Well-posedness and Time Discretization of a Nonlinear Volterra Integrodifferential Equation*

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Abstract

This paper deals with a nonlinear abstract evolution integrodifferential equation of Volterra type. This equation contains two nonlinearities, namely a maximal monotone operator and a Lipschitz continuous operator, and may arise from a Stefan problem in materials with memory. Uniqueness of a solution to the associated Cauchy problem is proved. Then, global existence is achieved via a semi-implicit time discretization procedure. Moreover, some estimates for the discretization error are established.

Key words: nonlinear Volterra integrodifferential equation, abstract Cauchy problem, time discretization, existence and uniqueness, convergence result, error estimate.

AMS (MOS) Subject Classification: 80A22, 45K05, 45L10, 65M15.

1 Introduction

The present analysis is concerned with a nonlinear Volterra integrodifferential equation governing the evolution of two unknown fields, u and ϑ. These must also satisfy a relation induced by a maximal monotone graph $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$. More precisely, letting Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with a smooth boundary $\partial\Omega$, and T be some final time, we deal with the following equation and inclusion in $Q := \Omega \times (0,T)$

$$u_t - \Delta(\vartheta + k * \vartheta) = g(\vartheta) + f \qquad \text{a.e. in } Q, \tag{1.1}$$

$$\vartheta + \gamma(\vartheta) \ni u \qquad \text{a.e. in } Q. \tag{1.2}$$

Here, k stands for a time dependent memory kernel and $*$ denotes the standard convolution product on $(0,t)$, namely $(k*\vartheta)(\cdot,t) := \int_0^t k(t-s)\vartheta(\cdot,s)\,ds$ for $t \in (0,T)$. Moreover, g and f represent source terms while γ is a completely arbitrary maximal monotone graph in $\mathbb{R} \times \mathbb{R}$.

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The model (1.1)-(1.2) is of relevant interest within applications. Referring to the theory of heat conduction in materials with memory (see, e.g., [9, 10] and references therein), we may interpret u and ϑ as the enthalpy and the relative temperature, respectively. Then, we introduce the constitutive assumptions on the internal energy and the heat flux

$$\begin{aligned} e(x,t) &= u(x,t), \\ \boldsymbol{q}(x,t) &= -\nabla\vartheta(x,t) - \int_{-\infty}^{t} k(t-s)\,\nabla\vartheta(x,s)\,ds \end{aligned}$$

for any $(x,t)\in\Omega\times(0,T)$, and assume that the past history of ϑ is known up to time $t=0$. Within this setting, we may consider the energy balance equation

$$e_t + \operatorname{div}\boldsymbol{q} = \widetilde{f} \quad \text{in } \ Q,$$

where $\widetilde{f}$ stands for the heat supply that can be assumed to depend on the actual value of the temperature as well. At a first glance, it appears that (1.1) is nothing but the previous equation where the above constitutive assumptions are plugged in, but we include in f also the past history on ϑ and deal with a nonlinear contribution $g(\vartheta)$.

Regarding the memory kernel, we may ask k to be smooth enough and such that the inequality

$$\int_0^t \big(v(s) + (k*v)(s)\big)\,v(s)\,ds \ge \omega \int_0^t v^2(s)\,ds, \tag{1.3}$$

is fulfilled for a suitable $\omega>0$, and for all $v\in L^2(0,T)$ and $t\in(0,T)$. It is worth recalling that condition (1.3) makes the model consistent with the Second Principle of Thermodynamics (see, for instance, [14]). However, we stress that the property (1.3) is not assumed in our analysis. Indeed, we just ask for a suitably smooth kernel

$$k\in W^{1,1}(0,T).$$

On the other hand, the inclusion (1.2) accounts for the phase transition occurring in the material. In particular, a significant example of graph γ is $\gamma(r)=\mathcal{H}(r)$, where $\mathcal{H}$ denotes the Heaviside graph, namely $\mathcal{H}(r)=0$ if $r<0$, $\mathcal{H}(r)=[0,1]$ if $r=0$, and $\mathcal{H}(r)=1$ if $r>0$. Indeed, this choice corresponds to the Stefan problem for materials with memory and has been investigated for instance in [2, 9, 10].

The problem (1.1)-(1.2) has to be supplied with initial and boundary conditions. To this end, we prescribe

$$-\partial_\nu(\vartheta + k*\vartheta) = \lambda(\vartheta-\vartheta_b) + h \quad \text{on } \ \partial\Omega\times(0,T), \tag{1.4}$$

$$u(\cdot,0) = u^0 \qquad \text{a.e. on } \ \Omega, \tag{1.5}$$

where ∂_ν indicates the outward normal derivative on the boundary $\partial\Omega$, λ is a positive constant, and $\vartheta_b, h:\partial\Omega\times(0,T)\longrightarrow\mathbb{R}$, $u^0:\Omega\longrightarrow\mathbb{R}$ are given functions. Note that (1.4) comes from the law stating that the normal component of the heat flux $\boldsymbol{q}$ on the boundary is proportional to the difference of internal ($\vartheta|_{\partial\Omega}$) and external ($\vartheta_b$) temperatures. In this context, the supplementary datum h accounts for

$$\partial_\nu\left(\int_{-\infty}^{0} k(t-s)\,\nabla\vartheta(x,s)\,ds\right).$$

Various initial and boundary value problems concerning systems close to (1.1)-(1.2) have been examined. Indeed, the related results often deal with more general integral equations, possibly including a memory effect also on the enthalpy u, that is, $e := u+\varphi*u$ for a suitable kernel φ.

Of course, in the case when $k \equiv 0$, the model reduces to a well-known equation (see, for instance, [13, 21]). Moreover, accounting for a non-vanishing kernel k but neglecting the nonlinearity g in (1.1), existence and longtime behavior of solutions have been analyzed for a wide class of problems; we refer, in particular, to [1, 6, 11] and the references therein. Actually, an existence result for (1.1)-(1.2) (thus accounting for its doubly nonlinear character) has already been obtained in [2], where, nevertheless, the authors impose restrictions on the possible choice of the graph γ. Namely, the latter is asked to be at most linear at infinity.

In order to specify our results and for the sake of convenience, let us put problem (1.1)-(1.2), (1.4)-(1.5) in an abstract form from the very beginning. To this end, set

$$H := L^2(\Omega) \quad \text{and} \quad V := H^1(\Omega),$$

and identify H with its dual space H', so that

$$V \subset H \subset V'$$

with dense and continuous embeddings. Denoting by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V, and assuming the function g to be Lipschitz continuous, we introduce the operators A, B, and G defined by

$$\begin{aligned}
&A : V \longrightarrow V' \qquad \langle Av, w \rangle := \int_\Omega \nabla v \cdot \nabla w + \lambda \int_{\partial\Omega} v\, w \quad \forall v, w \in V,\\
&B : V \longrightarrow V' \qquad \langle Bv, w \rangle := \int_\Omega \nabla v \cdot \nabla w \quad \forall v, w \in V,\\
&G : H \longrightarrow H \qquad G(u)(\cdot) := g(u(\cdot)) \quad \forall u \in H.
\end{aligned}$$

Observe that A is coercive, while B is not. Next, we include the boundary data into the function $F : (0,T) \to V'$ defined by

$$\langle F(t), v \rangle := \int_\Omega f(\cdot, t)\, v + \int_{\partial\Omega} (\lambda \vartheta_b + h)(\cdot, t)\, v, \quad \text{for a.e. } t \in (0,T), \quad \forall v \in V.$$

Moreover, let $j : \mathbb{R} \longrightarrow]-\infty, +\infty]$ be a convex, proper, and lower semicontinuous function such that $\gamma = \partial j$. Then, we associate to j the functionals J_H and J_V on H and V as follows

$$J_H(u) := \begin{cases} \displaystyle\int_\Omega j(u(x))\, dx & \text{if } u \in H \text{ and } j(u) \in L^1(\Omega) \\ +\infty & \text{if } u \in H \text{ and } j(u) \notin L^1(\Omega) \end{cases} \tag{1.6}$$

$$J_V(v) := J_H(v) \quad \text{if } v \in V. \tag{1.7}$$

As is well known, both J_H and J_V are convex, proper, and lower semicontinuous on H and V, respectively. Thus, the subdifferential $\partial_{V,V'} J_V : V \longrightarrow 2^{V'}$ turns out to be a maximal monotone operator. Assuming also $u^0 \in V'$ and $F \in L^2(0,T;V')$, and asking for the regularities $\vartheta \in L^2(0,T;V)$, $u \in H^1(0,T;V')$, problem (1.1)-(1.2), (1.4)-(1.5) can be set as

$$u' + A\vartheta + k * B\vartheta = G(\vartheta) + F \quad \text{in } V', \text{ a.e. in } (0,T), \tag{1.8}$$

$$\vartheta + \partial_{V,V'} J_V(\vartheta) \ni u \quad \text{in } V', \quad \text{a.e. in } (0,T), \tag{1.9}$$

$$u(0) = u^0, \tag{1.10}$$

where the prime obviously stands for the derivative with respect to time.

The main novelty of this paper is that of dealing with (1.9) (instead of (1.2)) together with the nonlinear source term g. Note that in this framework (1.9) is actually an extension of (1.2). Indeed, (1.9) entails (1.2) whenever, for instance, $u \in L^2(0,T;H)$ (cf. [12]) or γ is linearly bounded (cf. [2]) as will be made precise later.

Hence, our existence result provides a solution to the problem above for a completely arbitrary graph γ (thus, not requiring any linear boundedness). Let us point out that this extension turns out to include interesting applications. Indeed, we actually prove the existence of a solution to some *pseudo-parabolic* problems of the form

$$(\vartheta - \Delta\vartheta)_t - \Delta(\vartheta + k * \vartheta) = f - g(\vartheta) \tag{1.11}$$

which may occur in several diffusion processes [8, 15].

In addition, this paper provides an approximation of the abstract Cauchy problem. Indeed, the existence proof for the continuous problem is carried out by making use of a semi-implicit time discretization of (1.1)-(1.2). The key point of this procedure is the approximation of the term $k * B\vartheta$. In this direction, the reader is referred to the papers [4, 17, 18, 19], where the authors carefully analyze the numerical aspects of some discretization of memory terms. In particular, some quadrature procedures are devised and the related error estimates are deduced.

Here, the convolution product $k * B\vartheta$ is treated in a natural way. Namely, letting $\tau := T/N \, (N \in \mathbb{N})$ denote the time step and $\{k_i\}_{i=1}^N \in \mathbb{R}^N$, and $\{B\vartheta_i\}_{i=1}^N \in V'$ be approximations of k and $B\vartheta$, respectively, we replace $(k * B\vartheta)(t), t \in (0,T)$, by the quantities

$$\tau \sum_{j=1}^{i} k_{i-j+1} \, B\vartheta_j, \qquad i = 1, \ldots, N.$$

This choice turns out to be adequate for our case, since this kind of approximation leads to a discrete version of inequality (1.3) (see, e.g., [17, Sec. 4]), together with some interesting properties such as a discrete Young's inequality for convolutions. Moreover, there is a suitable convergence to the continuous counterpart as τ tends to zero (cf. (4.14)).

As a by-product of our analysis, we provide some convergence results of the discrete solution to the continuous one as the time step goes to 0. Moreover, an *a priori* estimate for the discretization error is recovered. This estimate depends solely on the data and requires no additional regularity for the solution. Namely, denoting by $\overline{\vartheta}_\tau$ the piecewise constant solution of the discretized problem with time step τ, we obtain (see (5.36))

$$\|\vartheta - \overline{\vartheta}_\tau\|_{L^2(0,T;L^2(\Omega))} + \|1 * (\vartheta - \overline{\vartheta}_\tau)\|_{C^0([0,T];H^1(\Omega))} \le C\sqrt{\tau}.$$

The remainder of the paper is organized as follows. In Section 2 the general assumptions on the data are stated and the main result is rigorously established. Then, Section 3 is devoted to prove the uniqueness of the solution, while Section 4 describes the details of the time discretization. Finally, the global existence of a solution is proved in Section 5, where we provide also the estimate for the discretization error.

2 Main result

We start by listing our assumptions on the data.

(A1) H and V are real Hilbert spaces such that $V \subset H$ densely and continuously. Moreover, H is identified with its dual space H', whence $V \subset H \subset V'$ with dense and continuous embeddings. The symbols $\langle \cdot, \cdot \rangle$, $(\cdot, \cdot)$, and $|\cdot|$ will denote the duality pairing between V' and V, the inner product in H, and the related norm in H, respectively.

(A2) $A : V \longrightarrow V'$ is a linear continuous symmetric operator. Moreover A is also coercive in the sense that

$$\langle Av, v \rangle \geq \alpha \|v\|_V^2 \qquad \forall v \in V,$$

for some positive constant α, where $\|\cdot\|_V$ stands for the natural norm in V. We define the following

$$((u, v)) := \langle Au, v \rangle \quad \forall u, v \in V \tag{2.1}$$

as an inner product in V which is equivalent to the natural one, and denote by $\|\cdot\|$, $((\cdot,\cdot))_*$, and $\|\cdot\|_*$ the corresponding norm in V, the induced inner product in V', and the related norm in V', respectively. Hence, A may serve for the Riesz isomorphism between V and V'.

(A3) $B : V \longrightarrow V'$ is a linear continuous operator.

(A4) $J : H \longrightarrow]-\infty, +\infty]$ is a convex, proper, and lower semicontinuous function.

Now, let $J_V : V \longrightarrow]-\infty, +\infty]$ be the restriction of J to V. It is a standard matter to verify that J_V is still convex, proper, and lower semicontinuous. Thus, the subdifferential $\partial_{V,V'} J_V : V \longrightarrow 2^{V'}$ is defined. For the reader's convenience, we recall here its definition

$$\begin{gathered} u \in \partial_{V,V'} J_V(v) \quad \text{if and only if} \\ u \in V', \quad v \in D(J_V), \text{ and } J_V(v) \leq \langle u, v - w \rangle + J_V(w) \quad \forall w \in V, \end{gathered}$$

where $D(J_V)$ represent the effective domain of J_V. Indeed, $\partial_{V,V'} J_V$ turns out to be a maximal monotone operator from V to V' (for the theory of maximal monotone operators and convex functions we refer, e.g., to [5]).

Next, let us introduce the conjugate function of J_V, namely $J_V^* : V' \longrightarrow]-\infty, +\infty]$, defined as

$$J_V^*(w^*) := \sup_{v \in V} \big(\langle w^*, v \rangle - J_V(v) \big) \quad \forall w^* \in V'. \tag{2.2}$$

Basic results on conjugate functions (see, e.g., [5, Sec. II.2.2]) ensure that J_V^* is still convex, proper and lower semicontinuous, and that, denoting by $\partial_{V',V} J_V^* : V' \longrightarrow 2^V$ the subdifferential of J_V^*, we have the following

$$\begin{gathered} v \in \partial_{V',V} J_V^*(u) \qquad \text{if and only if} \\ v \in V,\ u \in D(J_V^*),\ \text{and}\ \ J_V^*(u) \le \langle u - w^*, v\rangle + J_V^*(w^*) \quad \forall w^* \in V' \\ \text{if and only if} \qquad u \in \partial_{V,V'} J_V(v). \end{gathered} \tag{2.3}$$

Finally, set

$$\Gamma := \partial_{V,V'} J_V. \tag{2.4}$$

Moreover, we assume that

(A5) $G : H \longrightarrow H$ is a Lipschitz continuous operator. That is, there exists a positive constant C_G such that

$$|G(u) - G(v)| \le C_G |u - v| \quad \forall u, v \in H. \tag{2.5}$$

(A6) $k \in W^{1,1}(0,T)$.

(A7) $F \in L^2(0,T;H) + W^{1,1}(0,T;V')$.

(A8) $u^0 \in V'$ and there exists $\vartheta_0 \in D(\Gamma)\,(\subseteq V)$ such that $u^0 \in (I + \Gamma)(\vartheta_0)$.

where I stands for the embedding of V into V'.

Remark 2.1. We notice that, actually, our analysis does not require the strong coerciveness property for A stated in (A2). Indeed, it suffices to assume A weakly coercive, that is

$$\exists\, \lambda, \alpha > 0 \ \text{ such that } \ \langle Av, v\rangle + \lambda |v|^2 \ge \alpha \|v\|_V^2 \quad \forall v \in V.$$

In this case, we can replace A by $\tilde{A} := A + \lambda\, Id$ (where Id stands for the identity in H) and incorporate the outcome into G (see (A5)). This would allow us to treat other boundary conditions, different from (1.4), including Neumann boundary conditions.

Remark 2.2. Observe that our set of assumptions does not contain the property (1.3) for k. Indeed, such property, although physically motivated, can be neglected in the forthcoming theory.

Remark 2.3. As regards the operator G, we point out that our results hold for a wider class of nonlinearities. Namely, we may ask for a $G : L^2(0,T;H) \longrightarrow L^2(0,T;H)$ which is *causal*, i.e.,

$$\begin{gathered} \text{if } \ u_1, u_2 \in L^2(0,T;H),\ t \in (0,T),\ \text{and} \ \ u_1 = u_2 \ \text{ a.e. in } \ (0,t) \\ \text{then } \ G(u_1) = G(u_2) \ \text{ a.e. in } \ (0,t), \end{gathered}$$

and Lipschitz continuous, that is, there exists a positive constant C_G, fulfilling

$$\|G(u_1) - G(u_2)\|_{L^2(0,t;H)} \le C_G \|u_1 - u_2\|_{L^2(0,t;H)}$$

for any $t \in (0,T)$ and all $u_1, u_2 \in L^2(0,T;H)$. Indeed, it would be possible to adjust the argument devised here to operators G of the type above which could, in particular, represent non-local (in time) nonlinearities, e.g., including convolution terms.

We are now able to state the main result of the paper, which reads as follows.

Theorem 2.4. *Let assumptions (A1)-(A8) hold. Then, there exists a unique pair* (ϑ, u) *fulfilling*

$$\vartheta \in H^1(0,T;H) \ \cap \ L^\infty(0,T;V), \tag{2.6}$$
$$u \in H^1(0,T;V'), \tag{2.7}$$
$$u' + A\vartheta + k * B\vartheta = G(\vartheta) + F \quad \text{in} \ \ V', \ \text{a.e. in} \ \ (0,T), \tag{2.8}$$
$$\vartheta + \Gamma(\vartheta) \ni u \quad \text{in} \ \ V', \ \text{a.e. in} \ \ (0,T), \tag{2.9}$$
$$u(0) = u^0. \tag{2.10}$$

We stress that, under suitable assumptions on the graph γ (cf. (1.2)), relation (2.9) entails the pointwise inclusion (1.2) as well. In this context, let us take a convex, proper, and lower semicontinuous function $j : \mathbb{R} \longrightarrow]-\infty, +\infty]$ such that $\gamma = \partial j$ and consider the related functionals J_H, J_V and J_V^* defined as in (1.6)-(1.7) and (2.2), respectively. Referring to [6] for the details, we note that one can deduce the equivalence between

$$u \in (I + \Gamma)(\vartheta)$$

and

$$u \in D(J_V^*) \quad \text{and} \quad u \in \vartheta + \gamma(\vartheta) \quad \text{a.e. in} \ \ \Omega$$

for all $\vartheta \in V$, $u \in V'$, under the further assumption

$$D(j) = \mathbb{R}, \tag{2.11}$$

namely, $j(\xi) < +\infty$ for any $\xi \in \mathbb{R}$. We point out that the assumption (2.11) is equivalent to either $D(\gamma) = \mathbb{R}$ or γ^{-1} is surjective or

$$\lim_{|\xi| \to +\infty} \frac{j^*(\xi)}{|\xi|} = +\infty,$$

where j^* denotes the conjugate of j. In particular, (2.11) is fulfilled whenever the graph γ is sublinear, namely

$$|\eta| \ \leq \ C_0\big(1 + |\xi|\big) \qquad \forall \xi \in \mathbb{R}, \quad \forall \eta \in \gamma(\xi), \tag{2.12}$$

for a suitable constant C_0. This is, actually, the case of the Stefan problem, where γ reduces to the Heaviside graph. Moreover, if (2.12) holds, then (2.6) and (2.12) imply that $u \in L^\infty(0,T;H)$. In this connection, we underline that the fact that the additional property $u \in L^2(0,T;H)$ (that we cannot prove in our framework) along with (2.9) would yield that

$$u \ \in \ \vartheta + \gamma(\vartheta) \qquad \text{a.e. in} \ \ Q$$

whatever the maximal monotone graph γ is, as has been pointed out in [12].

On the other hand, our assumption on the maximal monotone operator Γ allows us to treat a wide range of problems including variational boundary conditions or nonlinear parabolic equations with the additional term $-\partial_t \Delta \vartheta$, which are known to arise in several diffusion processes [8, 15] and are usually referred to as *pseudo-parabolic* equations.

Indeed, we point out that the previous term may, for instance, be replaced by the time derivative of an arbitrary elliptic operator acting on ϑ.

Moreover, it is worth remarking that our result still holds when we replace $\Gamma = \partial_{V,V'} J_V$ by a general maximal monotone operator $\widetilde{\Gamma} : V \longrightarrow 2^{V'}$, thus avoiding assumption (A4). In this way, one obtains a significantly larger class of applications to partial differential equations, especially to systems.

The forthcoming sections will provide both the theory for the approximation of the system (2.8)-(2.10) and the proof of the previous result.

3 Uniqueness

In this section, we prove the uniqueness result contained in Theorem **2.4**, by reasoning by contradiction. Let (ϑ_1, u_1) and (ϑ_2, u_2) be two pairs satisfying (2.6)-(2.10), and set

$$\widetilde{\vartheta} := \vartheta_1 - \vartheta_2, \quad \widetilde{u} := u_1 - u_2, \quad \widetilde{G} := G(\vartheta_1) - G(\vartheta_2).$$

Taking the difference between equation (2.8), written for (ϑ_1, u_1), and the same equation for (ϑ_2, u_2), one infers that

$$\widetilde{u}' + A\widetilde{\vartheta} + k * B\widetilde{\vartheta} = \widetilde{G} \quad \text{in } V', \quad \text{a.e. in } (0,T).$$

Next, we integrate the previous equation over $(0,t)$, and have

$$\widetilde{u}(t) + (1 * A\widetilde{\vartheta})(t) + (1 * k * B\widetilde{\vartheta})(t) = (1 * \widetilde{G})(t) \quad \text{in } V', \tag{3.1}$$

for all $t \in (0,T)$. If we test (3.1) by $\widetilde{\vartheta}(t) \in V$ (cf. (2.6)), integrate once more in time, and recall (A5) and (2.1), we obtain

$$\begin{aligned} &\int_0^t \langle \widetilde{u}(s), \widetilde{\vartheta}(s) \rangle ds + \int_0^t (((1 * \widetilde{\vartheta})(s), \widetilde{\vartheta}(s)))ds \\ &= -\int_0^t \langle (1 * k * B\widetilde{\vartheta})(s), \widetilde{\vartheta}(s) \rangle ds + \int_0^t \big((1 * \widetilde{G})(s), \widetilde{\vartheta}(s)\big) ds. \end{aligned} \tag{3.2}$$

Due to the monotonicity of Γ, by (2.9) one immediately deduces that

$$\int_0^t \langle \widetilde{u}(s), \widetilde{\vartheta}(s) \rangle ds + \int_0^t (((1 * \widetilde{\vartheta})(s), \widetilde{\vartheta}(s)))ds \geq \int_0^t |\widetilde{\vartheta}(s)|^2 ds + \frac{1}{2} \|(1 * \widetilde{\vartheta})(t)\|^2.$$

In order to control the first term in the right hand side of (3.2), we recall the equality (cf. (A6))

$$(k * \sigma)' = k(0)\sigma + k' * \sigma \qquad \forall \sigma \in L^1(0,T), \tag{3.3}$$

and use (3.3) in an integration by parts. Owing to the elementary inequality (which will be used in the sequel without any explicit mention)

$$ab \;\leq\; \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2 \qquad \forall a, b \in \mathbb{R}, \; \forall \varepsilon > 0,$$

and by virtue of Young's theorem, Hölder's inequality, and assumption (A3), we deduce that

$$
\begin{aligned}
&- \int_0^t \langle (1 * k * B\widetilde{\vartheta})(s), \widetilde{\vartheta}(s)\rangle ds \ \le \ \big|\langle (k * 1 * B\widetilde{\vartheta})(t), (1 * \widetilde{\vartheta})(t)\rangle\big| \\
&+ \left| k(0) \int_0^t \langle B(1 * \widetilde{\vartheta})(s), (1 * \widetilde{\vartheta})(s)\rangle ds \right| + \left| \int_0^t \langle B(k' * 1 * \widetilde{\vartheta})(s), (1 * \widetilde{\vartheta})(s)\rangle ds \right| \\
&\le \ \frac{1}{4}\|(1 * \widetilde{\vartheta})(t)\|^2 + \|k\|^2_{L^2(0,T)} \|B\|^2_{\mathcal{L}(V,V')} \int_0^t \|(1 * \widetilde{\vartheta})(s)\|^2 ds \\
&+ |k(0)|\,\|B\|_{\mathcal{L}(V,V')} \int_0^t \|(1 * \widetilde{\vartheta})(s)\|^2 ds + \|k'\|_{L^1(0,T)} \|B\|_{\mathcal{L}(V,V')} \int_0^t \|(1 * \widetilde{\vartheta})(s)\|^2 ds,
\end{aligned}
$$

where $\|\cdot\|_{\mathcal{L}(V,V')}$ denotes the standard norm in the space of linear continuous operators from V to V', and $|k(0)|$ stands for the absolute value of $k(0)$.

Concerning the last term in the right hand side of (3.2), owing to (2.5) we easily obtain

$$
\begin{aligned}
\int_0^t \big((1 * \widetilde{G})(s), \widetilde{\vartheta}(s)\big) ds \ &\le \ \frac{1}{2}\int_0^t |(1 * \widetilde{G})(s)|^2 ds + \frac{1}{2}\int_0^t |\widetilde{\vartheta}(s)|^2 ds \\
&\le \ \frac{1}{2}\int_0^t |\widetilde{\vartheta}(s)|^2 ds + \frac{C_G^2\, T}{2} \int_0^t \|\widetilde{\vartheta}\|^2_{L^2(0,s;H)} ds,
\end{aligned}
$$

where the last inequality is ensured by the relation

$$
\begin{aligned}
&\|a * b\|^2_{L^2(0,t;H)} \ \le \ t\,\|a\|^2_{L^\infty(0,t)} \int_0^t \|b\|^2_{L^2(0,s;H)} ds \\
&\text{for any} \quad a \in L^\infty(0,T), \quad b \in L^2(0,T;H), \quad \text{and} \quad t \in (0,T).
\end{aligned} \tag{3.4}
$$

Hence, choosing a proper positive constant C_1, for instance

$$
C_1 := \max\left\{ \frac{C_G^2\, T}{2} ; \|B\|_{\mathcal{L}(V,V')} \Big(\|B\|_{\mathcal{L}(V,V')} \|k\|^2_{L^2(0,T)} + |k(0)| + \|k'\|_{L^1(0,T)} \Big) \right\},
$$

and taking into account (3.2), one infers that

$$
\frac{1}{2}\int_0^t |\widetilde{\vartheta}(s)|^2 ds + \frac{1}{4}\|(1 * \widetilde{\vartheta})(t)\|^2 \le \ C_1 \left(\int_0^t \|\widetilde{\vartheta}\|^2_{L^2(0,s;H)} ds + \int_0^t \|(1 * \widetilde{\vartheta})(s)\|^2 ds \right).
$$

Finally, an application of Gronwall's lemma (see, e.g., the version reported in [3, Thm. 2.1]) ensures that $\widetilde{\vartheta} = 0$ a.e. in Q, whence, by comparison in (3.1), it follows at once that $\widetilde{u} = 0$ as well and the uniqueness proof is complete.

4 Time discretization

In this section, we present a semi-implicit time discretization of (2.8)-(2.9). As a first step, we prepare some results in the direction of a discrete convolution procedure. Then, we state the discrete scheme and provide the existence and uniqueness of a discrete solution.

We start by fixing a partition of the time interval $[0,T]$. To this end, we choose a constant time step $\tau = T/N$, $N \in \mathbb{N}$.

4.1 Discrete convolution.

Our next aim is to introduce a discrete version of the convolution product in $(0,t)$ for $t \in (0,T)$. In this context, we refer the reader to [4, 17, 18, 19] and references therein for an exhaustive analysis of discrete convolution procedures. Nevertheless, for the sake of clarity, here we recall the following

Definition 4.1. *Let $\underline{a} = \{a_i\}_{i=1}^N$ be a real vector and let $\underline{b} = \{b_i\}_{i=1}^N \in \mathcal{W}^N$, where $\mathcal{W}$ stands for a real linear space. Then, we define the vector $\{(\underline{a} *_\tau \underline{b})_i\}_{i=0}^N \in \mathcal{W}^{N+1}$ as*

$$(\underline{a} *_\tau \underline{b})_i := \begin{cases} 0 & \text{if } i = 0 \\ \tau \sum_{j=1}^i a_{i-j+1} b_j & \text{if } i = 1, \dots, N. \end{cases} \tag{4.1}$$

In the sequel, we will refer to the previous vector as the discrete convolution product of vectors $\underline{a}$ and $\underline{b}$ with respect to the time step τ.

We stress that, in the definition of $(\underline{a} *_\tau \underline{b})_i$, only the values $\{a_j\}_{j=1}^i$ and $\{b_j\}_{j=1}^i$ are involved (this is usually known as the causality property). Other properties of our discrete convolution product are listed in the following lemma, whose proof is straightforward.

Lemma 4.2. *Let $\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N \in \mathbb{R}^N$, $\{c_i\}_{i=1}^N \in \mathcal{W}^N$, and the discrete convolution product with respect to the time step τ be defined as in (4.1). Then, we have*

$$(\underline{a} *_\tau \underline{b}) = (\underline{b} *_\tau \underline{a}),$$
$$\big((\underline{a} *_\tau \underline{b}) *_\tau \underline{c}\big) = \big(\underline{a} *_\tau (\underline{b} *_\tau \underline{c})\big).$$

Now, let us introduce some convenient notation. For the $(N+1)-$tuple $\{w_i\}_{i=0}^N \in \mathcal{W}^{N+1}$, let the functions $\overline{w}_\tau, w_\tau : (0,T) \longrightarrow \mathcal{W}$ be specified by

$$\overline{w}_\tau(t) := w_i, \quad w_\tau(t) := \alpha_i(t) w_i + \big(1 - \alpha_i(t)\big) w_{i-1},$$
$$\text{where } \alpha_i(t) := \big(t - (i-1)\tau\big)/\tau, \quad \text{for } t \in \big((i-1)\tau, i\tau\big], \ i = 1, \dots, N. \tag{4.2}$$

Let us also set

$$\delta w_i := \frac{w_i - w_{i-1}}{\tau} \quad \text{for } i = 1, \dots, N. \tag{4.3}$$

Owing to the previous notation, it is not difficult to check the following equality

$$\overline{(\underline{a} *_\tau \underline{b})}_\tau(t) = (\overline{a}_\tau * \overline{b}_\tau)(i\tau) \quad \text{for } t \in \big((i-1)\tau, i\tau\big]. \tag{4.4}$$

For the sake of reproducing a discrete version of relation (3.3), it suffices to observe that, given $\{a_i\}_{i=0}^N \in \mathbb{R}^{N+1}$ and $\{b_i\}_{i=1}^N \in \mathcal{W}^N$, we have

$$\begin{aligned} \delta(\underline{a} *_\tau \underline{b})_i &= \sum_{j=1}^{i} a_{i-j+1} b_j - \sum_{j=1}^{i-1} a_{i-j} b_j = a_1 b_i + \sum_{j=1}^{i-1} \tau\, \delta a_{i-j+1} b_j \\ &= a_1 b_i + (\underline{\delta a} *_\tau \underline{b})_i - \tau\, \delta a_1 b_i = a_0 b_i + (\underline{\delta a} *_\tau \underline{b})_i, \end{aligned} \tag{4.5}$$

for $i = 1, \dots, N$.

Finally, we state a discrete Young theorem.

Lemma 4.3. (Discrete Young theorem) *Let* $\{a_i\}_{i=1}^N \in \mathbb{R}^N$, $\{b_i\}_{i=1}^N \in E^N$, *where* E *denotes a linear space endowed with the norm* $\|\cdot\|_E$. *Then the following inequalities hold*

$$\sum_{i=1}^N \tau \|(\underline{a} *_\tau \underline{b})_i\|_E \le \left(\sum_{i=1}^N \tau |a_i|\right)\left(\sum_{i=1}^N \tau \|b_i\|_E\right), \tag{4.6}$$

$$\left(\sum_{i=1}^N \tau \|(\underline{a} *_\tau \underline{b})_i\|_E^2\right)^{1/2} \le \left(\sum_{i=1}^N \tau |a_i|\right)\left(\sum_{i=1}^N \tau \|b_i\|_E^2\right)^{1/2}. \tag{4.7}$$

Proof. Let us recall definition (4.1) and write

$$\sum_{i=1}^N \tau \|(\underline{a} *_\tau \underline{b})_i\|_E \le \sum_{i=1}^N \tau \sum_{j=1}^i \tau |a_{i-j+1}|\, \|b_j\|_E = \tau^2 \sum_{\substack{i,j=1\\ j\le i}}^N |a_{i-j+1}|\, \|b_j\|_E. \tag{4.8}$$

On the other hand, we have that

$$\left(\sum_{i=1}^N \tau |a_i|\right)\left(\sum_{i=1}^N \tau \|b_i\|_E\right) = \tau^2 \sum_{k,j=1}^N |a_k|\, \|b_j\|_E.$$

It turns out that every element in the sum in the right hand side of (4.8) appears in the sum above as well. Thus, inequality (4.6) holds.

Moreover, arguing similarly and accounting for some additional intricacy, one can also verify relation (4.7). □

Let us note that, given a real vector $\{k_i\}_{i=0}^N$ and a vector $\{\sigma_i\}_{i=1}^N \in E^N$, where E stands for a normed space, and according to definitions (4.1)-(4.3), we have that

$$\overline{k}_\tau * \overline{\sigma}_\tau \quad \text{is a piecewise linear continuous function.} \tag{4.9}$$

Indeed, in view of (4.2), it is a standard matter to check that

$$(\overline{k}_\tau * \overline{\sigma}_\tau)(t) = \alpha_i(t)(\underline{k} *_\tau \underline{\sigma})_i + \big(1-\alpha_i(t)\big)(\underline{k} *_\tau \underline{\sigma})_{i-1},$$

for $t \in ((i-1)\tau, i\tau]$ and $i = 1, \ldots, N$, since we have

$$(\overline{k}_\tau * \overline{\sigma}_\tau)(t) = \sum_{j=2}^i \Big(k_{i-j+1}\sigma_j\big(t-(i-1)\tau\big) + k_{i-j+1}\sigma_{j-1}\big(i\tau - t\big)\Big) + k_i\sigma_1\big(t-(i-1)\tau\big)$$

$$= \big(t-(i-1)\tau\big)/\tau \sum_{j=1}^i \tau k_{i-j+1}\sigma_j \; + \; \big(i\tau - t\big)/\tau \sum_{j=1}^{i-1} \tau k_{i-j}\sigma_j.$$

Note that the second sum in the above right hand side contributes only if $i > 1$. As a by-product, owing to (4.5)-(4.6) and (4.9), one easily infers that

$$\begin{aligned} Var_{[0,T];E}[\overline{k}_\tau * \overline{\sigma}_\tau] &= \sum_{i=1}^N \|(\underline{k} *_\tau \underline{\sigma})_i - (\underline{k} *_\tau \underline{\sigma})_{i-1}\|_E \\ &\le \sum_{i=1}^N \tau\Big(\|(\underline{\delta k} *_\tau \underline{\sigma})_i\|_E + |k_0|\|\sigma_i\|_E\Big) \\ &\le \Big(\|\overline{\delta k}_\tau\|_{L^1(0,T)} + \|\overline{k}_\tau\|_{L^\infty(0,T)}\Big)\|\overline{\sigma}_\tau\|_{L^1(0,T;E)}, \end{aligned} \tag{4.10}$$

where $Var_{[0,T];E}[f]$ denotes the total variation on the interval $[0,T]$ of the function $f:[0,T]\longrightarrow E$, where E is a normed space (see, for instance, [7]).

4.2 Approximation.

Now it is worth introducing our approximation of equations (2.8)-(2.9). Let us set

$$k_i := k(i\tau) \quad \text{for} \ \ i=0,1,\dots,N, \tag{4.11}$$

whence it is a standard matter to verify that (cf. (A6))

$$\|k-\overline{k}_\tau\|_{L^1(0,T)} \ \le\ \tau\, Var_{[0,T];\mathbb{R}}[k], \tag{4.12}$$

$$\text{and} \quad \|\overline{\delta k}_\tau\|_{L^1(0,T)} \ \le\ Var_{[0,T];\mathbb{R}}[k]. \tag{4.13}$$

Moreover, we point out an estimate which will play a crucial role in the next section.

Proposition 4.4. *Let (A6) hold and* $\{\sigma_i\}_{i=1}^N \in E^N$*, where* E *denotes a linear space endowed with the norm* $\|\cdot\|_E$*. Moreover, let* $\{k_i\}_{i=0}^N$*,* $\overline{\sigma}_\tau$*, and* $\{(\underline{k} *_\tau \underline{\sigma})_i\}_{i=1}^N$ *be defined as in (4.11), (4.2), and (4.1), respectively. Then, there exists a positive constant* C_2 *which depends only on* k *and fulfills*

$$\|\overline{(\underline{k} *_\tau \underline{\sigma})}_\tau - k * \overline{\sigma}_\tau\|_{L^1(0,T;E)} \ \le\ \tau\, C_2 \|\overline{\sigma}_\tau\|_{L^1(0,T;E)}. \tag{4.14}$$

Proof. Easy calculations ensure that

$$\begin{aligned} &\|\overline{(\underline{k} *_\tau \underline{\sigma})}_\tau - k * \overline{\sigma}_\tau\|_{L^1(0,T;E)} \\ \le\ &\|\overline{(\underline{k} *_\tau \underline{\sigma})}_\tau - \overline{k}_\tau * \overline{\sigma}_\tau\|_{L^1(0,T;E)} + \|(\overline{k}_\tau - k) * \overline{\sigma}_\tau\|_{L^1(0,T;E)}. \end{aligned} \tag{4.15}$$

The first term in the right hand side above may be easily controlled by virtue of relation (4.4) as follows

$$\begin{aligned} \|\overline{(\underline{k} *_\tau \underline{\sigma})}_\tau - \overline{k}_\tau * \overline{\sigma}_\tau\|_{L^1(0,T;E)} \ &=\ \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \|(\overline{k}_\tau * \overline{\sigma}_\tau)(i\tau) - (\overline{k}_\tau * \overline{\sigma}_\tau)(t)\|_E \, dt \\ \le\ \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} Var_{[(i-1)\tau,i\tau];E}[\overline{k}_\tau * \overline{\sigma}_\tau] \ &=\ \tau\, Var_{[0,T];E}[\overline{k}_\tau * \overline{\sigma}_\tau]. \end{aligned}$$

According to (4.10)-(4.11) and (4.13), the previous inequality yields

$$\|\overline{(\underline{k} *_\tau \underline{\sigma})}_\tau - \overline{k}_\tau * \overline{\sigma}_\tau\|_{L^1(0,T;E)} \ \le\ \tau\Big(Var_{[0,T];\mathbb{R}}[k] + \|k\|_{L^\infty(0,T)}\Big) \|\overline{\sigma}_\tau\|_{L^1(0,T;E)}. \tag{4.16}$$

As regards the second term in the right hand side of (4.15), the Young theorem along with relation (4.12) ensures that

$$\begin{aligned} \|(\overline{k}_\tau - k) * \overline{\sigma}_\tau\|_{L^1(0,T;E)} \ &\le\ \|\overline{k}_\tau - k\|_{L^1(0,T)} \|\overline{\sigma}_\tau\|_{L^1(0,T;E)} \\ &\le\ \tau\ Var_{[0,T];\mathbb{R}}[k]\, \|\overline{\sigma}_\tau\|_{L^1(0,T;E)} \end{aligned}$$

Therefore relation (4.14) is satisfied with, for instance, the choice

$$C_2 := 2\, Var_{[0,T];\mathbb{R}}[k] \ +\ \|k\|_{L^\infty(0,T)}. \quad \square$$

Remark 4.5. We stress that, if we are given a positive, non-increasing, and convex k, a positivity inequality, analogous to (1.3), may be simply deduced. The reader is referred to [17, Sect. 4] for the details.

Regarding F, we decompose $F = F_1 + F_2$, with $F_1 \in L^2(0,T;H)$, and $F_2 \in W^{1,1}(0,T;V')$, and set

$$F_{1,i} := \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} F_1(t)\,dt \in H \quad \text{for } i = 1,\dots,N, \tag{4.17}$$

$$F_{2,i} := F_2(i\tau) \in V' \quad \text{for } i = 0,1,\dots,N. \tag{4.18}$$

Moreover, we define

$$F_i := F_{1,i} + F_{2,i} \quad \text{for } i = 1,\dots,N. \tag{4.19}$$

Note that $\|\overline{F}_{1,\tau}\|_{L^2(0,T;H)} \le \|F_1\|_{L^2(0,T;H)}$ and the following convergence is straightforward

$$\overline{F}_{1,\tau} \longrightarrow F_1 \quad \text{strongly in } L^2(0,T;H). \tag{4.20}$$

In addition, we have that (cf. (4.12)-(4.13))

$$\|F_2 - \overline{F}_{2,\tau}\|_{L^1(0,T;V')} \le \tau\, Var_{[0,T];V'}[F_2], \tag{4.21}$$

$$\text{and} \quad \|\overline{\delta F}_{2,\tau}\|_{L^1(0,T;V')} \le Var_{[0,T];V'}[F_2]. \tag{4.22}$$

Then, the approximation scheme may be formulated by making use of an auxiliary unknown ξ_i as

$$\delta u_i + A\vartheta_i + (\underline{k} *_\tau \underline{B\vartheta})_i = G(\vartheta_{i-1}) + F_i \quad \text{in } V',\ \text{for } i = 1,\dots,N, \tag{4.23}$$

$$u_i = \vartheta_i + \xi_i \quad \text{in } V',\ \text{for } i = 1,\dots,N, \tag{4.24}$$

$$\xi_i \in \Gamma(\vartheta_i) \quad \text{for } i = 1,\dots,N, \tag{4.25}$$

$$u_0 = u^0, \tag{4.26}$$

with the obvious notation $\underline{B\vartheta} = (B\vartheta)_i := B\vartheta_i$.

Next, we state and prove an existence and uniqueness result for the solution to the scheme (4.23)-(4.26).

Theorem 4.6. *Let assumptions (A1)-(A8), (4.11), and (4.17)-(4.19) hold and let the time step* τ *be small enough. Then, there exists a unique triplet of vectors* $\{\vartheta_i,\, u_i,\, \xi_i\}_{i=0}^N \in (V \times V' \times V')^{N+1}$ *fulfilling relations (4.23)-(4.26).*

Proof. Recalling (A8), we set $\vartheta_0 := \vartheta^0 \in V$, and define $\xi_0 := u^0 - \vartheta^0$. Thus, it suffices to prove that, for any given triplet of vectors $\{\vartheta_j,\, u_j,\, \xi_j\}_{j=0}^{i-1} \in (V \times V' \times V')^i$, there exists a unique triplet $\{\vartheta_i,\, u_i,\, \xi_i\} \in V \times V' \times V'$ which fulfills (4.23)-(4.25). To this end, put

$$\varphi_i := \tau\left(F_i + G(\vartheta_{i-1}) - \tau\sum_{j=1}^{i-1} k_{i-j+1} B\vartheta_j\right) + u_{i-1}.$$

Then, our aim is to verify that, for any given $\varphi_i \in V'$, the equation

$$I\vartheta_i + \Gamma(\vartheta_i) + \tau A\vartheta_i \ni -\tau^2 k_1 B\vartheta_i + \varphi_i$$

has a unique solution $\vartheta_i \in V$. In fact, u_i and ξ_i will be then uniquely determined by (4.23)-(4.24). As I is monotone and continuous from V to V', the result in [5, Cor. 1.3, p. 48] ensures that $I+\Gamma$ is a maximal monotone operator from V to V'. Hence, thanks to (A2) it is possible to define the operator $S : V \longrightarrow V$, which maps ϑ into the unique solution $S(\vartheta) \in V$ to the equation

$$\left(A + \frac{1}{\tau}(I+\Gamma)\right)\big(S(\vartheta)\big) \ni -\tau k_1 B\vartheta + \varphi_i/\tau.$$

Now, let $\vartheta_1, \vartheta_2 \in V$. Then, by virtue of the Lipschitz continuity of the resolvent

$$\mathcal{J}_{1/\tau} := \left(A + \frac{1}{\tau}(I+\Gamma)\right)^{-1}$$

with Lipschitz constant equal to 1 (see [7, Prop. 2.2, p. 23]), and owing to assumption (A3), we have

$$\begin{aligned}\|S(\vartheta_1) - S(\vartheta_2)\| &= \|\mathcal{J}_{1/\tau}\big(-\tau k_1 B\vartheta_1 + \varphi_i/\tau\big) - \mathcal{J}_{1/\tau}\big(-\tau k_1 B\vartheta_2 + \varphi_i/\tau\big)\| \\ &\le \tau|k_1|\,\|B(\vartheta_1 - \vartheta_2)\|_* \le \tau|k_1|\,\|B\|_{\mathcal{L}(V,V')}\|\vartheta_1 - \vartheta_2\|.\end{aligned}$$

Finally, choosing τ small enough, S turns out to be a contraction mapping in V. Hence, Theorem **4.6** follows as a consequence of the contraction mapping principle. Moreover, note that relations (2.3), (A8), and (4.25) entail $\vartheta_i \in D(J_V)$ and $\xi_i \in D(J_V^*)$, for any $i = 0, 1, \ldots N$, as well. □

Next, let $\mathcal{T}_\tau$ be the translation operator defined on the piecewise constant functions

$$\psi(t) \;=\; \begin{cases} \psi_0 & \text{for } t \le 0 \\ \psi_i & \text{for } t \in \big((i-1)\tau, i\tau\big],\ i = 1, \ldots, N. \end{cases}$$

as follows

$$\mathcal{T}_\tau \psi(t) \;=\; \psi_{i-1} \quad \text{for } t \in \big((i-1)\tau,\ i\tau\big],\ i = 1, \ldots, N. \tag{4.27}$$

Then, for the sake of clarity and owing to Theorem **4.6**, we may rewrite (4.23)-(4.25) as

$$u_\tau' + A\overline{\vartheta}_\tau + \overline{(\underline{k} *_\tau \underline{B\vartheta})}_\tau \;=\; G(\mathcal{T}_\tau\overline{\vartheta}_\tau) + \overline{F}_\tau \quad \text{in } V',\ \text{a.e. in } (0,T), \tag{4.28}$$

$$\overline{u}_\tau \;=\; \overline{\vartheta}_\tau + \overline{\xi}_\tau \quad \text{in } V',\ \text{a.e. in } (0,T), \tag{4.29}$$

$$\overline{\xi}_\tau \;\in\; \Gamma\big(\overline{\vartheta}_\tau\big) \quad \text{in } V',\ \text{a.e. in } (0,T), \tag{4.30}$$

where the notation in (4.2), (4.4), and (4.27) has been used and the function $\overline{\vartheta}_\tau$ is defined to take the value ϑ_0 in $]-\infty, 0]$.

5 Existence

This section concludes the proof of Theorem **2.4**. As a first step, some boundedness estimates, uniform with respect to τ, are deduced. Then, passage to the limit in (4.28)-(4.30) is achieved via compactness, monotonicity, and a direct Cauchy argument. As a by-product of this analysis, an *a priori* error estimate for the discretization error is recovered.

5.1 A priori estimates.

Henceforth, let C denote any constant dependent on the data, but not on the time step τ. Of course, C may vary from line to line.

First estimate. Testing equation (4.23) by $\vartheta_i - \vartheta_{i-1} \in V$, one infers

$$\begin{aligned}\tau\langle \delta u_i, \delta\vartheta_i\rangle + ((\vartheta_i, \vartheta_i - \vartheta_{i-1})) &= -\langle (\underline{k} *_\tau \underline{B\vartheta})_i, \vartheta_i - \vartheta_{i-1}\rangle \\ &+ (G(\vartheta_{i-1}), \vartheta_i - \vartheta_{i-1}) + \langle F_i, \vartheta_i - \vartheta_{i-1}\rangle. \end{aligned} \tag{5.1}$$

By exploiting the monotonicity of Γ, it is straightforward to deduce that

$$\tau\langle \delta u_i, \delta\vartheta_i\rangle \geq \tau|\delta\vartheta_i|^2.$$

Moreover, the last two terms in the right hand side of (5.1) can be handled as follows (see (A5), (A7), and (4.19)).

$$\begin{aligned}(G(\vartheta_{i-1}), \vartheta_i - \vartheta_{i-1}) &\leq \frac{\tau}{4}|\delta\vartheta_i|^2 + 2\tau\big(C_G^2|\vartheta_{i-1}|^2 + |G(0)|^2\big), \\ \langle F_i, \vartheta_i - \vartheta_{i-1}\rangle &\leq (F_{1,i}, \vartheta_i - \vartheta_{i-1}) + \langle F_{2,i}, \vartheta_i - \vartheta_{i-1}\rangle \\ &\leq \frac{\tau}{4}|\delta\vartheta_i|^2 + \tau|F_{1,i}|^2 + \langle F_{2,i}, \vartheta_i - \vartheta_{i-1}\rangle.\end{aligned}$$

Hence, summing up for $i = 1, \ldots, m$ in (5.1), we have

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^m \tau|\delta\vartheta_i|^2 + \frac{1}{2}\|\vartheta_m\|^2 + \frac{1}{2}\sum_{i=1}^m \|\vartheta_i - \vartheta_{i-1}\|^2 - \frac{1}{2}\|\vartheta_0\|^2 \\ &\leq 2C_G^2 \sum_{i=0}^{m-1}\tau|\vartheta_i|^2 + 2T|G(0)|^2 + \sum_{i=1}^N \tau|F_{1,i}|^2 \\ &+ \sum_{i=1}^m \langle F_{2,i}, \vartheta_i - \vartheta_{i-1}\rangle - \sum_{i=1}^m \langle (\underline{k} *_\tau \underline{B\vartheta})_i, \vartheta_i - \vartheta_{i-1}\rangle,\end{aligned}$$

and, according to (A1), (A5), and (A7)-(A8), one infers that

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^m \tau|\delta\vartheta_i|^2 + \frac{1}{2}\|\vartheta_m\|^2 + \frac{1}{4}\sum_{i=1}^m \|\vartheta_i - \vartheta_{i-1}\|^2 \\ &\leq C\left(1 + \sum_{i=0}^{m-1}\tau\|\vartheta_i\|^2\right) + \sum_{i=1}^m \langle F_{2,i}, \vartheta_i - \vartheta_{i-1}\rangle - \sum_{i=1}^m \langle (\underline{k} *_\tau \underline{B\vartheta})_i, \vartheta_i - \vartheta_{i-1}\rangle.\end{aligned} \tag{5.2}$$

Our next aim is to control the right hand side above, in particular the last two terms.

To achieve this, we make use of a discrete integration by parts. Namely, note that

$$\sum_{i=1}^{m}\langle F_{2,i},\vartheta_i-\vartheta_{i-1}\rangle=\langle F_{2,m},\vartheta_m\rangle-\langle F_{2,1},\vartheta_0\rangle$$
$$+\sum_{i=1}^{m-1}\langle F_{2,i}-F_{2,i+1},\vartheta_i\rangle,$$
$$-\sum_{i=1}^{m}\langle(\underline{k}*_\tau\underline{B\vartheta})_i,\vartheta_i-\vartheta_{i-1}\rangle=-\langle(\underline{k}*_\tau\underline{B\vartheta})_m,\vartheta_m\rangle+\langle(\underline{k}*_\tau\underline{B\vartheta})_1,\vartheta_0\rangle$$
$$-\sum_{i=1}^{m-1}\langle(\underline{k}*_\tau\underline{B\vartheta})_i-(\underline{k}*_\tau\underline{B\vartheta})_{i+1},\vartheta_i\rangle.$$

Then, we obtain

$$\sum_{i=1}^{m}\langle F_{2,i},\vartheta_i-\vartheta_{i-1}\rangle\ \le\ \frac{1}{8}\|\vartheta_m\|^2\ +\ \frac{5}{2}\|\overline{F}_{2,\tau}\|^2_{L^\infty(0,T;V')}$$
$$+\ \frac{1}{2}\|\vartheta_0\|^2\ +\ \sum_{i=1}^{m-1}\tau\|\delta F_{2,i+1}\|_*\,\|\vartheta_i\|. \tag{5.3}$$

Moreover, applying the identity (4.5), one infers that

$$-\sum_{i=1}^{m}\langle(\underline{k}*_\tau\underline{B\vartheta})_i,\vartheta_i-\vartheta_{i-1}\rangle=-\langle(\underline{k}*_\tau\underline{B\vartheta})_m,\vartheta_m\rangle+\langle(\underline{k}*_\tau\underline{B\vartheta})_1,\vartheta_0\rangle$$
$$+\ \sum_{i=1}^{m-1}\tau\langle(\underline{\delta k}*_\tau\underline{B\vartheta})_{i+1},\vartheta_i\rangle+k_0\sum_{i=1}^{m-1}\tau\langle B\vartheta_{i+1},\vartheta_i\rangle.$$

Now, it is easy to check that (see (4.2))

$$\|(\underline{k}*_\tau\underline{B\vartheta})_m\|_*\ \le\ \|\overline{(\underline{k}*_\tau\underline{B\vartheta})}_\tau\|_{L^\infty(0,m\tau;V')}$$
$$\le\ \|B\|_{\mathcal{L}(V,V')}\sum_{j=1}^{i^*}\tau|k_{i^*-j+1}|\|\vartheta_j\|\ \le\ \|B\|_{\mathcal{L}(V,V')}\|k\|_{L^\infty(0,T)}\sum_{j=1}^{m}\tau\|\vartheta_j\|$$

for some $i^*\le m$. Then, easy calculations ensure that

$$-\sum_{i=1}^{m}\langle(\underline{k}*_\tau\underline{B\vartheta})_i,\vartheta_i-\vartheta_{i-1}\rangle$$
$$\le\ \frac{1}{4}\|\vartheta_m\|^2+\|B\|^2_{\mathcal{L}(V,V')}\left(\|k\|^2_{L^\infty(0,T)}T\sum_{i=1}^{m}\tau\|\vartheta_i\|^2+\frac{|k_1|^2}{2}\tau^2\|\vartheta_1\|^2\right)$$
$$+\ \frac{1}{2}\|\vartheta_0\|^2+\frac{1}{2}\|B\|_{\mathcal{L}(V,V')}\sum_{i=1}^{m-1}\tau\Big(\|(\underline{\delta k}*_\tau\underline{\vartheta})_{i+1}\|^2+\|\vartheta_i\|^2\Big)$$
$$+\ \frac{1}{2}|k_0|\,\|B\|_{\mathcal{L}(V,V')}\sum_{i=1}^{m-1}\tau\left(\|\vartheta_i\|^2+\|\vartheta_{i+1}\|^2\right),$$

and, recalling Lemma **4.3** and (4.13), we finally deduce that

$$-\sum_{i=1}^{m}\langle(\underline{k} *_\tau \underline{B\vartheta})_i, \vartheta_i - \vartheta_{i-1}\rangle \le \frac{1}{4}\|\vartheta_m\|^2 + \frac{1}{2}\|\vartheta_0\|^2 + C\sum_{i=1}^{m}\tau\|\vartheta_i\|^2. \tag{5.4}$$

Combining (5.3)-(5.4) with (5.2) and taking into account (A8) one achieves

$$\begin{aligned}&\sum_{i=1}^{m}\tau|\delta\vartheta_i|^2 \ + \ \|\vartheta_m\|^2 \ + \ \sum_{i=1}^{m}\|\vartheta_i - \vartheta_{i-1}\|^2 \\ &\le \ C\left(1 \ + \ \sum_{i=1}^{m-1}\tau\|\delta F_{2,i+1}\|_*\,\|\vartheta_i\| + \ \sum_{i=1}^{m}\tau\|\vartheta_i\|^2\right).\end{aligned}$$

Finally, upon choosing τ small enough, applying the discrete Gronwall lemma (see, e.g., the version reported in [16, Prop. 2.2.1]), and owing to (4.22), we conclude that

$$\|\vartheta_\tau\|_{H^1(0,T;H)} + \|\overline{\vartheta}_\tau\|_{L^\infty(0,T;V)} \le C, \tag{5.5}$$

$$\sum_{i=1}^{N}\|\vartheta_i - \vartheta_{i-1}\|^2 \le C. \tag{5.6}$$

Related estimates. Owing to **4.3**, assumptions (A3), (A5)-(A7), and bound (5.5), a comparison in (4.28) ensures that

$$\|u_\tau'\|_{L^2(0,T;V')} \le C. \tag{5.7}$$

Moreover, from (A8), (4.26), and (4.29) we recover

$$\|u_\tau\|_{H^1(0,T;V')} + \|\overline{\xi}_\tau\|_{L^\infty(0,T;V')} \le C \tag{5.8}$$

as well.

Before passing to the limit, we collect below other properties of the approximating sequences.

Lemma 5.1. *The following estimates hold*

$$\|\vartheta_\tau - \overline{\vartheta}_\tau\|_{L^2(0,T;H)} \le C\tau, \tag{5.9}$$

$$\|\vartheta_\tau - \overline{\vartheta}_\tau\|_{L^\infty(0,T;H)} \le C\sqrt{\tau}, \tag{5.10}$$

$$\|\overline{\vartheta}_\tau - \mathcal{T}_\tau\overline{\vartheta}_\tau\|_{L^2(0,T;H)} \le C\tau, \tag{5.11}$$

$$\|\overline{\vartheta}_\tau - \mathcal{T}_\tau\overline{\vartheta}_\tau\|_{L^\infty(0,T;H)} \le C\sqrt{\tau}, \tag{5.12}$$

$$\|u_\tau - \overline{u}_\tau\|_{L^2(0,T;V')} \le C\tau, \tag{5.13}$$

$$\|u_\tau - \overline{u}_\tau\|_{L^\infty(0,T;V')} \le C\sqrt{\tau}. \tag{5.14}$$

Proof. Regarding (5.9)-(5.11), it is straightforward to see that (4.2) and (5.5) yield

$$\|\vartheta_\tau - \overline{\vartheta}_\tau\|^2_{L^2(0,T;H)} = \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \big(\alpha_i(t)-1\big)^2 |\vartheta_i - \vartheta_{i-1}|^2 dt \le \frac{\tau^2}{3}\sum_{i=1}^N \tau|\delta\vartheta_i|^2 \le C\tau^2,$$

$$\|\vartheta_\tau - \overline{\vartheta}_\tau\|^2_{L^\infty(0,T;H)} = \sup_{1\le i\le N} |\vartheta_i - \vartheta_{i-1}|^2 \le \sum_{i=1}^N |\vartheta_i - \vartheta_{i-1}|^2 \le \tau \sum_{i=1}^N \tau|\delta\vartheta_i|^2 \le C\tau,$$

$$\|\overline{\vartheta}_\tau - \mathcal{T}_\tau\overline{\vartheta}_\tau\|_{L^2(0,T;H)} = \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} |\vartheta_i - \vartheta_{i-1}|^2 dt \le \tau^2 \sum_{i=1}^N \tau|\delta\vartheta_i|^2 \le C\tau^2.$$

Moreover, taking into account (5.5) and (5.8), estimates (5.12)-(5.14) may be proved in a similar way. □

5.2 Passage to the limit.

Thanks to assumption (A6), Lemma **4.3**, estimates (5.5), (5.8), (5.10), and well-known compactness results, one infers that there exist at least a sequence of time steps (still denoted by τ) and four functions ϑ, u, ξ, and φ such that

$$\vartheta_\tau \longrightarrow \vartheta \quad \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V), \tag{5.15}$$

$$\overline{\vartheta}_\tau \longrightarrow \vartheta \quad \text{weakly star in } L^\infty(0,T;V), \tag{5.16}$$

$$u_\tau \longrightarrow u \quad \text{weakly in } H^1(0,T;V'), \tag{5.17}$$

$$\overline{\xi}_\tau \longrightarrow \xi \quad \text{weakly star in } L^\infty(0,T;V'), \tag{5.18}$$

$$\overline{(\underline{k} *_\tau \underline{B\vartheta})}_\tau \longrightarrow \varphi \quad \text{weakly in } L^2(0,T;V'), \tag{5.19}$$

as τ tends to 0. Besides, from (A2) and (5.16), in particular one deduces that

$$A\overline{\vartheta}_\tau \longrightarrow A\vartheta \quad \text{weakly star in } L^\infty(0,T;V'). \tag{5.20}$$

In addition, the generalized Ascoli theorem (see, e.g., [20, Cor. 4]) and relations (5.10), (5.12) ensure that

$$\vartheta_\tau \longrightarrow \vartheta \quad \text{strongly in } C^0([0,T];H), \tag{5.21}$$

$$\overline{\vartheta}_\tau \longrightarrow \vartheta \quad \text{strongly in } L^\infty(0,T;H), \tag{5.22}$$

$$\mathcal{T}_\tau\overline{\vartheta}_\tau \longrightarrow \vartheta \quad \text{strongly in } L^\infty(0,T;H), \tag{5.23}$$

whence, owing to (A5), we have

$$G(\mathcal{T}_\tau\overline{\vartheta}_\tau) \longrightarrow G(\vartheta) \quad \text{strongly in } L^\infty(0,T;H). \tag{5.24}$$

Moreover, note that, due to relations (4.14), (5.5), (5.16), and (5.22), the following convergences hold

$$\overline{(\underline{k} *_\tau \underline{B\vartheta})}_\tau - k * B\overline{\vartheta}_\tau \longrightarrow 0 \quad \text{strongly in } L^1(0,T;V')$$

$$k * B\overline{\vartheta}_\tau \longrightarrow k * B\vartheta \quad \text{weakly in } L^2(0,T;V')$$

thus (cf. (5.19)) $\varphi = k * B\vartheta$. Therefore, recalling also (A7) and (4.20)-(4.21), we can take the weak limit in (4.28) and (4.29) obtaining (2.8) and

$$u \;=\; \vartheta + \xi \quad \text{in} \;\; V', \;\; \text{a.e. in} \;\; (0,T), \tag{5.25}$$

respectively.

Remark 5.2. We point out that, since the continuous problem (2.8)-(2.10) has a unique solution, the convergences listed above hold not only for a subsequence but for the whole family of partitions, as the time step τ tends to 0.

With the aim of concluding the proof of Theorem **2.4**, it suffices to show that

$$\xi(t) \in \Gamma(\vartheta(t)) \;\; \text{in} \;\; V', \; \text{for a.e.} \;\; t \in (0,T). \tag{5.26}$$

To this end, we recover some further strong convergence by virtue of a direct Cauchy argument. As a first step, take the sum over i in (4.23). By virtue of definitions (4.1) and (4.2), one easily obtains

$$\begin{aligned} &\overline{u}_\tau(t) - u^0 + \big(1 * A\overline{\vartheta}_\tau\big)(t) + \big(1 * k * B\overline{\vartheta}_\tau\big)(t) \\ &\quad = \big(1 * G(\mathcal{T}_\tau\overline{\vartheta}_\tau)\big)(t) + \big(1 * \overline{F}_\tau\big)(t) + \mathcal{R}_\tau(t), \end{aligned} \tag{5.27}$$

for all $t \in (0,T)$, where the *residual* term is specified by

$$\begin{aligned} \mathcal{R}_\tau(t) \;\; := \;\; & \big(1 * A\overline{\vartheta}_\tau - \overline{(\underline{1} *_\tau \underline{A\vartheta})}_\tau\big)(t) + \big(1 * k * B\overline{\vartheta}_\tau - \overline{(\underline{1} *_\tau \underline{k} *_\tau \underline{B\vartheta})}_\tau\big)(t) \\ + \;\; & \big(\overline{(\underline{1} *_\tau \underline{G(\mathcal{T}_\tau\vartheta)})}_\tau - 1 * G(\mathcal{T}_\tau\overline{\vartheta}_\tau)\big)(t) + \big(\overline{(\underline{1} *_\tau \underline{F})}_\tau - 1 * \overline{F}_\tau\big)(t), \end{aligned}$$

with the obvious notations $\underline{1} := (1,1,\ldots,1) \in \mathbb{R}^N$ and $\big(G(\mathcal{T}_\tau\vartheta)\big)_i := G(\vartheta_{i-1})$ for $i = 1,\ldots,N$. Due to assumption (A5), (A7), and estimates (4.14) and (5.5), it is a standard matter to verify that there exists a positive constant C_3 which depends only on the data and fulfills

$$\|\mathcal{R}_\tau\|_{L^1(0,T;V')} \le C_3\tau. \tag{5.28}$$

Indeed, arguing as in Proposition **4.4** and accounting for (A5), (A7), and (5.5), we easily check that

$$\begin{aligned} &\|1 * A\overline{\vartheta}_\tau - \overline{(\underline{1} *_\tau \underline{A\vartheta})}_\tau\|_{L^1(0,T;V')} \le \tau\|\overline{\vartheta}_\tau\|_{L^1(0,T;V)} \le C\tau, \\ &\|1 * G(\mathcal{T}_\tau\overline{\vartheta}_\tau) - \overline{(\underline{1} *_\tau \underline{G(\mathcal{T}_\tau\vartheta)})}_\tau\|_{L^1(0,T;V')} \le \tau C\|G(\mathcal{T}_\tau\overline{\vartheta}_\tau)\|_{L^1(0,T;H)} \le C\tau, \\ &\|1 * \overline{F}_\tau - \overline{(\underline{1} *_\tau \underline{F})}_\tau\|_{L^1(0,T;V')} \le \tau\|\overline{F}_\tau\|_{L^1(0,T;V')} \le C\tau. \end{aligned}$$

On the other hand, owing to Lemma **4.2**, we see that

$$\begin{aligned} &\|1 * k * B\overline{\vartheta}_\tau - \overline{(\underline{1} *_\tau \underline{k} *_\tau \underline{B\vartheta})}_\tau\|_{L^1(0,T;V')} \le \|k * 1 * B\overline{\vartheta}_\tau - k * \overline{(\underline{1} *_\tau \underline{B\vartheta})}_\tau\|_{L^1(0,T;V')} \\ &+\|(k - \overline{k}_\tau) * \overline{(\underline{1} *_\tau \underline{B\vartheta})}_\tau\|_{L^1(0,T;V')} + \|\overline{k}_\tau * \overline{(\underline{1} *_\tau \underline{B\vartheta})}_\tau - \overline{(\underline{k} *_\tau \underline{1} *_\tau \underline{B\vartheta})}_\tau\|_{L^1(0,T;V')}. \end{aligned}$$

Then, taking into account (4.12), (4.16), and exploiting the same argument developed in the proof of Proposition **4.4**, we have

$$\begin{aligned} &\|1 * k * B\overline{\vartheta}_\tau - \overline{(\underline{1} *_\tau \underline{k} *_\tau \underline{B\vartheta})}_\tau\|_{L^1(0,T;V')} \le \tau\|B\|_{\mathcal{L}(V,V')}\Big(\|k\|_{L^1(0,T)}\; Var_{[0,T];V}[1 * \overline{\vartheta}_\tau] \\ &\qquad +Var_{[0,T],\mathbb{R}}[k]\;\|\overline{(\underline{1} *_\tau \underline{\vartheta})}_\tau\|_{L^1(0,T;V)} + Var_{[0,T];V}[\overline{k}_\tau * \overline{(\underline{1} *_\tau \underline{\vartheta})}_\tau]\Big), \end{aligned}$$

and finally, due to (4.10) and (5.5) we obtain

$$\|1 * k * B\overline{\vartheta}_\tau - \overline{(\underline{1} *_\tau \underline{k} *_\tau \underline{B\vartheta})}_\tau\|_{L^1(0,T;V')} \;\leq\; C\tau.$$

Next, write (5.27) for two different choices of time steps, say τ and μ, take the difference between the two and test it by $\overline{\vartheta}_\tau - \overline{\vartheta}_\mu$. If we define

$$\widetilde{\vartheta} := \overline{\vartheta}_\tau - \overline{\vartheta}_\mu, \quad \widetilde{u} := \overline{u}_\tau - \overline{u}_\mu, \quad \widetilde{G} := G(\mathcal{T}_\tau\overline{\vartheta}_\tau) - G(\mathcal{T}_\mu\overline{\vartheta}_\mu),$$

and integrate the resulting equation in time, we may reproduce the same argument developed in Section 3. In particular, we handle the term containing $1 * (\overline{F}_\tau - \overline{F}_\mu)$ by making use of an integration by parts and of Young's theorem as

$$\begin{aligned}
\int_0^t \langle (1 * (\overline{F}_\tau - \overline{F}_\mu))(s), \widetilde{\vartheta}(s)\rangle ds &= \langle (1 * (\overline{F}_\tau - \overline{F}_\mu))(t), (1 * \widetilde{\vartheta})(t)\rangle \\
&- \int_0^t \langle (\overline{F}_\tau - \overline{F}_\mu)(s), (1 * \widetilde{\vartheta})(s)\rangle ds \\
&\leq \frac{1}{8}\|(1 * \widetilde{\vartheta})(t)\|^2 + 2\|(1 * (\overline{F}_\tau - \overline{F}_\mu))(t)\|_*^2 + \int_0^t \|(\overline{F}_\tau - \overline{F}_\mu)(s)\|_* \,\|(1 * \widetilde{\vartheta})(s)\|\, ds \\
&\leq \frac{1}{8}\|(1 * \widetilde{\vartheta})(t)\|^2 + 2\|\overline{F}_\tau - \overline{F}_\mu\|_{L^1(0,T;V')}^2 + \int_0^t \|(\overline{F}_\tau - \overline{F}_\mu)(s)\|_* \,\|(1 * \widetilde{\vartheta})(s)\|\, ds.
\end{aligned}$$

Then by virtue of (5.5), (5.11) and (5.28), we obtain

$$\begin{aligned}
\frac{1}{2}\int_0^t |\widetilde{\vartheta}(s)|^2 ds + \frac{1}{4}\|(1 * \widetilde{\vartheta})(t)\|^2 &\leq C_1 \left(\int_0^t \|\widetilde{\vartheta}\|_{L^2(0,s;H)}^2 ds + \int_0^t \|(1 * \widetilde{\vartheta})(s)\|^2 ds \right) \\
&+ \int_0^t \langle (1 * (\overline{F}_\tau - \overline{F}_\mu))(s), \widetilde{\vartheta}(s)\rangle ds + \int_0^t \langle (\mathcal{R}_\tau - \mathcal{R}_\mu)(s), \widetilde{\vartheta}(s)\rangle ds \\
&\leq C \Bigg(\int_0^t \|\widetilde{\vartheta}\|_{L^2(0,s;H)}^2 ds + \int_0^t \|(1 * \widetilde{\vartheta})(s)\|^2 ds + \|\overline{F}_\tau - \overline{F}_\mu\|_{L^1(0,T;V')}^2 \\
&+ \int_0^t \|(\overline{F}_\tau - \overline{F}_\mu)(s)\|_* \,\|(1 * \widetilde{\vartheta})(s)\|\, ds + \tau + \mu \Bigg).
\end{aligned}$$

Finally, an application of Gronwall's lemma yields

$$\frac{1}{2}\int_0^t |\widetilde{\vartheta}(s)|^2 ds + \frac{1}{4}\|(1 * \widetilde{\vartheta})(t)\|^2 \leq C_4\big(\|\overline{F}_\tau - \overline{F}_\mu\|_{L^1(0,T;V')}^2 + \tau + \mu\big), \tag{5.29}$$

for a proper positive constant C_4, depending only on the data. Since we have (4.20)-(4.21), $1 * \overline{\vartheta}_\tau$ turns out to be a Cauchy sequence in $C^0([0,T];V)$. Therefore, owing to (4.20)-(4.21), and (5.15), we conclude that

$$1 * \overline{\vartheta}_\tau \;\longrightarrow\; 1 * \vartheta \qquad \text{strongly in } \; C^0([0,T];V). \tag{5.30}$$

This convergence allows, in particular, the following (cf. (A2)-(A3))

$$1 * A\overline{\vartheta}_\tau \;\longrightarrow\; 1 * A\vartheta \qquad \text{strongly in } \; C^0([0,T];V'), \tag{5.31}$$

$$1 * k * B\overline{\vartheta}_\tau \;\longrightarrow\; 1 * k * B\vartheta \qquad \text{strongly in } \; C^0([0,T];V'). \tag{5.32}$$

Moreover, the argument devised above entails an *a priori* error estimate for the discretization error as well. Indeed, under a further assumption on time regularity of F_1, namely

$$F_1 \in BV([0,T];V'), \tag{5.33}$$

we achieve the result

Proposition 5.3. (Error estimate) *Let assumptions (A1)-(A8), and (5.33) hold and τ be small enough. Moreover, let ϑ as in Theorem* **2.4**, *$\{\vartheta_i\}_{i=0}^N$ as in Theorem* **4.6** *and $\overline{\vartheta}_\tau$ be defined as in (4.2). Then, there exists a positive constant C_5, depending only on data, such that the following estimate holds*

$$\|\vartheta - \overline{\vartheta}_\tau\|_{L^2(0,T;H)} + \|1 * (\vartheta - \overline{\vartheta}_\tau)\|_{C^0([0,T];V)} \le C_5\sqrt{\tau}. \tag{5.34}$$

Proof. Note that,

$$\begin{aligned}
\|F_1 - \overline{F}_{1,\tau}\|_{L^1(0,T;V')} &\le \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \left\| F_1(t) - \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} F_1(s)ds \right\|_* dt \\
&\le \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \left(\frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} \|F_1(t) - F_1(s)\|_* ds \right) dt \\
&\le \tau \sum_{i=1}^N Var_{[(i-1)\tau, i\tau];V'}[F_1] = \tau Var_{[0,T];V'}[F_1].
\end{aligned} \tag{5.35}$$

Then, passing to the limit in (5.29) as μ tends to 0 and taking into account (A7) and (4.21), one infers that

$$\|\vartheta - \overline{\vartheta}_\tau\|_{L^2(0,t;H)} + \|(1 * (\vartheta - \overline{\vartheta}_\tau))(t)\| \le C_5\sqrt{\tau}, \tag{5.36}$$

for $t \in (0,T)$ and a proper $C_5 > 0$. Whence, the assertion follows. □

Remark 5.4. We stress that the previous *a priori* estimate depends solely on the data, in particular, exponentially on T, since we prove (5.29) by using Gronwall's lemma. Referring to the present literature on this argument, in [4, 17, 18, 19] the authors deal with a simplified model since both γ and g vanish, and devise an error control procedure which relies on strong assumptions with respect to the regularity of solutions. Indeed, the estimate depends on the $H^2(0,T;H)$-norm of ϑ, which is actually not *a priori* bounded (cf. (2.6)). In contrast, we note that our estimate requires no further assumption on ϑ.

Moreover, let us observe that the above estimate is suboptimal with respect to the rate of convergence. Indeed, we only achieve the order $1/2$ instead of 1, which is the expected rate since we used the backward Euler's method to approximate the time derivative in (2.8). We believe the discrepancy can be attributed to the inner structure of the problem itself, particularly to its strongly nonlinear features. Relation (4.14) shows that neither the approximation used for the convolution product nor the regularity of the ingredients of the problem can be responsible for the low rate of convergence.

Taking advantage of (5.30) it is now possible to achieve the desired inclusion (5.26). We test (4.29) by $\overline{\vartheta}_\tau \in V$ and integrate on $(0,T)$. One has

$$\int_0^T \langle \overline{\xi}_\tau(t), \overline{\vartheta}_\tau(t)\rangle dt = \int_0^T \langle \overline{u}_\tau(t), \overline{\vartheta}_\tau(t)\rangle dt - \int_0^T |\overline{\vartheta}_\tau(t)|^2 dt.$$

Owing to (5.27) it is now possible to compute that

$$\int_0^T \langle \overline{\xi}_\tau(t), \overline{\vartheta}_\tau(t)\rangle dt = \int_0^T \langle (u^0 - 1 * A\overline{\vartheta}_\tau - 1 * k * B\overline{\vartheta}_\tau)(t), \overline{\vartheta}_\tau(t)\rangle dt$$
$$+ \int_0^T \langle (1 * G(\mathcal{T}_\tau \overline{\vartheta}_\tau) + 1 * \overline{F}_\tau + \mathcal{R}_\tau)(t), \overline{\vartheta}_\tau(t)\rangle dt - \int_0^T |\overline{\vartheta}_\tau(t)|^2 dt. \tag{5.37}$$

Now, we take the $\limsup$ as τ tends to 0 on both sides of (5.37). Since we have (4.20)-(4.21), (5.15), (5.24), (5.28), and (5.31)-(5.32), it is straightforward to check that

$$\limsup_{\tau \searrow 0} \int_0^T \langle \overline{\xi}_\tau(t), \overline{\vartheta}_\tau(t)\rangle dt$$
$$\leq \int_0^T (u^0 - 1 * A\vartheta - 1 * k * B\vartheta + 1 * G(\vartheta) + 1 * F)(t), \vartheta(t)\rangle dt - \int_0^T |\vartheta(t)|^2 dt.$$

Then, by considering the integral in time of (2.8), it is a standard matter to deduce that the previous inequality entails, in particular, the following

$$\limsup_{\tau \searrow 0} \int_0^T \langle \overline{\xi}_\tau(t), \overline{\vartheta}_\tau(t)\rangle dt \leq \int_0^T \langle \xi(t), \vartheta(t)\rangle dt,$$

and the inclusion (5.26) is ensured by (5.15), (5.18), and the well-known result [7, Prop. 2.5, p. 27]. Finally, the proof of Theorem **2.4** is complete.

References

[1] S. Aizicovici, P. Colli, and M. Grasselli. On a class of degenerate nonlinear Volterra equations. *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, 132:1–18, 1998.

[2] S. Aizicovici, P. Colli, and M. Grasselli. Doubly nonlinear evolution equations with memory. *Preprint no. 345/P, Dipartimento di Matematica, Politecnico di Milano, Italy*, 1999.

[3] C. Baiocchi. Sulle equazioni differenziali astratte lineari del primo e del secondo ordine negli spazi di Hilbert. *Ann. Mat. Pura Appl. (4)*, 76:233–304, 1967.

[4] N.Yu. Bakaev, S. Larsson, and V. Thomée. Backward Euler type methods for parabolic integro-differential equations in Banach space. *RAIRO Modél. Math. Anal. Numér.*, 32(1):85–99, 1998.

[5] V. Barbu. *Nonlinear semigroups and differential equations in Banach spaces.* Noordhoff International Publishing, 1976.

[6] V. Barbu, P. Colli, G. Gilardi, and M. Grasselli. Existence, uniqueness, and longtime behavior for a nonlinear Volterra integrodifferential equation. *Differential Integral Equations*, to appear.

[7] H. Brezis. *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert.* Number 5 in North Holland Math. Studies. North-Holland, Amsterdam, 1973.

[8] R. W. Carrol and R. E. Showalter. Singular and degenerate Cauchy problems. Number 27 in Mathematics in Science and Engineering. Academic Press, New York, 1976.

[9] J. M. Chadam and H. M. Yin. The two-phase Stefan problem in materials with memory. In J. M. Chadam and H. Rasmussen, editors, *Free boundary problems involving solids*, number 281 in Pitman Res. Notes Math. Ser., pages 117–123. Longman, Harlow, 1993.

[10] P. Colli and M. Grasselli. Phase transition problems in materials with memory. *J. Integral Equations Appl.*, 5:1–22, 1993.

[11] P. Colli and M. Grasselli. Degenerate nonlinear Volterra integrodifferential equations. In C. Corduneanu and I.W. Sandberg, editors, *Volterra equations and applications*, Stability Control Theory Methods Appl. Gordon and Breach, Lausanne, to appear.

[12] A. Damlamian and N. Kenmochi. Evolution equations generated by subdifferentials in the dual space of $H^1(\Omega)$. *Discrete Contin. Dynam. Systems*, 5(2):269–278, 1999.

[13] E. Di Benedetto and R. E. Showalter. Implicit degenerate evolution equations and applications. *SIAM J. Math. Anal.*, 12(5):731–751, 1981.

[14] G. Gentili and C. Giorgi. Thermodynamic properties and stability for the heat flux equation with linear memory. *Quart. Appl. Math.*, 51:343–362, 1993.

[15] M. E. Gurtin and P. J. Chen. On a theory of heat conduction involving two temperatures. *Z. Angew. Math. Phys.*, 19:614–627, 1968.

[16] J.W. Jerome. *Approximation of nonlinear evolution systems.* Number 164 in Math. Sci. Engrg. Academic Press, Orlando, 1983.

[17] W. Mc Lean and V. Thomée. Numerical solution to an evolution equation with a positive-type memory term. *J. Austral. Math. Soc. Ser. B*, 35:23–70, 1993.

[18] Ch. Lubich, I.H. Sloan, and V. Thomée. Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term. *Math. Comp.*, 65(213):1–17, 1996.

[19] W. McLean, V. Thomée, and L.B. Wahlbin. Discretization with variable time steps of an evolution equation with a positive-type memory term. *J. Comput. Appl. Math.*, 69(1):49–69, 1996.

[20] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[21] A. Visintin. General free boundary evolution problems in several space dimensions. *J. Math. Anal. Appl.*, 95:117–143, 1983.