following proposition, that can be seen as a simplification (in a particular case) of a result of [9].

Proposition 3. *For $n = 2$, for any couple of spaces $\Sigma_h \times U_h$ satisfying conditions* (3.26) *and* (2.11)*, and any couple $V_h \times Q_h$ stable for the Stokes problem and satisfying* (3.31)*, the triplet*

$$\Sigma_h \times U_h \times X_h \quad \textit{with } X_h = \mathcal{S}^2(Q_h) \tag{3.34}$$

satisfies the conditions of Theorem 3.1.

This analysis through a Stokes problem was first introduced in [13]. There are several examples in the literature of approximations that could be inserted in the above theory. These include the PEERS element of Arnold-Brezzi-Douglas [2], or its variant by Brezzi-Douglas-Marini [7]. These include as well the Amara-Thomas element [1] and other possible elements that could be developed along these lines. In the next section we shall see other two-dimensional elements whose convergence can be proved with this strategy.

3.4. **Relaxed symmetry, the three-dimensional case.** Building stable elements with relaxed symmetry is somewhat more tricky for the three-dimensional case. The basic idea is again to start from elements satisfying the first part (3.29) of the inf-sup condition and then to correct for symmetry. We can still use the same trick as in the two-dimensional case and rely, following the continuous case, on solving three Stokes problems. In more details, we start from a pair $\Sigma_h \times U_h$ (in three dimensions this time) such that (3.26) is satisfied, and moreover the "first part"of the inf-sup condition holds: that is

$$\inf_{\underline{v}_h \in U_h} \sup_{\underline{\underline{\tau}}_h^1 \in \Sigma_h} \frac{b(\underline{\underline{\tau}}_h^1, \underline{v}_h)}{\|\underline{\underline{\tau}}_h^1\|_\Sigma \|\underline{v}_h\|_U} \ge C > 0. \tag{3.35}$$

We recall once more that this is equivalent to be able to build, for any $\underline{\underline{\tau}}$, a tensor $\underline{\underline{\tau}}_h^1$ such that,

$$\begin{aligned} &b(\underline{\underline{\tau}} - \underline{\underline{\tau}}_h^1, \underline{v}_h) = 0\, \forall \underline{v}_h \in U_h, \\ &\|\underline{\underline{\tau}}_h^1\|_\Sigma \le C \|\underline{\underline{\tau}}\|_\Sigma. \end{aligned} \tag{3.36}$$

To get the full inf-sup condition, we have again to work on the symmetry condition. Let us suppose that we have a stable approximation $V_h \times Q_h$ for the Stokes problem in three dimensions. For any triplet of vectors $\underline{\phi}^{(1)}$, $\underline{\phi}^{(2)}$, and $\underline{\phi}^{(3)}$, we denote by $\underline{\underline{\boldsymbol{\Phi}}}$ the tensor having them as columns, and we denote by $[V_h|V_h|V_h]$ the tensor space containing all possible tensors built in his way. Our requirement (in place of the two-dimensional (3.31)) is now that

$$\mathrm{curl}(\mathcal{A}([V_h|V_h|V_h])) \subset \Sigma_h. \tag{3.37}$$

where $\mathcal{A}$ is still defined as in (3.20). If this is satisfied, we can then proceed as we did for the continuous case. We do not repeat the procedure here. We just point out that, instead of $\mathcal{S}^2(Q_h)$, in three dimensions we define, for every vector $\underline{q}$, the tensor $\mathcal{S}^3(\underline{q})$ given by

$$\mathcal{S}^3(\underline{q}) = \begin{pmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{pmatrix}, \tag{3.38}$$

with $(q_1, q_2, q_3) \in Q_h$

We summarize the result for the three-dimensional case.

Proposition 4. *For $n = 3$, for any couple of spaces $\Sigma_h \times U_h$ satisfying conditions* (3.26) *and* (3.35)*, and any couple $V_h \times Q_h$ stable for the Stokes problem and satisfying* (3.37)*, the triplet*

$$\Sigma_h \times U_h \times X_h \quad \textit{with } X_h = \mathcal{S}^3(Q_h) \tag{3.39}$$

satisfies the conditions of Theorem 3.1.

Remark 8. We already noted that using a Stokes problem was not the most general way of obtaining the continuous inf-sup condition. In the same way, the above result will enable us to obtain some useful constructions of relaxed symmetry tensors. However, it does not yield all constructions.

Example 1. As examples of applications of the above strategy we could consider the three-dimensional version of the PEERS element. We define Σ_h as the space of tensors (in 3 dimensions) where each line is an element of the lowest order Raviart-Thomas space on tetrahedra. Using for U_h a space of piecewise constant vectors, it is immediate that we have (3.26) and (3.35). For the Stokes problem we use the three-dimensional MINI element:

$$\begin{aligned} V_h &= (\mathfrak{L}_1^1)^3 \oplus (B_4)^3 \\ Q_h &= \mathfrak{L}_1^1, \end{aligned} \tag{3.40}$$

where B_4 is the space generated by the elementwise *quartic bubbles* b_4, obtained as the product of the equations of the four faces. We can then augment the space Σ_h by adding, in each element, $\underline{\underline{\mathrm{curl}}}(\mathcal{A}([V_h|V_h|V_h])$. This will leave (3.26) and (3.35) still holding true, and (3.37) will now hold as well (we hammered it into the method). All the assumptions of Proposition 4 will then be satisfied.

Example 2. Consider, for each face f of each tetrahedron K, the cubic function b_3^f that vanishes identically on the three other faces and has value 1 at the barycenter of f. Then consider the vector valued function $b_3^f \mathbf{n}^f$ where $\mathbf{n}^f$ is the unit normal vector to f. Let finally B_{faces} be the space generated by all these vector valued functions all over the domain, and consider a Stokes element where $V_h = (\mathfrak{L}_1^1)^3 \bigoplus B_{faces}$ and Q_h is made by piecewise constant functions. We consider now Σ_h as the tensor space obtained using a lowest-order Raviart-Thomas element per line (as above), augmented with $\underline{\underline{\mathrm{curl}}}(\mathcal{A}(B_{faces}))$, and U_h made by piecewise constant vectors. Then all the assumptions of Proposition 4 will be satisfied. The space of tensors thus obtained is unfortunately not easy to make explicit.

Example 3. The three-dimensional equivalent of the Amara–Thomas element. As in the two-dimensional case, it is easy to obtain a suitable construction based on the Crouzeix–Raviart element for the Stokes problem. This construction yields a very rich space of tensors which is not likely to be of practical use and that we shall not try to make explicit here.

Example 4. A simple second order element We define Σ_h as the space of tensors (in 3 dimensions) where each line is an element of the Raviart-Thomas RT_1 space on tetrahedra. We can then use $U_h = \mathfrak{L}_1^0$, the space of discontinuous piecewise linear vectors; it is immediate that we have (3.26) and (3.35). For the Stokes problem we consider the Taylor–Hood element in which velocity is approximated by a space of quadratic elements $(\mathfrak{L}_2^1)^3$ and where pressure is continuous piecewise linear $(\mathfrak{L}_1^1)$. This immediately yields a second order elasticity element in which we have

$$\begin{aligned} \Sigma_h &= (RT_1)^3, \\ U_h &= (\mathfrak{L}_1^0)^3, \\ X_h &= \mathcal{S}^3((\mathfrak{L}_1^1)^3). \end{aligned} \tag{3.41}$$

Symmetry is enforced as in the PEERS element but we now have second order accuracy. This construction is obviously a member of an infinite family, using higher order Raviart–Thomas elements and corresponding generalized Taylor–Hood elements.

4. **An alternative approach.**

4.1. **Notation and useful formulae.**

In order to explain the following variant of the above strategy, it is convenient to recall the definition of the *permutation tensor* (or pseudo-tensor) in two and three dimensions: for $n = 2$ the double tensor $\underline{\underline{\mathbb{P}}}$ is given by

$$\underline{\underline{\mathbb{P}}} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.1}$$

while for $n = 3$ the triple tensor $\underline{\underline{\underline{\mathbb{P}}}}$ is given by

$$\mathbb{P}_{ijk} = \begin{cases} 1 & \text{if } \{\mathrm{i,j,k}\} = \{1,2,3\} \text{ or } \{3,1,2\} \text{ or } \{2,3,1\} \\ -1 & \text{if } \{\mathrm{i,j,k}\} = \{3,2,1\} \text{ or } \{1,3,2\} \text{ or } \{2,1,3\} \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Note that we could have summarized (4.1) and (4.2) in one formula (as in most textbooks on tensors), using the concept of *even* and *odd* permutations. We chose the above presentation for the sake of clarity, considering that the cases $n = 2$ and $n = 3$ will be the only ones of interest for us.

We point out that, in two dimensions, a tensor $\underline{\underline{\tau}}$ is symmetric if and only if the *scalar* $\underline{\underline{\tau}} : \underline{\underline{\mathbb{P}}}$ verifies

$$\underline{\underline{as}}(\underline{\underline{\tau}}) \equiv \frac{1}{2}\mathcal{S}^2(\underline{\underline{\tau}} : \underline{\underline{\mathbb{P}}}) = 0. \tag{4.3}$$

Similarly, in three dimensions, a tensor $\underline{\underline{\tau}}$ is symmetric if and only if the *vector* $\underline{\underline{\tau}} : \underline{\underline{\underline{\mathbb{P}}}}$ satisfies

$$\underline{\underline{as}}(\underline{\underline{\tau}}) \equiv \frac{1}{2}\mathcal{S}^3(\underline{\underline{\tau}} : \underline{\underline{\underline{\mathbb{P}}}}) = \underline{0}. \tag{4.4}$$

Then we denote by

$$\underline{x} \wedge \underline{y}$$

the external product of two vectors. In two dimensions, this is a scalar given by

$$\underline{x} \wedge \underline{y} := (\underline{\underline{\mathbb{P}}} \cdot \underline{y}) \cdot \underline{x} = (\mathbb{P}_{ij} y_j) x_i \tag{4.5}$$

and in three dimensions it is a vector given by

$$\underline{x} \wedge \underline{y} := (\underline{\underline{\underline{\mathbb{P}}}} \cdot \underline{y}) \cdot \underline{x} \qquad \text{that is} \qquad (\underline{x} \wedge \underline{y})_i = (\mathbb{P}_{ijk} y_k) x_j \tag{4.6}$$

where in (4.5) and (4.6) (and in all the rest of the paper) the Einstein convention of summation of repeated indices is employed.

In a similar way we can define the wedge product of a double tensor $\underline{\underline{\tau}}$ with a vector $\underline{v}$ as

$$(\underline{\underline{\tau}} \wedge \underline{v})_r := \mathbb{P}_{ij} v_j \tau_{ir} \quad \text{ and } \quad (\underline{\underline{\tau}} \wedge \underline{v})_{ir} := \mathbb{P}_{ijk} v_k \tau_{jr} \tag{4.7}$$

in two and three dimensions, respectively. We recall now a useful property of tensor calculus. We denote by $\underline{x} \equiv (x_1, x_2)^T$ or $\underline{x} \equiv (x_1, x_2, x_3)^T$ (respectively in two or three dimensions) the vector containing the independent variables. Let $\underline{\underline{\tau}}$ be given in $\underline{\underline{H}}(\underline{\mathrm{div}};\Omega)$. In two dimensions we have

$$\mathrm{div}(\underline{\underline{\tau}} \wedge \underline{x}) = (\underline{\mathrm{div}}\underline{\underline{\tau}}) \wedge \underline{x} + \underline{\underline{\mathbb{P}}} : \underline{\underline{\tau}}$$

as it can be easily seen by

$$(\mathbb{P}_{ij} x_j \tau_{ir})_{/r} = \mathbb{P}_{ij} \delta_{jr} \tau_{ir} + \mathbb{P}_{ij} x_j (\tau_{ir})_{/r} = \underline{\underline{\mathbb{P}}} : \underline{\underline{\tau}} + (\underline{\mathrm{div}}\underline{\underline{\tau}}) \wedge \underline{x}.$$

Similarly, in three dimensions we have

$$\underline{\mathrm{div}}(\underline{\underline{\tau}} \wedge \underline{x}) = (\underline{\mathrm{div}}\underline{\underline{\tau}}) \wedge \underline{x} + \underline{\underline{\underline{\mathbb{P}}}} : \underline{\underline{\tau}}$$

as it can be easily seen by

$$(\mathbb{P}_{ijk} x_k \tau_{jr})_{/r} = \mathbb{P}_{ijk} \delta_{kr} \tau_{jr} + \mathbb{P}_{ijk} x_k (\tau_{ir})_{/r} = \underline{\underline{\underline{\mathbb{P}}}} : \underline{\underline{\tau}} + (\underline{\mathrm{div}}\underline{\underline{\tau}}) \wedge \underline{x}.$$

In particular we have

$$\underline{\mathrm{div}}(\underline{\underline{\tau}} \wedge \underline{x}) = \underline{\underline{\tau}} : \underline{\underline{\underline{\mathbb{P}}}} \qquad \text{when } \underline{\mathrm{div}}\, \underline{\underline{\tau}} = \underline{0} \tag{4.8}$$

with its obvious analogue in dimension $n = 2$.

Multiplying (4.8) times a vector $\underline{p} \in (H^1(K))^n$, we get

$$\int_K \underline{\mathrm{div}}(\underline{\underline{\tau}} \wedge \underline{x}) \cdot \underline{p}\, \mathrm{d}x = \int_K \underline{\underline{\tau}} : \underline{\underline{\underline{\mathbb{P}}}} \cdot \underline{p}\, \mathrm{d}x \qquad \text{when } \underline{\mathrm{div}}\, \underline{\underline{\tau}} = \underline{0}$$

that integrated by parts reads

$$\int_K \underline{\underline{\tau}} : \underline{\underline{\underline{\mathbb{P}}}} \cdot \underline{p}\ \mathrm{d}x = \int_{\partial K} \underline{p} \cdot (\underline{\underline{\tau}} \wedge \underline{x}) \cdot \underline{n}_K\, \mathrm{d}x - \int_K (\underline{\underline{\tau}} \wedge \underline{x}) : \underline{\underline{\nabla}}(\underline{p})\, \mathrm{d}x \quad \text{when } \underline{\mathrm{div}}\, \underline{\underline{\tau}} = \underline{0} \tag{4.9}$$

where $\underline{n}_K$ is the outward unit normal vector to ∂K. In two dimensions, instead, (4.9) obviously becomes

$$\int_K \underline{\underline{\tau}} : \underline{\underline{\mathbb{P}}}\, p\ \mathrm{d}x = \int_{\partial K} (\underline{\underline{\tau}} \wedge \underline{x}) \cdot \underline{n}_K\, p\ \mathrm{d}x - \int_K (\underline{\underline{\tau}} \wedge \underline{x}) \cdot \underline{\nabla} p\ \mathrm{d}x \quad \text{when } \underline{\mathrm{div}}\, \underline{\underline{\tau}} = \underline{0}. \tag{4.10}$$

Using now the following identities

$$(\underline{\underline{\tau}} \wedge \underline{x}) : \underline{\underline{\nabla}}\, \underline{p} = -\underline{\underline{\tau}}\ :\ (\underline{x} \wedge \underline{\underline{\nabla} \underline{p}}), \tag{4.11}$$

$$\underline{p} \cdot (\underline{\underline{\tau}} \wedge \underline{x}) \cdot \underline{n}_K = (\underline{x} \wedge \underline{p}) \cdot \underline{\underline{\tau}} \cdot \underline{n}_K, \tag{4.12}$$

we can write (4.9) in the form

$$\int_K \underline{\underline{\tau}} : \underline{\underline{\mathbb{P}}} \cdot \underline{p} \, \mathrm{d}x = \int_{\partial K} (\underline{x} \wedge \underline{p}) \cdot \underline{\underline{\tau}} \cdot \underline{n}_K \, \mathrm{d}s + \int_K \underline{\underline{\tau}} : (\underline{x} \wedge \underline{\underline{\nabla}} \underline{p}) \, \mathrm{d}x \quad \text{ when } \underline{\mathrm{div}} \, \underline{\underline{\tau}} = \underline{0} \tag{4.13}$$

which will be more suitable to define degrees of freedom for a tensor $\underline{\underline{\tau}}$.

To deal with the two-dimensional case, we denote by $\underline{x}^{\perp} := (x_2, -x_1)$ the orthogonal of $\underline{x}$. The starting point is now (4.10). A simple computation shows that instead of (4.11) we now have:

$$(\underline{\underline{\tau}} \wedge \underline{x}) \cdot \underline{\nabla} p = \underline{\underline{\tau}} : (\underline{x}^{\perp} \otimes \underline{\nabla} p),$$

where we used the notation

$$\underline{a} \otimes \underline{b} := \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$$

that in our case gives

$$\underline{x}^{\perp} \otimes \underline{\nabla} p := \begin{pmatrix} x_2 \partial p / \partial x_1 & x_2 \partial p / \partial x_2 \\ -x_1 \partial p / \partial x_1 & -x_1 \partial p / \partial x_2 \end{pmatrix}.$$

On the boundary, instead of (4.12) we have

$$p \, (\underline{\underline{\tau}} \wedge \underline{x}) \cdot \underline{n}_K = p \, \underline{x}^{\perp} \cdot \underline{\underline{\tau}} \cdot \underline{n}_K,$$

and from (4.3) we immediately have

$$\int_K \underline{\underline{\tau}} : \underline{\underline{\mathbb{P}}} p \, \mathrm{d}x = \int_K \underline{\underline{as}}(\underline{\underline{\tau}}) : \mathcal{S}^2(p) \, \mathrm{d}x,$$

so that the two-dimensional version of (4.13) is

$$\int_K \underline{\underline{as}} \, (\underline{\underline{\tau}}) : \mathcal{S}^2(p) \, \mathrm{d}x = \int_{\partial K} p \, \underline{x}^{\perp} \cdot \underline{\underline{\tau}} \cdot \underline{n}_K \, \mathrm{d}s + \int_K \underline{\underline{\tau}} : (\underline{x}^{\perp} \otimes \underline{\nabla} p) \, \mathrm{d}x, \tag{4.14}$$

whenever $\underline{\mathrm{div}} \, \underline{\underline{\tau}} = \underline{0}$.

The idea, now, is to use (4.13) or (4.10) with $\underline{\underline{\tau}} = \underline{\underline{\sigma}} - \Pi_h \underline{\underline{\sigma}}$. Indeed, this is what we shall do in the sequel. We will deal first with the two-dimensional case, and then we will briefly present the corresponding three-dimensional results.

4.2. **The two-dimensional case.** We consider first the two dimensional case that, as usual, is easier and allows several types of strategies. We have the following basic theorem.

Theorem 4.1. *For $n = 2$ assume that $\Sigma_h \times U_h \times W_h$ are such that:*

$$\underline{\mathrm{div}}(\Sigma_h) \subseteq U_h \tag{4.15}$$

and that there exists a piecewise polynomial space Ξ_h such that $((\Xi_h)^2, W_h)$ is a stable Stokes element. Assume moreover that there exists a mapping Π_h from $(H^1(\Omega))^{n \times n}_S + \underline{\underline{\mathrm{curl}}}((\Xi_h)^2)$ into Σ_h satisfying for each element K:

$$\|\Pi_h \underline{\underline{\tau}}\|_{\underline{\underline{H}}(\underline{\mathrm{div}};K)} \leq C \, \|\underline{\underline{\tau}}\|_{(H^1(K))^{n \times n}_S} \tag{4.16}$$

together with the following properties:

$$\int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : (\underline{\underline{\nabla}} \, \underline{v}_h + \underline{x}^{\perp} \otimes \underline{\nabla} w_h) \, \mathrm{d}x = 0 \qquad \forall \, \underline{v}_h \in U_h, \, \forall w_h \in W_h, \tag{4.17}$$

$$\int_{\ell} (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) \cdot \underline{n}_K \cdot (\underline{v}_h + \underline{x}^{\perp} w_h) \, \mathrm{d}s = 0 \qquad \forall \text{ edge } \ell, \, \forall \, \underline{v}_h \in U_h, \, \forall w_h \in W_h. \tag{4.18}$$

Then the triplet

$$\Sigma_h \times U_h \times X_h \quad \textit{with } X_h = \mathcal{S}^2(W_h) \tag{4.19}$$

satisfies the conditions of Theorem 3.1.

Proof. The inclusion of kernels (3.26) follows easily from (4.15). From the convexity assumption, we easily have that for every $\underline{u}_h \in U_h \subseteq (L^2(\Omega))^n$ we can find a $\underline{\underline{\tau}} = \underline{\underline{\tau}}(\underline{u}_h) \in (H^1(\Omega))^{n\times n}_S$ such that

$$\underline{\mathrm{div}}\underline{\underline{\tau}} = \underline{u}_h \quad \text{and} \quad \|\underline{\underline{\tau}}\|_{(H^1(\Omega))^{n\times n}_S} \le C\|\underline{u}_h\|_{(L^2(\Omega))^n}. \tag{4.20}$$

Then using (4.15), (4.17) with $\underline{w}_h = 0$, (4.18) with $w_h = 0$ and (4.16) we easily have that

$$\underline{\mathrm{div}}\Pi_h\underline{\underline{\tau}} = \underline{u}_h \quad \text{and} \quad \|\Pi_h\underline{\underline{\tau}}\|_{\underline{\underline{H}}(\underline{\mathrm{div}};\Omega)} \le C\|\underline{u}_h\|_{(L^2(\Omega))^n}. \tag{4.21}$$

Hence the pair of spaces (Σ_h, U_h) satisfy condition (3.24), just by taking $\underline{\underline{\tau}}^1_h := \Pi_h\underline{\underline{\tau}}$. In particular from (4.17), (4.18), and again (4.15) we have that $\underline{\mathrm{div}}(\underline{\underline{\tau}} - \Pi_h\underline{\underline{\tau}}) = 0$, so that we can apply formula (4.14). At this point, using (3.3) and (4.19), then (4.3), and finally employing (4.17) and (4.18)in (4.14), gives

$$c(\underline{\underline{\tau}}-\Pi_h\underline{\underline{\tau}}, \underline{\underline{\phi}}_h) \equiv \int_K \underline{\underline{\mathrm{as}}}(\underline{\underline{\tau}}-\Pi_h\underline{\underline{\tau}}) : \mathcal{S}^2(v_h)\,\mathrm{d}x = 0 \qquad \forall \underline{\underline{\phi}}_h \equiv \mathcal{S}^2(v_h) \in X_h \equiv \mathcal{S}^2(W_h).$$

Now for every $\underline{\underline{\gamma}}_h = \mathcal{S}^2(v_h) \in X_h$ we can use the assumption that $((\Xi_h)^2, W_h)$ is a stable element for Stokes, and find a $\underline{\Psi}_h \in (\Xi_h)^2$ such that

$$\|\underline{\Psi}_h\|_{(H^1(\Omega))^2} \le C\,\|v_h\|_{L^2(\Omega)}$$

and

$$\int_\Omega \mathrm{div}\underline{\Psi}_h\, q_h\,\mathrm{d}x = \int_\Omega v_h\, q_h\,\mathrm{d}x \qquad \forall q_h \in W_h.$$

Now we take, as in (3.9),

$$\underline{\underline{\tau}}^2(h) = \begin{pmatrix} -\partial_2\psi^1_h & \partial_1\psi^1_h \\ -\partial_2\psi^2_h & \partial_1\psi^2_h \end{pmatrix}. \tag{4.22}$$

One sees immediately that $\underline{\mathrm{div}}\underline{\underline{\tau}}^2(h) = 0$ and

$$\int_\Omega \underline{\underline{as}}(\underline{\underline{\tau}}^2(h)) : \mathcal{S}^2(q_h)\,\mathrm{d}x = \int_\Omega \mathrm{div}\underline{\Psi}_h\, q_h\,\mathrm{d}x = \int_\Omega v_h\, q_h\,\mathrm{d}x \qquad \forall q_h \in W_h. \tag{4.23}$$

It is clear, from the previous discussion, that $\Pi_h\underline{\underline{\tau}}^2(h)$ will satisfy

$$\underline{\mathrm{div}}\Pi_h\underline{\underline{\tau}}^2(h) = 0 \quad \text{and} \quad c(\underline{\underline{\tau}}^2(h) - \Pi_h\underline{\underline{\tau}}^2(h), \underline{\underline{\phi}}_h) = 0\ \forall \underline{\underline{\phi}}_h \in X_h. \tag{4.24}$$

We can therefore take $\underline{\underline{\tau}}_h := \Pi_h(\underline{\underline{\tau}}+\underline{\underline{\tau}}^2(h))$ to show that (3.23), and therefore (3.22), hold true. □

Remark 9. It is important to point out that, in general, conditions (4.17) and (4.18) *do not* define the interpolation operator Π_h, meaning that the number of degrees of freedom (4.17) and (4.18) (always in general) is smaller than the dimension of Σ_h, so that other degrees of freedom need to be added in order to define Π_h in a unique way. However, to apply the theorem we do not need to exhibit an interpolation operator that satisfies (4.16) (4.17), and(4.18). It will be enough to show that such an operator **exists**.

Remark 10. It is clear that the convexity assumption is unnecessarily strong. Indeed, as is well known, a little bit of regularity *more* than $\underline{\underline{H}}(\underline{\mathrm{div}};\Omega)$ will be sufficient, in applications, to build Π_h. See e.g. [8] or [5]. On the other hand, the use of the space Ξ_h is motivated by the *lack of regularity* that we would have proceeding as in (3.9)-(3.10). Indeed, if you look for a vector $\underline{\Psi}$ with divergence equal to q you cannot have anything better than

$$\|\underline{\Psi}\|_{(H^1(\Omega))^2} \leq C\,\|q\|_{L^2(\Omega)}$$

and the corresponding $\underline{\underline{\tau}}^2$ given by (3.9) will not be smooth enough to apply the interpolation operator Π_h. To the contrary, the solution of a *discrete problem* provides a piecewise polynomial whose interpolant can be defined without regularity problems.

Theorem 4.1 opens new ways of getting reduced symmetry. In particular we can recover, with a simpler proof, the convergence of the elements with reduced symmetry of Arnold–Falk–Winther [4].

Example 5. The lowest degree element of the Arnold–Falk–Winther family in two dimensions. Our first example (and main example of this sub section) will be the Arnold-Falk-Winther element [4]. Here we consider the two-dimensional case with the simplest (and possibly most interesting) case $k=1$, given by the choice

$$\begin{aligned}&\Sigma_h = (\mathbf{BDM}_1)^2 \equiv ((\mathfrak{L}^1_1)^2)^2\\&U_h = (\mathfrak{L}^0_0)^2\\&X_h = \mathcal{S}^2(W_h) = \mathcal{S}^2(\mathfrak{L}^0_0).\end{aligned} \tag{4.25}$$

For this the interpolation operator defined in (4.17) and(4.18) is simply

$$\int_\ell (\underline{\underline{\tau}} - \Pi_h\underline{\underline{\tau}})\cdot\underline{n}_K\cdot\underline{p}_1\,\mathrm{d}s = 0 \qquad \textit{for all edge } \ell \textit{ and for all } \underline{p}_1 \in (P_1(\ell))^2, \tag{4.26}$$

that corresponds to apply two times the $\mathbf{BDM}_1$ interpolation operator (one for each row). Theorem 4.1 then applies immediately and gives immediately the result.

Remark 11. In the two-dimensional case, the case $k=1$ was introduced in [13]. The proof was based on the stability of the $P_2 - P_0$ element for the Stokes problem.

Example 6. The general case of the Arnold–Falk–Winther family in two dimensions. We consider now the two dimensional Arnold-Falk-Winther element [4] of degree $k>1$. The spaces are:

$$\begin{aligned}&\Sigma_h = (\mathbf{BDM}_k)^2 \equiv ((\mathfrak{L}^1_k)^2)^2\\&U_h = (\mathfrak{L}^0_{k-1})^2\\&X_h = \mathcal{S}^2(W_h) = \mathcal{S}^2(\mathfrak{L}^0_{k-1})\end{aligned} \tag{4.27}$$

and the interpolation operator Π_h is defined on each element K by

$$\int_K (\underline{\underline{\tau}} - \Pi_h\underline{\underline{\tau}}) : (\underline{\underline{v}}_h + \underline{w}_h \otimes \underline{x}^\perp)\,\mathrm{d}x = 0 \qquad \forall\,\underline{\underline{v}}_h \in (P_{k-2})^{2\times 2},\ \forall \underline{w}_h \in (P_{k-2})^2, \tag{4.28}$$

$$\int_\ell (\underline{\underline{\tau}} - \Pi_h\underline{\underline{\tau}})\cdot\underline{n}_K\cdot\underline{v}_h\,\mathrm{d}s = 0 \qquad \forall \textit{ edge } \ell, \qquad \forall\,\underline{v}_h \in (P_k)^2. \tag{4.29}$$

Note that in (4.28) the order of $\underline{w}_h \otimes \underline{x}^\perp$ is different from the one required in (4.17) (that would be $\underline{x}^\perp \otimes \underline{w}_h$) and then we cannot use directly the interpolation operator defined by (4.28) and (4.29) to apply Theorem 4.1. However in the sequel, among other results, we will provide four different proofs (apart from the original one in [4]) of the optimal convergence of the element.

The following Theorem provides a different interpolation operator, that allows the immediate application of Theorem 4.1.

Theorem 4.2. . *For the element described in* (4.27) *the interpolation operator defined on each* K *by*

$$\int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : (\underline{\underline{v}}_h + \underline{x}^\perp \otimes \underline{w}_h)\,\mathrm{d}x = 0 \qquad \forall\, \underline{\underline{v}}_h \in (P_{k-2})^{2\times 2},\ \forall \underline{w}_h \in (P_{k-2})^2, \tag{4.30}$$

and

$$\int_\ell (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) \cdot \underline{n}_K \cdot \underline{v}_h\,\mathrm{d}s = 0 \qquad \forall \text{ edge } \ell, \qquad \forall\, \underline{v}_h \in (P_k)^2 \tag{4.31}$$

is well defined, and satisfies (4.16) *with a constant* C *independent of* h.

Proof. We prove the unisolvence of (4.30) and (4.31) (restricting to the case $k \ge 2$). To start with we observe that the number of degrees of freedom of (4.30)-(4.31) is the same as that of (4.28)-(4.28) (and hence equal to the dimension of Σ_h). Then we only have to show the injectivity. For this we assume that $\underline{\underline{\tau}} = \underline{\underline{0}}$ and we want to deduce that $\Pi_h \underline{\underline{\tau}} = \underline{\underline{0}}$. We first use $\underline{\underline{v}}_h = \underline{\underline{\nabla}}(\underline{\mathrm{div}}\underline{\underline{\tau}})$ with $\underline{w_h} = \underline{0}$ in (4.30) and $\underline{v}_h = \Pi_h \underline{\underline{\tau}} \cdot \underline{n}$ with $w_h = 0$ in (4.31) to show (as usual for **BDM**) that $\Pi_h \underline{\underline{\tau}}$ has zero divergence and zero normal component (meaning: every row is a vector having zero divergence and zero normal component on ∂K). Hence we have that $\Pi_h \underline{\underline{\tau}}$ has the form

$$\Pi_h \underline{\underline{\tau}} = \begin{pmatrix} \phi^1_y & -\phi^1_x \\ \phi^2_y & -\phi^2_x \end{pmatrix} \tag{4.32}$$

where the ϕ^i $(i = 1, 2)$ have the form

$$\phi^i = b_3\, r^i_{k-2} \tag{4.33}$$

and where both r^i_{k-2} $(i = 1, 2)$ are polynomials of degree $\le k - 2$, and b_3 is the cubic bubble. Imposing (4.30) with $\underline{\underline{v}}_h = \underline{\underline{0}}$ and $\underline{w_h} = (w^1, 0)$ to the form (4.32) and integrating by parts we have

$$-\int_K r^1_{k-2} b_3 \partial_2 (x_2 w^1)\,\mathrm{d}x + \int_K r^2_{k-2} b_3 \partial_2 (x_1 w^1)\,\mathrm{d}x = 0 \qquad \forall w^1 \in P_{k-2} \tag{4.34}$$

and imposing (4.30) with $\underline{\underline{v}}_h = \underline{\underline{0}}$ and $\underline{w_h} = (0, w^2)$ we have

$$\int_K r^1_{k-2} b_3 \partial_1 (x_2 w^2)\,\mathrm{d}x - \int_K r^2_{k-2} b_3 \partial_1 (x_1 w^2)\,\mathrm{d}x = 0 \qquad \forall w^2 \in P_{k-2}. \tag{4.35}$$

We claim that (4.34) and (4.35) imply

$$\int_K r^1_{k-2}\, b_3 t^1\,\mathrm{d}x = \int_K r^2_{k-2}\, b_3 t^2\,\mathrm{d}x = 0 \qquad \forall \underline{t} \in (P_{k-2})^2 \tag{4.36}$$

and hence $r^i_{k-2} = 0$ $(i = 1, 2)$ and finally, using (4.32) and (4.33), we have $\Pi_h \underline{\underline{\tau}} = \underline{\underline{0}}$. To see (4.36) we show that for every $\underline{t} \in (P_{k-2})^2$ we can find $\underline{w} \in (P_{k-2})^2$ in (4.34) and (4.35), respectively, such that

$$-\partial_2 (x_2 w^1) + (\partial_1 x_2 w^2) = t^1 \quad \text{ and } \partial_2 (x_1 w^1) - \partial_1 (x_1 w^2) = t^2 \tag{4.37}$$

that is

$$-w^1 - x_2 \partial_2 w^1 + x_2 \partial_1 w^2 = t^1 \quad \text{ and } x_1 \partial_2 w^1 - w^2 - x_1 \partial_1 w^2 = t^2. \tag{4.38}$$

We set

$$d = \partial_1 w^2 - \partial_2 w^1 \tag{4.39}$$

so that (4.38) becomes

$$-w^1 + x_2 d = t^1 \quad \text{and} \quad -x_1 d - w^2 = t^2. \tag{4.40}$$

Taking the derivative of the first equation with respect to x_2 and subtracting the derivative of the second equation with respect to x_1 and substituting (4.39) into it, we obtain

$$3d + x_1\partial_1 d + x_2\partial_2 d = \partial_2 t^1 - \partial_1 t^2. \tag{4.41}$$

It is easy to check that for every $\underline{t}$ (4.41) has a unique solution $d \in P_{k-3}$ and from d and $\underline{t}$, using (4.40), we get $\underline{w}$. Finally, the proof of (4.16) with C independent of h follows with the usual arguments. □

Remark 12. In the case $k = 2$ the above proof becomes trivial, as (4.34) and (4.35) will give immediately the desired result.

Before going to other strategies, we introduce a further notation. For every integer $s \geq 0$ and for every element K we define

$$\begin{aligned} &P^h_s := \{\text{homogeneous polynomials of degree } s\} \\ &\tilde{P}_s(K) := \{\text{polynomials of degree} \leq s \text{ having zero mean value on } K\} \end{aligned} \tag{4.42}$$

and we note, once and for all, that for $s > 0$ we have

$$P_s = P_{s-1} \oplus P^h_s$$
$$P_s(K) = P_0 \oplus \tilde{P}_s(K).$$

Our alternative strategies will be based on the following simple observation..

Proposition 5. *Consider the element described in* (4.27). *If an interpolation operator satisfies*

$$\int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : \underline{\underline{\nabla}}\, \underline{v}_h \,\mathrm{d}x = 0 \qquad \forall\, \underline{v}_h \in (P_{k-1})^2, \tag{4.43}$$

and

$$\int_\ell (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) \cdot \underline{n}_K \cdot \underline{v}_h \,\mathrm{d}s = 0 \qquad \forall \text{ edge } \ell \text{ and } \forall\, \underline{v}_h \in (P_k)^2, \tag{4.44}$$

then the requirement

$$\int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : \underline{x}^\perp \otimes \underline{\nabla}(\phi_h) \,\mathrm{d}x = 0 \qquad \forall\, \phi_h \in P^h_{k-2} \tag{4.45}$$

is **equivalent** *to the requirement that*

$$\int_K \underline{\underline{as}}(\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : \mathcal{S}^2(\phi_h) \,\mathrm{d}x = 0 \qquad \forall\, \phi_h \in P^h_{k-1}. \tag{4.46}$$

The proof is an easy exercise based on formula (4.14). □

Note that (4.43), in particular, will be implied by

$$\int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : \underline{\underline{v}}_h \,\mathrm{d}x = 0 \qquad \forall\, \underline{\underline{v}}_h \in (P_{k-2})^{2\times 2}. \tag{4.47}$$

In the next proposition we consider a new, different set of d.o.f. to be added to the ones contained in (4.47), (4.44), and (4.45) (or (4.46) that, as we have seen, is equivalent). To this purpose we define first the space P^{h*}_{k-1} as follows

- for s odd (s=2j-1)

$$P_s^{h*} := span\{x^s, x^{s-1}y, .., x^{j+1}y^{j-2}, x^{j-2}y^{j+1}, ..., xy^{s-1}, y^s\}$$

$$Ex: \ P_9^{h*} = span\{x^9, x^8y, x^7y^2, x^6y^3, x^3y^6, x^2y^7, xy^8, y^9\}$$

- for s even (s=2j)

$$P_s^{h*} := span\{x^s, x^{s-1}y, .., x^{j+1}y^{j-1} + x^{j-1}y^{j+1}, ..., xy^{s-1}, y^s\}$$

$$Ex: \ P_8^{h*} = span\{\ x^8, x^7y, x^6y^2, x^5y^3 + x^3y^5, x^2y^6, xy^7, y^8\}$$

(note that in both cases the dimension of P_s^{h*} equals $s-1$). We then define the space P_s^* as

$$P_s^* := P_{s-1} \oplus P_s^{h*}.$$

We also need the following Lemma.

Lemma 4.3. *Let s be an integer ≥ 2. For all polynomial q of degree $s-2$ there exists a $p \in P_s^*$ such that $\Delta p = q$.*

Proof. For the sake of simplicity we will just show that we can reconstruct all q that are monomials of degree 7 or 8. For the degree 7 we use

$$x^9, x^8y, \frac{9\times 8}{1\times 2}x^7y^2 - x^9, \frac{8\times 7}{2\times 3}x^6y^3 - x^8y,$$
$$\frac{8\times 7}{2\times 3}y^6x^3 - y^8x, \frac{9\times 8}{1\times 2}y^7x^2 - y^9, y^8x, y^9$$

whose Laplacians (apart from a multiplicative factor) are, respectively

$$x^7, x^6y, x^5y^2, x^4y^3, y^4x^3, y^5x^2, y^6, y^7,$$

and for the degree 8

$$x^{10}, x^9y, \frac{10\times 9}{1\times 2}x^8y^2 - x^{10}, \frac{9\times 8}{2\times 3}x^7y^3 - x^9y,$$
$$\frac{8\times 7}{3\times 4}(x^6y^4 + x^4y^6) - x^8y^2 - x^2y^8,$$
$$\frac{9\times 8}{2\times 3}y^7x^3 - y^9x, \frac{10\times 9}{1\times 2}y^8x^2 - y^{10}, y^9x, y^{10}$$

whose Laplacians (apart from a multiplicative factor) are, respectively

$$x^8, x^7y, x^6y^2, x^5y^3, x^4y^4, y^5x^3, y^6x^2, y^7x, y^8.$$

It is then elementary to see how to proceed in the general case. □

We are finally ready to consider our new set of degrees of freedom.

Theorem 4.4. *Consider the element described in* (4.27)*. Then the interpolation operator defined by* (4.47),(4.44)*,* (4.46)*, and*

$$\int_K [tr(\underline{\tau} - \Pi_h\underline{\tau}]\, p_{k-1}^{h*} = 0 \qquad \forall p_{k-1}^{h*} \in P_{k-1}^{h*} \tag{4.48}$$

is unisolvent and verifies (4.16).

Proof. First we check the number of degrees of freedom: the space $(P_k)^{2\times 2}$ has dimension $4(k+1)(k+2)/2 = 2k^2+6k+4$. Conditions (4.47) are $4(k-1)k/2 = 2k^2-2k$. Conditions (4.44) are $3\times 2\times (k+1) = 6k+6$. conditions (4.46) are k and conditions (4.48) are $k-2$. Total conditions

$$(6k+6)+(2k^2-2k)+k+(k-2) = 2k^2+6k+4$$

as wanted.

Then we start by assuming that $\underline{\underline{\tau}}$ is zero, and we want to show that $\Pi_h\underline{\underline{\tau}} = \underline{\underline{0}}$. From (4.47) and (4.44) we proceed as in the proof of Proposition 5 and we deduce that

$$\Pi_h\underline{\underline{\tau}}\cdot\underline{n} = \underline{0} \qquad \text{on } \partial K.$$

and

$$\underline{\mathrm{div}}\Pi_h\underline{\underline{\tau}} = \underline{0}.$$

We have therefore as in (4.32) that $\Pi_h\underline{\underline{\tau}}$ has the form

$$\Pi_h\underline{\underline{\tau}} = \begin{pmatrix} -\partial_2\phi^1 & \partial_1\phi^1 \\ -\partial_2\phi^2 & \partial_1\phi^2 \end{pmatrix} \tag{4.49}$$

where ϕ^1 and ϕ^2 are polynomials of degree $\le k+1$ vanishing on ∂K. Imposing (4.46) to the form (4.49) and also using (4.44) we have

$$\int_K (\partial_1\phi^1+\partial_2\phi^2)p_{k-1} = 0 \qquad \forall p_{k-1}\in P_{k-1} \tag{4.50}$$

and using (4.48)

$$\int_K (-\partial_2\phi^1+\partial_1\phi^2)p^*_{k-1} = 0 \qquad \forall p_{k-1}\in P^*_{k-1}. \tag{4.51}$$

Integrating (4.50) and (4.51) by parts, and setting $\underline{\Phi} := (\phi^1,\phi^2)$, we have

$$\int_K \underline{\Phi}\cdot\underline{\nabla} p_{k-1} = 0 \qquad \forall p_{k-1}\in P_{k-1} \tag{4.52}$$

and

$$\int_K \underline{\Phi}\cdot \underline{\mathrm{curl}} p^*_{k-1} = 0 \qquad \forall p_{k-1}\in P^*_{k-1}, \tag{4.53}$$

where clearly $\underline{\mathrm{curl}} p = (\partial_2 p, -\partial_1 p)$. In order to deduce

$$\int_K \underline{\Phi}\cdot\underline{p}_{k-2} = 0 \qquad \forall \underline{p}_{k-2}\in (P_{k-2})^2, \tag{4.54}$$

(giving finally $\underline{\Phi} = \underline{0}$ and then $\underline{\underline{\tau}} = 0$ as desired), we should show that every $\underline{p}_{k-2}$ in $(P_{k-2})^2$ can be written as $\underline{\nabla}\, p_{k-1} + \underline{\mathrm{curl}} p^*_{k-1}$. For this, taken a $\underline{p}_{k-2}$ in $(P_{k-2})^2$ we can first find a p^* in P^*_{k-1} such that $-\Delta p^* = \mathrm{rot}\underline{p}_{k-2}$ (where $\mathrm{rot}\underline{p} := -\partial_2 p^1 + \partial_1 p^2$). Then we observe that $\mathrm{rot}(\underline{p}_{k-2} - \underline{\mathrm{curl}} p^*) = 0$ and hence $\underline{p}_{k-2} - \underline{\mathrm{curl}} p^*$ is a gradient.

Finally the proof of (4.16) with C independent of h follows by the usual arguments. ☐

It is not difficult to check that with this interpolation operator one can easily show that the assumptions of Theorem 3.1 are satisfied, and hence the element has optimal convergence properties.

However, in order to show that in our case the assumptions of Theorem 3.1 are satisfied, we only need to check **the existence** of an interpolation operator Π_h, uniformly bounded in h, such that

$$\int_\Omega \underline{\mathrm{div}}(\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) \cdot \underline{v}_h \, \mathrm{d}x = 0 \qquad \forall \underline{v}_h \in U_h \tag{4.55}$$

$$\int_\Omega \underline{\underline{as}}(\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : \mathcal{S}^2(v_h) \, \mathrm{d}x = 0 \qquad \forall v_h \in W_h. \tag{4.56}$$

To show **the existence** of such an interpolation operator we could then follow a different path. The idea is to consider first the subset $\mathbb{H}_k$ made of tensors $\underline{\underline{\kappa}}$ in $(P_k)^{2\times 2}$ such that, for each element K:

$$\int_\ell [\underline{\underline{\kappa}} \cdot \underline{n}] \cdot \underline{p}_k = 0 \qquad \forall \text{ edge } \ell \text{ and } \forall\, \underline{v}_h \in (P_k)^2, \tag{4.57}$$

$$\int_K \underline{\underline{\kappa}} : \underline{\underline{\nabla}}\, \underline{p}_{k-1} = 0 \qquad \forall \underline{p}_{k-1} \in (P_{k-1})^2. \tag{4.58}$$

We remark that the tensors in $\mathbb{H}_k$ will automatically satisfy a weak symmetry condition, namely

$$\int_K as(\underline{\underline{\kappa}}) = 0.$$

We consider then the degrees of freedom

$$\int_K as(\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}})\, \tilde{p}_{k-1} = 0 \qquad \forall \tilde{p}_{k-1} \in \tilde{P}_{k-1}(K), \tag{4.59}$$

and we consider the subset $\mathbb{K}_k$ of $\mathbb{H}_k$ made of those $\underline{\underline{\kappa}} \in \mathbb{H}_k$ such that

$$\int_K as(\underline{\underline{\kappa}})\, \tilde{p}_{k-1} = 0 \qquad \forall \tilde{p}_{k-1} \in \tilde{P}_{k-1}(K). \tag{4.60}$$

Finally we complete the set of degrees of freedom with

$$\int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : \underline{\underline{\kappa}}_k = 0 \qquad \forall \underline{\underline{\kappa}}_k \in \mathbb{K}_k. \tag{4.61}$$

It is clear that if $\underline{\underline{\tau}} = 0$ the only $\Pi_h \underline{\underline{\tau}} \in (P_k)^{2\times 2}$ that satisfies (4.43), (4.44), (4.59), and (4.61) is $\Pi_h \underline{\underline{\tau}} = \underline{\underline{0}}$. This however is not enough to prove that the interpolator defined by (4.43)-(4.44), (4.59), and (4.61) is well defined. Indeed, we should show that the degrees of freedom (4.43), (4.44),and (4.59) are *linearly independent*.

Theorem 4.5. *Consider the element described in* (4.27)*. Then the interpolation operator defined by* (4.43)*,*(4.44)*,* (4.59) *and* (4.61) *is unisolvent and verifies* (4.16).

Proof. As we know (from the classical **BDM** interpolation on each row) that (4.43) and (4.44) *are* linearly independent we have only to show that for every element K

$$\forall \tilde{p}_{k-1} \in \tilde{P}_{k-1}(K): \qquad \Big[\int_K as(\underline{\underline{\kappa}})\, \tilde{p}_{k-1}(K) = 0 \;\forall \underline{\underline{\kappa}} \in \mathbb{H}_k \Big] \Rightarrow \tilde{p}_{k-1}(K) = 0, \tag{4.62}$$

where, as before, $\mathbb{H}_k$ is defined as the set of $\underline{\underline{\kappa}} \in (P_k)^{2\times 2}$ that satisfy (4.57) and (4.58).

The proof of (4.62) however is quite easy. Indeed, using the structure (4.32) for the elements in $\mathbb{H}_k$ we have to prove that for every polynomial $\tilde{p}_{k-1}$ in P_{k-1} having zero mean value on K, **if**

$$\int_K (\partial_1 \phi^1 + \partial_2 \phi^2) \tilde{p}_{k-1} = 0 \tag{4.63}$$

for all polynomials ϕ^1 and ϕ^2 of degree $\leq k+1$ vanishing on ∂K, **then** $\tilde{p}_{k-1} = 0$. Integrating by parts, we have from (4.63) that (both)

$$\int_K \phi \partial_1 \tilde{p}_{k-1} = 0 \quad \text{and} \quad \int_K \phi \partial_2 \tilde{p}_{k-1} = 0$$

for every ϕ of degree $k+1$ vanishing on ∂K. This easily implies that $\partial_1 \tilde{p}_{k-1} = \partial_2 \tilde{p}_{k-1} = 0$ and hence $\tilde{p}_{k-1} = 0$ since it has zero mean value. Finally (4.16) with C independent of h follows, as before, by the usual arguments. □

Remark 13. Considering the lowest order case, we observe that on each edge the space Σ_h will have *four* degrees of freedom. For the general case $k > 1$ we would have $2k+2$ degrees of freedom per edge. We can try to reduce this number introducing on each edge e a local basis with normal and tangential vectors $\underline{n}$ and $\underline{t}$. Then on every edge e we denote by $\hat{P}_{k-1}(e)$ the homogeneous polynomials of degree $k-1$ on e. Taking into account that x_n is constant on e and forgetting redundant conditions, we get from (4.18):

$$\int_e (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}})_{nn}\, x_t\, \hat{q}\, \mathrm{d}s = 0 \quad \forall \hat{q} \in \hat{P}_{k-1}(e)$$

and

$$\int_e (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) \cdot \underline{n} \cdot \underline{q}\, \mathrm{d}s = 0 \quad \forall \underline{q} \in (P_{k-1}(e))^2.$$

We thus see that the normal component must be one degree higher than the tangential component.

Remark 14. As we have seen, the case k=1 of Example 8 was introduced in [13]. The reduced case was also considered there, and it was shown that it was sufficient to have, on each edge e, $\tau_{nn} \in P_1(e)$ for the normal component but only $\tau_{nt} \in P_0(e)$ for the tangential component.

Remark 15. It must be noted that the stability of all the elements of the family could be proved applying Proposition 3, and using the fact that $(\mathcal{L}^1_{k+1})^2 - \mathcal{L}^0_{k-1}$ is a stable Stokes element (this is indeed the *fourth* of the proofs we announced). However this last result is false in the three-dimensional case and this line of proof will not be applicable.

4.3. **The three-dimensional case.** In three dimensions, the analogue of Theorem 4.1 is the following one.

Theorem 4.6. *For $n = 3$, assume that $\Sigma_h \times U_h$ are such that*

$$\underline{\mathrm{div}}(\Sigma_h) \subseteq U_h. \tag{4.64}$$

We now introduce a space W_h of vector functions so that $X_h = \mathcal{S}^3(W_h)$ defines the reduced symmetry. We assume that there exists a piecewise polynomial space Ξ_h such that $((\Xi_h)^3, W_h)$ is a stable Stokes element. Finally we suppose that there exists a mapping Π_h from $(H^1(\Omega))^{3\times 3}_S + (\underline{\mathrm{curl}}(\Xi_h)^3)^3$ into Σ_h satisfying for each element K:

$$\|\Pi_h \underline{\underline{\tau}}\|_{\underline{\underline{H}}(\underline{\mathrm{div}};K)} \leq C\, \|\underline{\underline{\tau}}\|_{(H^1(K))^{n\times n}_S} \tag{4.65}$$

together with the following properties:

$$\int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : (\underline{\underline{\nabla}}\, \underline{v}_h + \underline{x} \wedge \underline{\underline{\nabla}}\, \underline{w}_h)\, \mathrm{d}x = 0 \qquad \forall\, \underline{v}_h \in U_h,\ \forall \underline{w}_h \in W_h, \tag{4.66}$$

$$\int_f (\underline{\underline{\tau}} - \Pi_h\underline{\underline{\tau}})\cdot \underline{n}_K \cdot (\underline{v}_h + \underline{x}\wedge \underline{w}_h)\,\mathrm{d}s = 0 \quad \forall\ \textit{face } f, \quad \forall\, \underline{v}_h \in U_h,\ \forall \underline{w}_h \in W_h. \tag{4.67}$$

Then the triplet

$$\Sigma_h \times U_h \times X_h \quad \textit{with } X_h = \mathcal{S}^3(W_h) \tag{4.68}$$

satisfies the conditions of Theorem 3.1.

Proof. The inclusion of kernels (3.26) follows easily from (4.64). From the convexity assumption, we easily have that for every $\underline{u}_h \in U_h \subseteq (L^2(\Omega))^n$ we can find a $\underline{\underline{\tau}} = \underline{\underline{\tau}}(\underline{u}_h) \in (H^1(\Omega))^{n\times n}_S$ such that

$$\mathrm{d\underline{i}v}\underline{\underline{\tau}} = \underline{u}_h \quad \text{and} \quad \|\underline{\underline{\tau}}\|_{(H^1(\Omega))^{n\times n}_S} \le C\|\underline{u}_h\|_{(L^2(\Omega))^n}.$$

Then using (4.64), (4.66) with $\underline{w}_h = 0$ and (4.67) with $w_h = 0$ we easily have that

$$\mathrm{d\underline{i}v}\Pi_h\underline{\underline{\tau}} = \underline{u}_h \quad \text{and} \quad \|\Pi_h\underline{\underline{\tau}}\|_{\underline{\underline{H}}(\mathrm{d\underline{i}v};\Omega)} \le C\|\underline{u}_h\|_{(L^2(\Omega))^n}. \tag{4.69}$$

Hence the pair of spaces (Σ_h, U_h) satisfy condition (3.24), just by taking $\underline{\underline{\tau}}^1_h := \Pi_h\underline{\underline{\tau}}$. In particular from (4.66) and (4.67) we have that $\mathrm{d\underline{i}v}(\underline{\underline{\tau}} - \Pi_h\underline{\underline{\tau}}) = 0$, so that we can apply formula (4.13). At this point using (3.3) and (4.68), then (4.4), and finally employing (4.66) and (4.67) in (4.13), gives

$$c(\underline{\underline{\tau}}-\Pi_h\underline{\underline{\tau}}, \underline{\underline{\phi}}_h) \equiv \int_K \underline{\underline{\mathrm{as}}}(\underline{\underline{\tau}}-\Pi_h\underline{\underline{\tau}}) : \mathcal{S}^3(\underline{w}_h)\,\mathrm{d}x = 0 \qquad \forall \underline{\underline{\phi}}_h \equiv \mathcal{S}^3(\underline{w}_h) \in X_h \equiv \mathcal{S}^3(W_h).$$

Now for every $\underline{\underline{\gamma}}_h \in X_h$ we can repeat the construction of the previous section (and in particular in the three-dimensional case of Proposition 2) to construct a $\underline{\underline{\tau}}^2(h) \in (\mathrm{c\underline{\underline{ur}}l}(\Xi_h)^3)^3$ such that $\mathrm{d\underline{i}v}\underline{\underline{\tau}}^2(h) = 0$ and

$$(\underline{\underline{as}}(\underline{\underline{\tau}}^2(h)), \underline{\underline{\phi}}_h) = (\underline{\underline{\gamma}}_h, \underline{\underline{\phi}}_h) \qquad \forall \underline{\underline{\phi}}_h \in X_h.$$

It is clear, from the previous discussion, that $\Pi_h\underline{\underline{\tau}}^2(h)$ will satisfy

$$\mathrm{d\underline{i}v}\Pi_h\underline{\underline{\tau}}^2(h) = 0 \quad \text{and} \quad c(\underline{\underline{\tau}}^2(h) - \Pi_h\underline{\underline{\tau}}^2(h), \underline{\underline{\phi}}_h) = 0\ \forall \underline{\underline{\phi}}_h \in X_h.$$

We can finally take $\underline{\underline{\tau}}_h := \Pi_h(\underline{\underline{\tau}}+\underline{\underline{\tau}}^2(h))$ to show that (3.23), and therefore (3.22), hold true. ☐

Remark 16. It is clear that the discussions presented in Remark 9 and in Remark 10 apply (with few obvious and minimal changes) to the three dimensional case as well

Example 7. The lowest degree element of the Arnold–Falk–Winther family in three dimensions. Our first example will be the three-dimensional Arnold-Falk-Winther element citeArnold-Falk-Winther of the lowest degree. The element is given by the choice of spaces

$$\begin{aligned} &\Sigma_h = (\mathbf{BDM}_1)^3 \equiv ((\mathfrak{L}^1_1)^3)^3 \\ &U_h = (\mathfrak{L}^0_0)^3 \\ &X_h = \mathcal{S}^3((\mathfrak{L}^0_0)^3). \end{aligned} \tag{4.70}$$

For these spaces the interpolation operator defined in (4.66), and(4.67) is simply

$$\int_f (\underline{\underline{\tau}} - \Pi_h\underline{\underline{\tau}})\cdot \underline{n}_K \cdot \underline{p}_1\,\mathrm{d}s = 0 \qquad \text{for all face } f \text{ and for all } \underline{p}_1 \in (P_1(f))^3, \tag{4.71}$$

that corresponds to apply three times the $\mathbf{BDM}_1$ interpolation operator (one for each row). Theorem 4.6 applies easily and gives immediately the result.

Example 8. The general case of the Arnold–Falk–Winther family. The general case $k > 1$ of the Arnold–Falk–Winther family is given by

$$\begin{array}{l} \Sigma_h = (\mathbf{BDM}_k)^3 \equiv ((\mathfrak{L}_k^1)^3)^3 \\ U_h = (\mathfrak{L}_{k-1}^0)^3 \\ X_h = \mathcal{S}^3((\mathfrak{L}_{k-1}^0)^3). \end{array} \tag{4.72}$$

The corresponding interpolation operator is given on each K by

$$\int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : (\underline{\underline{v}}_h + \underline{\underline{w}}_h \wedge \underline{x})\,\mathrm{d}x = 0 \qquad \forall\, \underline{\underline{v}}_h \in (P_{k-2})^{3\times 3}\, \forall \underline{\underline{w}}_h \in (P_{k-2}^h)^{3\times 3}, \tag{4.73}$$

$$\int_f (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) \cdot \underline{n}_K \cdot \underline{v}_h \,\mathrm{d}s = 0 \qquad \forall \text{ face } f, \forall\, \underline{v}_h \in (P_k)^3. \tag{4.74}$$

Again, note the difference between $\underline{x} \wedge \underline{\underline{\nabla}}\, \underline{w}_h$ in (4.66) and $\underline{\underline{w}}_h \wedge \underline{x}$ in (4.73) that forbids the use of the AFW interpolator in Theorem 4.6 for $k > 1$. However we will be able to prove the optimal convergence of the element by using the three dimensional version of Theorem 4.5.

Similarly to Proposition 5 we have, in any case, the following equivalence result.

Proposition 6. *Consider the element described in* (4.72). *If an interpolation operator satisfies*

$$\int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : \underline{\underline{\nabla}}\, \underline{v}_h \,\mathrm{d}x = 0 \qquad \forall\, \underline{v}_h \in (P_{k-1})^3, \tag{4.75}$$

and

$$\int_f (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) \cdot \underline{n}_K \cdot \underline{v}_h \,\mathrm{d}s = 0 \qquad \forall \textit{ face } f, \forall\, \underline{v}_h \in (P_k)^3, \tag{4.76}$$

then the requirement

$$\int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : \underline{x} \wedge \underline{\underline{\nabla}}(\underline{\phi}_h)\,\mathrm{d}x = 0 \qquad \forall\, \underline{\phi}_h \in (P_{k-2}^h)^3 \tag{4.77}$$

is **equivalent** *to the requirement that*

$$\int_K \underline{\underline{as}}(\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : \mathcal{S}^3(\underline{\phi}_h)\,\mathrm{d}x = 0 \qquad \forall\, \underline{\phi}_h \in (P_{k-2}^h)^3. \tag{4.78}$$

The proof is an easy exercise based on formula (4.13).

To present the three-dimensional version of Theorem 4.5, we define now, in analogy with the two-dimensional case, $\mathbb{H}_k$ as the subset of $(P_k)^{3\times 3}$ made of tensors $\underline{\underline{\kappa}}$ such that on each K

$$\int_f [\underline{\underline{\kappa}} \cdot \underline{n}] \cdot \underline{p}_k = 0 \qquad \forall \text{ face } f \text{ and } \forall\, \underline{v}_h \in (P_k)^3, \tag{4.79}$$

$$\int_K \underline{\underline{\kappa}} : \underline{\underline{\nabla}}\, \underline{p}_{k-1} = 0 \qquad \forall \underline{p}_{k-1} \in (P_{k-1})^3. \tag{4.80}$$

We then define $\mathbb{K}_k$ as the subset of $\mathbb{H}_k$ made of tensors that satisfy also

$$\int_K \underline{\underline{\kappa}} : \mathcal{S}^3(\underline{\tilde{p}}_{k-1}) = 0 \qquad \forall \underline{\tilde{p}}_{k-1} \in (\tilde{P}_{k-1}(K))^3. \tag{4.81}$$

The correspondent of Theorem 4.5 is now

Theorem 4.7. *Consider the element described in* (4.72). *Then the interpolation operator defined on each* K *by* (4.75), (4.76), (4.78) *and*

$$\int_K (\underline{\underline{\tau}} - \Pi_h \underline{\underline{\tau}}) : \underline{\underline{\kappa}}_k \, dx = 0 \qquad \forall \underline{\underline{\kappa}}_k \in \mathbb{K}_k \tag{4.82}$$

is unisolvent and verifies (4.65)

Proof. As for Proposition 4.5 we must prove that *for every* $\underline{\tilde{p}} \in \underline{\tilde{P}}_{k-1}$, **if**

$$\int_T [(\kappa_{12} - \kappa_{21})\tilde{p}_3 - (\kappa_{13} - \kappa_{31})\tilde{p}_2 + (\kappa_{23} - \kappa_{32})\tilde{p}_1] = 0 \quad \forall \underline{\underline{\kappa}} \in \mathbb{H}_k \tag{4.83}$$

then $\underline{\tilde{p}} = \underline{0}$. We prove it first on the reference element. Assume therefore that

$$K = \{(x, y, z) | x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}.$$

It is not difficult to see that the elements $\underline{\underline{\tau}}$ that satisfy (4.79) and (4.80) have zero divergence and zero trace of the normal component on ∂K. It is not difficult either to see that this includes all tensors of the form

$$\underline{\underline{\tau}} = \begin{pmatrix} \phi^3_y - \phi^2_z & \phi^1_z - \phi^3_x & \phi^2_x - \phi^1_y \\ \psi^3_y - \psi^2_z & \psi^1_z - \psi^3_x & \psi^2_x - \psi^1_y \\ \chi^3_y - \chi^2_z & \chi^1_z - \chi^3_x & \chi^2_x - \chi^1_y \end{pmatrix} \tag{4.84}$$

where **each vector** ($\underline{\Phi}$, $\underline{\Psi}$, or $\underline{\mathbf{X}}$) has the form

$$yz\pi q^1_{k-2}\underline{n}^1 + xz\pi q^2_{k-2}\underline{n}^2 + xy\pi q^3_{k-2}\underline{n}^3 + xyz q^4_{k-2}\underline{n}^4 \tag{4.85}$$

and where:

- the q^i_{k-2} are arbitrary polynomials of degree $\leq\ k-2$
- $\pi := 1 - (x + y + z)$,
- for $i = 1, 2, 3$ we denoted by $\underline{n}^i$ the outward unit normal to the face $x_i = 0$
- $\underline{n}^4 \equiv -3^{-1/2}(\underline{n}^1 + \underline{n}^2 + \underline{n}^3)$ is the unit outward normal to the last face.

Inserting (4.84) into (4.83) we have

$$\begin{aligned} \int_T [(\phi^1_z - \phi^3_x - (\psi^3_y - \psi^2_z))\tilde{p}_3 \\ - \int_T (\phi^2_x - \phi^1_y - (\chi^3_y - \chi^2_z))\tilde{p}_2 \\ + \int_T (\psi^2_x - \psi^1_y - (\chi^1_z - \chi^3_x))\tilde{p}_1] = 0 \end{aligned} \tag{4.86}$$

for all vectors $\underline{\Phi}$, $\underline{\Psi}$, or $\underline{\mathbf{X}}$ having the form (4.85). Now note that if we take

$$\underline{\Phi} = xy\pi q^3_{k-2}\underline{n}^3 \text{ with } \underline{\Psi} = \underline{\mathbf{X}} = \underline{0}$$

and integrate by parts, then (4.86) reduces to

$$\int_T xy\pi q^3_{k-2}\partial_1 \tilde{p}_3 = 0 \qquad \forall q^3_{k-2}$$

easily implying

$$\partial_1 \tilde{p}_3 = 0. \tag{4.87}$$

With a similar argument taking successively

$$\underline{\Psi} = xy\pi q^3_{k-2}\underline{n}^3 \text{ and } \underline{\Phi} = \underline{\mathbf{X}} = \underline{0},$$
$$\underline{\Phi} = xz\pi q^2_{k-2}\underline{n}^2 \text{ and } \underline{\Psi} = \underline{\mathbf{X}} = \underline{0},$$
$$\underline{\mathbf{X}} = xz\pi q^2_{k-2}\underline{n}^2 \text{ and } \underline{\Phi} = \underline{\Psi} = \underline{0},$$
$$\underline{\Psi} = yz\pi q^1_{k-2}\underline{n}^1 \text{ and } \underline{\Phi} = \underline{\mathbf{X}} = \underline{0},$$
$$\underline{\mathbf{X}} = yz\pi q^1_{k-2}\underline{n}^1 \text{ and } \underline{\Phi} = \underline{\Psi} = \underline{0},$$

we also get *respectively*

$$\partial_2\tilde{p}_3 = 0, \quad \partial_1\tilde{p}_2 = 0, \quad \partial_3\tilde{p}_2 = 0, \quad \partial_2\tilde{p}_1 = 0, \text{ and } \partial_3\tilde{p}_1 = 0. \tag{4.88}$$

If instead we take

$$\underline{\Phi} = yz\pi q^1_{k-2}\underline{n}^1 \quad \text{and } \underline{\Psi} = \underline{\mathbf{X}} = \underline{0},$$

then (4.86) reduces to

$$\int_T yz\pi q^1_{k-2}(-\partial_3\tilde{p}_3 - \partial_2\tilde{p}_2) = 0 \qquad \forall q^1_{k-2}$$

that gives

$$\partial_3\tilde{p}_3 + \partial_2\tilde{p}_2 = 0. \tag{4.89}$$

With a similar argument, taking successively

$$\underline{\Psi} = xz\pi q^2_{k-2}\underline{n}^2 \quad \text{and } \underline{\Phi} = \underline{\mathbf{X}} = \underline{0},$$
$$\underline{\mathbf{X}} = xy\pi q^3_{k-2}\underline{n}^3 \quad \text{and } \underline{\Phi} = \underline{\Psi} = \underline{0},$$

we obtain

$$\partial_3\tilde{p}_3 + \partial_1\tilde{p}_1 = 0 \text{ and } \partial_2\tilde{p}_2 + \partial_1\tilde{p}_1 = 0. \tag{4.90}$$

Joining (4.89) with (4.90) we easily obtain

$$\partial_3\tilde{p}_3 = \partial_2\tilde{p}_2 = \partial_1\tilde{p}_1 = 0,$$

that together with (4.87) and (4.88) (and recalling that the $\tilde{p}$'s have zero mean value on K) yield $\tilde{p}^1 = \tilde{p}^2 = \tilde{p}^3 = 0$. The proof on the current element is then obtained by the usual second Piola-Kirchhoff tensor mapping

$$\underline{\underline{\tau}} = \underline{\underline{F}}\,\underline{\underline{\hat{\tau}}}\,\underline{\underline{F}}^t$$

where F is the Jacobian of the transformation that maps the reference element $\hat{K}$ on the current element K. Finally (4.65) follows by the usual arguments. □

Remark 17. It is clear that the form (4.85) is redundant. For instance for $k = 2$ it is immediate to see that the gradient of the quartic bubble $b_4 := xyz\pi$ has exactly the form (4.85) but its curl is obviously zero. In particular, for $k = 2$, we could use only the first three terms in (4.85) with the q^i constant.

Remark 18. Always in the case $k = 2$, it is not difficult to see that $\mathbb{K}_k = \underline{\underline{0}}$. As a hint, take for instance $\underline{p} = (y - y_B, 0, 0)$ (where (x_B, y_B, z_B) is the barycenter of K) in (4.81), and use the form (4.86) to reach

$$\int_K \psi^1 \,\mathrm{d}x = 0,$$

that using the form (4.85) (using only three terms, as for Remark 17) gives easily

$$\psi^1 = 0,$$

With a procedure that resembles closely the proof of Theorem 4.7 we then reach

$$\underline{\Phi} = \underline{\Psi} = \underline{\mathbf{X}} = \underline{0}.$$

This, in a certain sense, could be seen as the natural extension of Theorems 4.2 and 4.4 to the three-dimensional case, at least for $k = 2$.

Remark 19. We recall that the first member of the the Arnold–Falk–Winther family is given by (4.72). To specify the tensor $\underline{\underline{\tau}}_h$ we need 9 d.o.f. on each face. This is obviously not optimal in regard of (4.66) and (4.67) where only 6 conditions are generated by $\underline{v}_h$ and $\underline{w}_h$ in $U_h = (\mathfrak{L}^0_0)^3$.

It is indeed stated in [4] that is possible to reduce the number of degrees of freedom on each face from nine to 6. This is not as good as in the two-dimensional case where the tangential part can be taken constant. The proper choice of space, using the right number of degrees of freedom will be a consequence of the following general result.

Let f be a face of some tetrahedron of a mesh. Let $P_k(f)$ and $\hat{P}_k(f)$ denote respectively the set of polynomials and homogeneous polynomials of degree k on f. Let $\underline{n}$ be the normal to the face and denote τ_{nn} and $\underline{\tau}_{nT} = \underline{\tau}_n - \tau_{nn}\underline{n}$ the normal and tangential part of the vector $\underline{\tau}_n$. We also denote $\underline{x}_T^{\perp} := \underline{x} \wedge \underline{n}$. With an abuse of language, we shall consider $\underline{\tau}_{nT}$ and $\underline{x}_T^{\perp}$ as two-dimensional vectors, and more generally we will identify, whenever convenient, all vectors tangential to the face f with their two-dimensional projection on f.

Proposition 7. *In order to satisfy* (4.67)*, we need on each face* f

$$\tau_{nn} \in P_k(f)$$

and for the tangential part

$$\underline{\tau}_{nT} \in N_{k-1}(f) := \underline{P}_{k-1}(f) + \underline{x}_T^{\perp}\hat{\underline{P}}_{k-1}(f)$$

Proof. Let $U_h = (\mathfrak{L}^0_{k-1})^3$ and consider the space $U_h + \underline{x} \wedge U_h$. It is easy to see that in order to generate this space, it is sufficient to consider functions of the form

$$\underline{p} + \underline{x} \wedge \hat{\underline{q}}$$

where $\underline{p} \in \underline{P}_{k-1}$ is a *general* vector valued polynomial of degree $k-1$ but $\hat{\underline{q}} \in \hat{\underline{P}}_{k-1}$ is a vector valued *homogeneous* polynomial of degree $k-1$. Now we want to evaluate on a face f of some tetrahedron

$$(\underline{p} + (\underline{x} \wedge \hat{\underline{q}}_h)) \cdot \underline{\underline{\tau}} \cdot \underline{n}.$$

To do so, we use on the face a set of orthogonal co-ordinates defined by the normal $\underline{n}$ and two tangential vectors, $\underline{s}$ and $\underline{t}$. We then write,

$$\begin{aligned}
\underline{x} &= x_n\underline{n} + x_s\underline{s} + x_t\underline{t}\\
\underline{p} &= p_n\underline{n} + p_s\underline{s} + p_t\underline{t}\\
\hat{\underline{q}} &= \hat{q}_n\underline{n} + \hat{q}_s\underline{s} + \hat{q}_t\underline{t}\\
\underline{\underline{\tau}} \cdot \underline{n} &= \tau_{nn}\underline{n} + \tau_{ns}\underline{s} + \tau_{nt}\underline{t}
\end{aligned}$$

An elementary computation then yields

$$\begin{aligned}
(\underline{p} + (\underline{x} \wedge \hat{\underline{q}})) \cdot \underline{\underline{\tau}} \cdot \underline{n} = (p_n - x_t\hat{q}_s + x_s\hat{q}_t)\tau_{nn}&\\
+ (p_s - x_n\hat{q}_t + x_t\hat{q}_n)\tau_{ns} + (p_t + x_n\hat{q}_s - x_s\hat{q}_n)\tau_{nt}&
\end{aligned}$$

Now we see that τ_{nn} is multiplied by a full polynomial of degree k and we need it to be of the same order. For the tangential terms, we recall that x_n is constant on the face so that the terms $x_n\hat{q}_t$ and $x_n\hat{q}_s$ can be absorbed by p_s and p_t respectively. That leaves us with something of the form

$$\begin{pmatrix} \tau_{ns} \\ \tau_{nt} \end{pmatrix} \cdot \begin{pmatrix} p_s + x_t\hat{q}_n \\ p_t - x_s\hat{q}_n \end{pmatrix}$$

which shows that the tangential part of $\underline{\tau}_n$ needs only to be in $N_{k-1}(f)$ which is a space smaller than $\underline{P}_k(f)$. □

Remark 20. A simple count shows that the number of degrees of freedom on each face is then

$$(k+1)(k+2)/2 + k(k+1) + k,$$

while for the whole $(\mathbf{BDM}_k)^3$ we would have

$$3(k+1)(k+2)/2.$$

For $k = 1$ we have 6 instead of 9 and for $k = 2$, 14 instead of 18.

5. **Conclusion.** We have been able to obtain by elementary methods many constructions of tensors with relaxed symmetry for mixed elasticity problems. Some of them have a large number of degrees of freedom which makes them difficult to use. However it seems that the price of symmetry is high, especially if inclusion of kernels, which implies strong conservation of momentum, is required.

Acknowledgements. We thank D. N. Arnold and R. S. Falk for the fruitful discussions. The work of the second author has been partially supported by Italian PRIN 2006.

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Received April 2008; revised August 2008.

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