

Well-posedness and Approximation for a One-Dimensional Model for Shape Memory Alloys

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Abstract

This note addresses a one-dimensional differential model describing the super-elastic effect in shape memory alloys. After a preliminary discussion on the model, we focus our attention on the study of the material constitutive relation and prove well-posedness and approximation results for some related initial value problem. Then, we turn our attention to the coupling of the constitutive law with a quasi-static momentum relation and solve the full PDE boundary value problem in one space dimension. In particular, we provide an existence and continuous dependence result and fully develop a space-time discretization procedure.

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1 Introduction

Shape memory alloys (SMAs) are metallic materials with an intrinsic ability of undergoing a thermo-elastic solid-solid (martensitic) transformation between two crystallographic structures with different physical and mechanical properties: the *austenite* (or parent phase), characterized by a more ordered unit cell and stable at temperatures above A_f , and the *martensite* (or product phase), characterized by a less ordered unit cell and stable at temperatures below M_f , with $M_f < A_f$ [10].

At the macroscopic level the ability of undergoing such a reversible phase transformation results in the so-called stress-driven *super-elastic effect*. When a specimen in the austenitic

phase is loaded at a temperature greater than A_f , the material presents a nonlinear behavior due to a stress-induced conversion of austenite into martensite. Upon unloading, a reverse transformation from martensite to austenite occurs as a result of the instability of the martensite at zero stress for temperatures higher than A_f . At the end of the loading-unloading process no permanent strains are present and the stress-strain path is a closed hysteresis loop.

This unusual macroscopic mechanical effect is nowadays exploited in several innovative devices, ranging from orthodontic appliances to self-expanding structures for the treatment of hollow-organ or duct-system occlusions as well as from dissipative devices for earthquake engineering applications to control system for airplane wings [15, 16].

Stated the clear commercial value of such applications, researchers have been extensively involved in experimental investigations as well as in the development of numerical tools. The literature on the mathematical study of differential models for shape memory alloys is accordingly quite rich. Indeed it is far beyond our purposes to present here an extensive report on the contributions in the direction of a mathematical treatment of differential models for SMAs. The reader is in particular referred to the monographs [6, 9], the papers [7, 8, 11, 18], and the references therein.

Accordingly, the present paper focuses on the study of a constitutive model, previously introduced in the literature [2] and able to describe, the super-elastic behavior of shape-memory alloys in a small deformation realm. The model under consideration is constructed under the assumption that the martensitic phase transition is associated to an always active reorientation process described through (a suitable generalization of) the following position

$$\varepsilon^{tr} = \kappa \xi \frac{\partial F(\sigma)}{\partial \sigma}, \quad (1.1)$$

where ε^{tr} is the inelastic strain induced by the martensitic transformation (in the following briefly indicated as *transformation strain*), κ is a positive material constant representing a measure of the maximum strain obtainable through alignment of the martensite variants, ξ is the martensite volume fraction, and F is a loading function driving the phase transformation process and depending on the stress σ .

The temperature of the medium is considered to be fixed throughout the phase transformation. This assumption of course restricts the applicability of this model to those situations where the mechanical evolution may be approximated by an isothermal process.

The aim of this paper is to present a full analysis of this model, both from the analytical and the numerical viewpoint. In particular, we focus our attention on the physical justification of the model, on the proof of well-posedness results for the related initial and boundary value problems, and their effective discretization.

In Section 2 we review the phenomenological ground-bases of the model. Moreover, we complement the analysis of [2] by proving the *dissipativity character* of the model.

Section 3 is devoted to the analysis of the so-called *constitutive relation problem* in three space dimensions. Namely, we assume to be given an initial state and a displacement field and solve the constitutive relation by determining the corresponding phase proportion and stress. The upcoming differential problem shows a *quasivariational character* [3] and is addressed by means of an *ad hoc* problem transformation. In particular, we prove a global well-posedness result for the continuous problem (global existence, continuous dependence) as well as some qualitative behavior of the solutions (monotonicity). Moreover, we discuss an efficient variable time-step discretization scheme which is the equivalent in this setting of the well-known *return mapping* algorithm in plasticity. The approximating solution is effectively computable

and converges to its continuous counterpart when the diameter of the partition goes to zero. Moreover, some optimal *a priori* error bounds are provided. Finally, whenever restricted to some special yet experimentally relevant situations the latter scheme turns out to be *exact* on the time partition.

Section 4 focuses on the study of the full *PDE boundary value problem* in one space dimension. The latter problem consists in the determination of the stress, the phase proportion, and the displacement from the load and the boundary conditions. Namely, we couple the quasivariational constitutive material relation with the momentum equilibrium equation. In particular, we investigate an *imposed boundary displacement problem*, provide a continuous dependence result, and address its global solvability by means of a variable time-step discretization technique. Eventually, we present some discussion in the direction of a possible full discretization scheme and state some a priori bounds on the discretization error.

2 Modeling

This section shall bring to the detailed introduction of the model and a brief discussion on its thermodynamic consistency.

2.1 Notations and preliminaries

We start by fixing some notations. Henceforth Ω denotes the interval $(0, 1)$ which represent the reference (equilibrium) configuration of the shape memory wire under consideration. We shall be interested in the evolution of the material in the time interval $[0, T]$ with $T > 0$. For later convenience, we define $Q_t := \Omega \times (0, t)$ for $t \in [0, T]$ and $Q := Q_T$. In the forthcoming analysis we will make use of classical Sobolev function spaces. The reader is referred at once to [13] for definitions and details. Let us just stress that the symbol $\|\cdot\|_E$ will denote the norm in the general normed space E while (\cdot, \cdot) stands for the standard scalar product in $L^2(\Omega)$.

Assume we are given a function $\varphi : \mathbb{R} \rightarrow (-\infty, +\infty]$ proper, convex, and lower semicontinuous. We will denote by $\partial\varphi : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ its (possibly multivalued) *subdifferential* defined as

$$\tau \in \partial\varphi(\sigma) \text{ iff } \sigma \in D(\varphi) \text{ and } \tau(\eta - \sigma) \leq \varphi(\eta) - \varphi(\sigma) \quad \forall \eta \in D(\varphi),$$

where $D(\varphi) := \{\sigma \in \mathbb{R} : \varphi(\sigma) < +\infty\}$ denotes the effective domain of φ . We remark that $\partial\varphi \subset \mathbb{R} \times \mathbb{R}$ turns out to be a maximal monotone graph. The reader is referred to [4] for a full discussion on this notion.

2.2 Basic Relations

We will denote by $\varepsilon := u_x = \partial u / \partial x$ the deformation (strain) and assume the additive decomposition

$$\varepsilon = \varepsilon^{el} + \varepsilon^{tr}, \tag{2.1}$$

where ε^{el} represents the elastic part of the strain and ε^{tr} represents an inelastic part induced by the solid-solid phase transformation (the so-called *transformation strain*). We assume an elastic material response, namely the stress σ is asked to fulfill

$$\sigma = C\varepsilon^{el}, \tag{2.2}$$

where $C > 0$ represents the *elasticity modulus*. For the sake of later purposes we will denote by $L = 1/C$ the so-called *compliance* of the medium. As for the constitutive relation for ε^{tr} we introduce an *auxiliary internal parameter* $\xi = \xi(x, t)$ that describes the local proportion of *product phase* (martensite) against *parent phase* (austenite) in the body (namely $\xi \in [0, 1]$). The parameter ξ will be referred to as *phase* for short. Moreover, let $F : \mathbb{R} \rightarrow [0, +\infty)$ be the *loading function* of the material. Namely, $F(\sigma)$ represents a suitable measure of the stress state of the medium that, together with the forthcoming relation g , drives the phase transformation. In particular, we ask for

(A1) F convex, locally Lipschitz continuous, and $0 = F(0) = \min F$.

We stress that the latter assumption reflects some natural physical situations. Indeed, a first choice for F could be $F(\sigma) := |\sigma|$ which corresponds in this setting with the Von Mises criterion. By assuming that the martensitic phase transition is associated to an always active reorientation process we prescribe

$$\varepsilon^{tr} \in \kappa \xi \partial F(\sigma), \quad (2.3)$$

where κ is a positive material constant representing a measure of the maximum strain obtainable through alignment of the martensitic variants and ∂ denotes the above defined subdifferential. In particular, by collecting (2.1)-(2.3) we obtain the constitutive relation

$$L\sigma + \kappa \xi \partial F(\sigma) \ni \varepsilon. \quad (2.4)$$

We fix for the moment $\xi = \xi(x, t)$. Then, we are in the position of defining the functional

$$\mathcal{F}_\xi(\sigma) := \frac{L}{2} \sigma^2 + \kappa \xi F(\sigma),$$

where ξ appears as a parameter. Of course, owing to (A1), the latter functional is still convex, locally Lipschitz continuous, with $0 = \mathcal{F}_\xi(0) = \min \mathcal{F}_\xi$. Moreover, relation (2.4) can be expressed at once as $\varepsilon \in \partial \mathcal{F}_\xi(\sigma)$ where of course the subdifferential is taken with respect to σ only. By introducing the *generalized elastic energy* functional \mathcal{F}_ξ^* as the Legendre conjugate of \mathcal{F}_ξ , we observe that we have in particular that

$$\sigma \varepsilon = \mathcal{F}_\xi(\sigma) + \mathcal{F}_\xi^*(\varepsilon) \quad \text{and} \quad \sigma \in \partial \mathcal{F}_\xi^*(\varepsilon).$$

We shall refer to $T_\xi := \partial \mathcal{F}_\xi^*$ as the *tension map* and to $T_\xi^{-1} := \partial \mathcal{F}_\xi$ as the *deformation map*. Finally, one observes that the Ω -distributed map $T_\xi : L^2(\Omega) \rightarrow L^2(\Omega)$ (we use the same notation) is Lipschitz continuous.

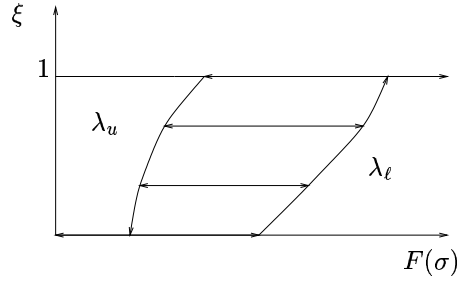
2.3 Evolution of the internal parameter

In order to close the above system we shall prescribe the evolution of ξ . To this end, we are given an initial phase configuration $\xi^0 = \xi^0(x)$ and we assume that ξ evolves as a function of the loading F as

$$\xi = g(\xi^0, F(\sigma)), \quad (2.5)$$

where g is a rate independent [19] hysteretic relation of the type depicted in Figure 1 below.

The latter is usually referred to as a *generalized play-type hysteresis operator* and its classical theory can be found, for instance, in [6, 19]. For the purpose of our analysis we shall make precise relation g as follows. Assume we are given

Figure 1: Diagram of relation g

(A2) $\lambda_\ell, \lambda_u : [0, +\infty) \rightarrow [0, 1]$ monotone and Lipschitz continuous with $\lambda_\ell \leq \lambda_u$,

a loading history $F(\sigma) : Q \rightarrow \mathbb{R}^+$, and an admissible initial state ξ^0 such that

$$\xi^0 \in [\lambda_\ell(F(\sigma(0))), \lambda_u(F(\sigma(0)))] \quad \text{almost everywhere in } \Omega.$$

Then we ask for a function $\xi : Q \rightarrow [0, 1]$ such that, almost everywhere,

$$\begin{aligned} \xi(0) &= \xi^0 \quad \text{in } \Omega & \xi &\in [\lambda_\ell(F(\sigma)), \lambda_u(F(\sigma))] \quad \text{in } Q, \\ \xi &\text{ is constant if } \xi \neq \lambda_\ell(F(\sigma)), \lambda_u(F(\sigma)), \\ \xi &\text{ is non-decreasing if } \xi = \lambda_\ell(F(\sigma)), \\ \xi &\text{ is non-increasing if } \xi = \lambda_u(F(\sigma)). \end{aligned}$$

Whenever some regularity is assumed for ξ , the above condition may be reformulated in the following complementary form

$$\begin{aligned} \xi(0) &= \xi^0 \quad \text{a.e. in } \Omega, & \xi &\in [\lambda_\ell(F(\sigma)), \lambda_u(F(\sigma))] \quad \text{a.e. in } Q, \\ \dot{\xi}(\xi - w) &\leq 0 \quad \text{a.e. in } Q, & \forall w &\in [\lambda_\ell(F(\sigma)), \lambda_u(F(\sigma))] \quad \text{a.e. in } Q, \end{aligned} \quad (2.6)$$

where the dot of course denotes derivation with respect to time.

Suppose now we are given $\varepsilon : Q \rightarrow \mathbb{R}$. Then the latter problem (2.6) can be equivalently rewritten in the more compact form

$$\xi(0) = \xi^0 \quad \text{a.e. in } \Omega, \quad (2.7)$$

$$\dot{\xi} + \partial I_{K(\varepsilon, \xi)}(\xi) \ni 0 \quad \text{a.e. in } Q, \quad (2.8)$$

where, for $(\varepsilon, \xi) \in \mathbb{R} \times [0, 1]$, we set

$$\begin{aligned} k_*(\varepsilon, \xi) &:= \lambda_\ell(F(T_\xi \varepsilon)), & k^*(\varepsilon, \xi) &:= \lambda_u(F(T_\xi \varepsilon)), \\ K(\varepsilon, \xi) &:= [k_*(\varepsilon, \xi), k^*(\varepsilon, \xi)]. \end{aligned}$$

Here, we used the notation $I_{K(\varepsilon, \xi)}$ for the *indicator function* of $K(\varepsilon, \xi)$ namely $I_{K(\varepsilon, \xi)}(x) = 0$ if $x \in K(\varepsilon, \xi)$ and $I_{K(\varepsilon, \xi)}(x) = +\infty$ otherwise. In particular, since the constraining convex set K depends on the phase ξ itself, one usually refers to problems in this class as of *quasivariational type* [3].

We stress that the ξ –dependence of K is a direct consequence of our overall construction. In particular, the current situation cannot be reconducted to the ξ –independent case without restricting the possible choice of the loading function F . Some details in this direction are discussed in the paper [1].

Let us now present some preparatory results.

Lemma 2.1. *Under the assumptions (A1)-(A2) one has that*

- i) k_*, k^* are continuous and $k_* \leq k^*$,
- ii) for all $\xi \in [0, 1]$, the functions $k_*(\cdot, \xi), k^*(\cdot, \xi)$ are Lipschitz continuous, uniformly with respect to ξ ,
- iii) for all $\varepsilon \in \mathbb{R}$, the functions $k_*(\varepsilon, \cdot), k^*(\varepsilon, \cdot)$ are non-increasing with ξ ,
- iv) for all $\xi \in [0, 1]$, one has that $\varepsilon_1(k(\varepsilon_1, \xi) - k(\varepsilon_2, \xi))(\varepsilon_1 - \varepsilon_2) \geq 0$ for any $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ such that $\varepsilon_1 \varepsilon_2 \geq 0$ and for $k = k_*, k^*$. In particular, the functions $k_*(\cdot, \xi), k^*(\cdot, \xi)$ are non-decreasing in \mathbb{R} with respect to the partial order \lesssim defined for all $a_1, a_2 \in \mathbb{R}$ as

$$a_1 \lesssim a_2 \text{ iff } (a_2 - a_1)a_1 \geq 0 \text{ and } a_1 a_2 \geq 0.$$

Proof.

- i) Since λ_ℓ, λ_u , and F are continuous it suffices to prove that $T_\xi(\varepsilon)$ depends continuously on ξ and ε . The latter continuous dependence is straightforward owing to the local boundedness of ∂F .
- ii) Of course it suffices to recall that the transformation T_ξ is Lipschitz continuous, uniformly with respect to ξ .
- iii) Let us fix $\varepsilon \in \mathbb{R}$. taking into account the monotonicity of λ_ℓ, λ_u it suffices to prove that, whenever $\xi_1, \xi_2 \in [0, 1]$ and $\xi_1 \geq \xi_2$, one has that $F(\sigma_1) \leq F(\sigma_2)$ where $\sigma_i := T_{\xi_i}(\varepsilon)$, $i = 1, 2$. We reason by contradiction and suppose that $F(\sigma_1) > F(\sigma_2)$. Since ∂F is monotone, one readily checks that $\eta_1(\sigma_1 - \sigma_2) \geq \eta_2(\sigma_1 - \sigma_2)$ for all $\eta_i \in \partial F(\sigma_i)$. On the other hand, owing to the definition of subdifferential, we have that $\eta_1(\sigma_1 - \sigma_2) > 0$. Hence, we recall the definition of T_{ξ_i} and write that

$$L\sigma_1 + \kappa\xi_1\eta_1 = L\sigma_2 + \kappa\xi_2\eta_2,$$

multiply the latter relation by $\sigma_1 - \sigma_2$, and deduce that

$$L(\sigma_1 - \sigma_2)^2 = -\kappa(\xi_1\eta_1 - \xi_2\eta_2)(\sigma_1 - \sigma_2) \leq 0,$$

a contradiction.

- iv) Let us fix $\xi \in [0, 1]$. Since λ_u and T_ξ are monotone, it suffices to recall that $T_\xi(0) = 0$ and that F is non-decreasing on $[0, +\infty)$ and non-increasing on $(-\infty, 0]$ in order to claim that $k^*(\cdot, \xi)$ is monotone in the sense of \lesssim . The same holds for $k_*(\cdot, \xi)$.

□

We shall stress that the latter results, that for simplicity are here presented in the space-time independent framework, may be easily extended to their natural space-time distributed analogue.

2.4 Dissipation

The introduced model can be indeed justified by means of classical dissipation considerations. In order to present some discussion in this direction, we just refer to the quantities above and define the *Helmholtz-type stored energy* of the system as

$$\psi := \frac{C}{2}(\varepsilon^{el})^2.$$

Of course relation (2.2) is then deduced by the standard position $\sigma \in \partial\psi(\varepsilon^{el})$. We now turn to the *dissipation inequality* that, in the current isothermal framework, reads pointwise along trajectories as

$$\dot{\psi} - \sigma \dot{\varepsilon} \leq 0. \quad (2.9)$$

Taking into account (2.1), we readily check that (2.9) is equivalent to

$$\sigma \dot{\varepsilon}^{tr} \geq 0, \quad (2.10)$$

in perfect analogy with the theory of plasticity.

We will reduce ourselves in proving (2.10) pointwise in Ω but in an integrated form on (t_1, t_2) for $t_1, t_2 \in [0, T]$. Indeed, also using (2.3), and formally integrating by parts (later on we will make precise this procedure by providing the necessary regularity in time for the ingredients see Remark 4.4)

$$\int_{t_1}^{t_2} \sigma \dot{\varepsilon}^{tr} = (\sigma \varepsilon^{tr})(t_2) - (\sigma \varepsilon^{tr})(t_1) - \int_{t_1}^{t_2} \dot{\sigma} \varepsilon^{tr} = (\sigma \varepsilon^{tr})(t_2) - (\sigma \varepsilon^{tr})(t_1) - \kappa \int_{t_1}^{t_2} \xi \dot{F}(\sigma).$$

On the other hand, owing to (2.6) one deduces that

$$\begin{aligned} \kappa \int_{t_1}^{t_2} \xi \dot{F}(\sigma) - \kappa (\xi F(\sigma))(t_2) + \kappa (\xi F(\sigma))(t_1) &= -\kappa \int_{t_1}^{t_2} \dot{\xi} F(\sigma) \\ &= -\kappa \int_{t_1}^{t_2} \left(\dot{\xi}^+ \lambda_\ell^{-1}(\xi) - \dot{\xi}^- \lambda_u^{-1}(\xi) \right) \\ &= -\frac{\kappa}{2} \int_{t_1}^{t_2} \left((\lambda_\ell^{-1}(\xi) + \lambda_u^{-1}(\xi)) \dot{\xi} + (\lambda_\ell^{-1}(\xi) - \lambda_u^{-1}(\xi)) |\dot{\xi}| \right), \end{aligned}$$

where we used the standard notation for positive and negative part. Moreover, we remark that the notations above make sense whenever $\dot{\xi} \neq 0$. Finally, let us define

$$\begin{aligned} \mathcal{R}(t) &:= \frac{\kappa}{2} \int_{t_1}^t (\lambda_\ell^{-1}(\xi) + \lambda_u^{-1}(\xi)) \dot{\xi}, \\ \mathcal{S}(t) &:= \frac{\kappa}{2} \int_{t_1}^t (\lambda_\ell^{-1}(\xi) - \lambda_u^{-1}(\xi)) \dot{\xi}, \\ \mathcal{T}(t) &:= (\sigma \varepsilon^{tr})(t) - \kappa (\xi F(\sigma))(t) + \mathcal{R}(t), \end{aligned}$$

Hence, we have derived the (formal) relation

$$\int_{t_1}^{t_2} \sigma \dot{\varepsilon}^{tr} = \mathcal{T}(t_2) - \mathcal{T}(t_1) + \int_{t_1}^{t_2} |\dot{\mathcal{S}}|, \quad (2.11)$$

where $\mathcal{T}(t_2) - \mathcal{T}(t_1)$ represent some conserved energy while $\int_{t_1}^{t_2} |\dot{\mathcal{S}}|$ is the dissipated energy. In the particular case of a complete cycle, the latter obviously reduces to

$$\int_{t_1}^{t_2} \sigma e^{tr} = \int_{t_1}^{t_2} |\dot{\mathcal{S}}|.$$

Namely, the right hand side above measures the area of the region bounded by the hysteretic loop. We shall remark that the above discussion on the dissipativity of the model was not originally contained in our main reference [2].

3 Constitutive relation

This section is devoted to the study of the constitutive relation (2.8). In particular, in the following we will assume the strain ε and the initial phase configuration ξ^0 to be given and investigate the evolution of the phase ξ determined by (2.7)-(2.8).

3.1 Results

For the sake of clarity, we recast in the following claim some of the results of Lemma 2.1.

- (B1)** $k_*, k^* : \mathbb{R} \times [0, 1] \longrightarrow [0, 1]$ are continuous, $k_* \leq k^*$,
for any $\xi \in [0, 1]$, the functions $k_*(\cdot, \xi), k^*(\cdot, \xi)$ are equi-Lipschitz continuous,
for any $\varepsilon \in \mathbb{R}$, the functions $k_*(\varepsilon, \cdot), k^*(\varepsilon, \cdot)$ are non-increasing.

For later convenience, we will set from the very beginning C_K to be the maximum of the uniform Lipschitz constants of $k_*(\cdot, \xi), k^*(\cdot, \xi)$ above. Moreover, we will fix our assumptions on data as

(B2) $\varepsilon \in H^1(0, T; L^2(\Omega)),$

(B3) $\xi^0 \in L^2(\Omega), \xi^0 \in K(\varepsilon(0), \xi^0)$ almost everywhere in Ω .

One could argue that the requirement (B3) introduces some restriction on the possible choice of K . Indeed, this is not the case since (B1) ensures that the set $\{\xi^0 \in L^2(\Omega) : \xi^0 \in K(\varepsilon(0), \xi^0) \text{ a.e. in } \Omega\}$ is always non-empty (see the proof of the forthcoming Lemma 3.5). We have the following results.

Lemma 3.1 (Well-posedness). *Under the assumptions (B1)-(B3) there exists a unique $\xi \in H^1(0, T; L^2(\Omega))$ fulfilling (2.7)-(2.8) almost everywhere.*

Lemma 3.2 (Continuous dependence). *Assume (B1), let (ε_1, ξ_1^0) and (ε_2, ξ_2^0) fulfill (B2)-(B3), and ξ_1, ξ_2 be the respective solutions to (2.7)-(2.8), whose existence is stated above. Then, for all subintervals $[t_1, t_2] \subset [0, T]$ one has*

$$\max_{[t_1, t_2]} |\xi_1 - \xi_2| \leq \max \left\{ |(\xi_1 - \xi_2)(t_1)|, C_K \max_{[t_1, t_2]} |\varepsilon_1 - \varepsilon_2| \right\} \quad \text{a.e. in } \Omega. \quad (3.1)$$

The proof of the above lemmas will be provided in the forthcoming subsections by means of a suitable problem transformation and a discretization arguments.

We may exploit estimate (3.1) in order to possibly introduce some notion of generalized solution $\xi \in C([0, T]; L^2(\Omega))$ corresponding to generalized data $\varepsilon \in C([0, T]; L^2(\Omega))$ as limit of solutions to regular problems. In other words, denoting by

$$\begin{aligned} H : \{(\varepsilon, \xi^0) \in H^1(0, T; L^2(\Omega)) \times L^2(\Omega) : \xi^0 \in K(\varepsilon(0), \xi^0) \text{ a.e. in } \Omega\} \\ \longrightarrow H^1(0, T; L^2(\Omega)), \end{aligned}$$

the solution operator implicitly defined by Lemma 3.1, we have the following result.

Corollary 3.3. *The operator H extends in a unique way to a Lipschitz continuous operator*

$$\begin{aligned} H_0 : \{(\varepsilon, \xi^0) \in C([0, T]; L^2(\Omega)) \times L^2(\Omega) : \xi^0 \in K(\varepsilon(0), \xi^0) \text{ a.e. in } \Omega\} \\ \longrightarrow C([0, T]; L^2(\Omega)). \end{aligned}$$

In particular, the latter generalized concept of solution may be defined by the relation $\text{graph } H_0 := \text{cl}(\text{graph } H)$, where the closure is obviously referred to the strong topology of $(C([0, T]; L^2(\Omega)) \times L^2(\Omega)) \times C([0, T]; L^2(\Omega))$. Moreover, we shall remark that the framework of Lemma 3.2 has been chosen just for the sake of clarity. Indeed, one could also easily prove that H restricted to the Hölder space $C^{0,\alpha}$ for $\alpha \in (0, 1]$ is bounded in $C^{0,\alpha}$ as well (see [19, Thm. 2.2.ii, p. 67]). In this regard also the above construction of the extension H_0 may be adapted in order to obtain the unique extensions

$$\begin{aligned} H_\alpha : \{(\varepsilon, \xi^0) \in C^{0,\alpha}([0, T]; L^2(\Omega)) \times L^2(\Omega) : \xi^0 \in K(\varepsilon(0), \xi^0) \text{ a.e. in } \Omega\} \\ \longrightarrow C^{0,\alpha}([0, T]; L^2(\Omega)) \quad \forall \alpha \in (0, 1]. \end{aligned}$$

Let us just stress that the latter operators H_α of course are not a priori strongly continuous [19, Rem. (ii), p. 68]. Indeed, we remark that also the original choice $\varepsilon \in H^1(0, T; L^2(\Omega))$ may be properly changed into $\varepsilon \in W^{1,p}(0, T; L^2(\Omega))$ for $p \in (1, \infty]$ and the argument of our analysis could be accordingly tailored in order to obtain a weakly continuous (weakly star continuous for $p = \infty$) operator

$$\begin{aligned} H_p : \{(\varepsilon, \xi^0) \in W^{1,p}(0, T; L^2(\Omega)) \times L^2(\Omega) : \xi^0 \in K(\varepsilon(0), \xi^0) \text{ a.e. in } \Omega\} \\ \longrightarrow W^{1,p}(0, T; L^2(\Omega)) \quad \forall p \in [1, \infty]. \end{aligned}$$

Finally, all the above mentioned results and extensions may be rewritten with $L^q(\Omega)$, $q \in [1, \infty]$ instead of $L^2(\Omega)$.

We are now interested in establishing a comparison result among solutions to (2.7)-(2.8) in the sense of Lemma 3.1. Of course we shall require some extra monotonicity assumption. To this aim we set

(B4) For any $\xi \in [0, 1]$, the functions $k_*(\cdot, \xi)$, $k^*(\cdot, \xi) : (\mathbb{R}, \lesssim) \rightarrow \mathbb{R}$ are non-decreasing.

From the physical point of view, the above monotonicity with respect to the ordering \lesssim in \mathbb{R} translates the fact that, for all fixed phase configurations, the stress-strain relation is monotone. Its full physical justification is however restricted to the modeling discussion of the latter section, namely Lemma 2.1.iv.

In the framework of assumption (B4), the following result holds.

Lemma 3.4 (Comparison). *Let the assumptions of Lemma 3.2 hold. Moreover let (B4) hold and $\varepsilon_2 \lesssim \varepsilon_1$ almost everywhere in Q . Then, we have that, if $\xi_1^0 \geq \xi_2^0$ almost everywhere in Ω , then $\xi_1 \geq \xi_2$ almost everywhere in Q .*

Again the proof of the latter result is discussed in the forthcoming subsections. We stress that some monotonicity of the kind of (B4) seems mandatory in order to obtain comparison of solutions. Indeed, let us consider the situation where $T = 2$, $k^*(\varepsilon) := |\varepsilon - 1| =: k_*(\varepsilon) + 1$. In the latter case we may choose $\xi_1^0 = \xi_2^0 := 1$, $\varepsilon_1 \equiv 0 \lesssim \varepsilon_2(t) := t$, and get the ordered solutions $\xi_1 \equiv 1 \geq \xi_2(t) = (1 - t)^+$. However, if we fix $\xi_1^0 = \xi_2^0 = 0$, $\varepsilon_1 \equiv 1 \lesssim \varepsilon_2(t) = t + 1$, we obtain $\xi_1 \equiv 0 \leq \xi_2(t) = (t - 2)^+$. Hence, no comparison can be inferred when (B4) does not hold.

We shall now conclude this section by providing the sketch of a superposition construction of Prandtl-Ishlinskiĭ type [19]. In particular, referring indeed to the physical interpretation of our model, we may be interested in the coupling in parallel of a set (or even a continuum) of rheological elements of the type of (2.7)-(2.8). From the mathematical point of view, this construction may be realized by the introduction of a suitable parameter set Y where (Y, \mathcal{M}, μ) is a measure space with finite Borel measure. One introduces of course some parameterized data, for instance let $k_*, k^* : Y \times \mathbb{R} \times [0, 1] \rightarrow [0, 1]$ be Carathéodory functions such that $k_* \leq k^*$ in $\mathbb{R} \times [0, 1]$, μ -a.e. in Y , $k_*(y, \cdot, \xi), k^*(y, \cdot, \xi)$ are Lipschitz continuous, uniformly with respect to $(y, \xi) \in Y \times [0, 1]$, and $k_*(y, \varepsilon, \cdot), k^*(y, \varepsilon, \cdot)$ are non-increasing for any $\varepsilon \in \mathbb{R}$ and μ -a.e. $y \in Y$. Of course $K := [k_*, k^*]$. Moreover, let $\varepsilon \in H^1(0, T; L^2(\Omega \times Y))$, $\xi^0 \in L^2(\Omega \times Y)$, such that $\xi^0 \in K(\varepsilon(0), \xi^0)$ almost everywhere. Then, for μ -a.e. $y \in Y$, we may find $\xi(y) \in H^1(0, T; L^2(\Omega))$ fulfilling almost everywhere the relations

$$\dot{\xi}(y) + \partial I_{K(y, \varepsilon(y), \xi(y))}(\xi(y)) \ni 0, \quad \xi(y, 0) = \xi^0(y).$$

Hence, we may define the superposition operator

$$\begin{aligned} H_Y : \{(\varepsilon, \xi^0) \in H^1(0, T; L^2(\Omega \times Y)) \times L^2(\Omega \times Y) : \xi^0 \in K(\varepsilon(0), \xi^0) \text{ a.e.}\} \\ \longrightarrow H^1(0, T; L^2(\Omega)), \end{aligned}$$

as $H_Y(\varepsilon, \xi^0) := \int_Y \xi(y) d\mu$. Of course the latter operator is well-defined and fulfills, by superposition, all the continuity and boundedness properties mentioned above for the single element. In particular, it is Lipschitz continuous with respect to the uniform topology.

3.2 Problem transformation

We shall introduce an equivalent *variational* formulation of problem (2.7)-(2.8). To this aim let us consider the nonlinear equations

$$a^* = k^*(\varepsilon(t), a^*), \quad a_* = k_*(\varepsilon(t), a_*). \quad (3.2)$$

Owing to (B1)-(B2), it is straightforward to check that, for all $t \in [0, T]$, there exist unique almost everywhere solutions $a^*(t), a_*(t)$ to the above equations, respectively. We now fix $s, t \in [0, T]$ and exploit (B1) in order to deduce that

$$\begin{aligned} |a^*(t) - a^*(s)|^2 - (k^*(\varepsilon(t), a^*(t)) - k^*(\varepsilon(t), a^*(s)), a^*(t) - a^*(s)) \\ \leq C_K |\varepsilon(t) - \varepsilon(s)| |a^*(t) - a^*(s)|. \end{aligned} \quad (3.3)$$

In particular, it turns out that $a^* \in H^1(0, T; L^2(\Omega))$ and of course the same holds true for a_* .

Let us now define $A(t) =: \{u \in L^2(\Omega) : a_*(t) \leq u \leq a^*(t) \text{ a.e. in } \Omega\}$ (which is non-empty, convex, and closed for almost every $t \in (0, T)$) and consider the problem of finding $\xi_A \in H^1(0, T; L^2(\Omega))$ such that

$$\xi_A(0) = \xi^0 \text{ a.e. in } \Omega, \quad (3.4)$$

$$\dot{\xi}_A(t) + \partial I_{A(t)}(\xi_A(t)) \ni 0 \text{ for a.e. } t \in (0, T). \quad (3.5)$$

We are indeed in the position of proving the following.

Lemma 3.5. *Let $\xi, \xi_A \in H^1(0, T; L^2(\Omega))$ solve (2.7)-(2.8) and (3.4)-(3.5), respectively. Then $\xi \equiv \xi_A$ almost everywhere.*

Proof. First of all we observe that $\xi^0 \in A(0)$. We now exploit (2.8), (3.5) and deduce that

$$\begin{aligned} (\dot{\xi}, \xi - w) &\leq 0 \quad \text{a.e. in } (0, T), \quad \forall w \in K(\varepsilon, \xi) \quad \text{a.e. in } Q, \\ (\dot{\xi}_A, \xi_A - w_A) &\leq 0 \quad \text{a.e. in } (0, T), \quad \forall w_A \in A \quad \text{a.e. in } (0, T). \end{aligned} \quad (3.6)$$

We readily check that, whenever $\xi \in K(\varepsilon, \xi)$ almost everywhere in Q , one also has that $\xi \in A$ almost everywhere in $(0, T)$. On the other hand, let $\xi_A \in A$ almost everywhere in $(0, T)$ and $\xi \in K(\varepsilon, \xi)$ almost everywhere in Q . From the above lines we easily deduce that

$$a_* \leq \xi \leq a^* \quad \text{almost everywhere in } Q,$$

and, owing to (B1),

$$k_*(\varepsilon, \xi) \leq k_*(\varepsilon, a_*) = a_* \leq \xi_A \leq a^* = k^*(\varepsilon, a^*) \leq k^*(\varepsilon, \xi) \quad \text{a.e. in } Q. \quad (3.7)$$

Namely, $\xi_A \in K(\varepsilon, \xi)$ almost everywhere in Q . Hence we are allowed to choose $w = \xi_A$, $w_A = \xi$ in the above inequalities, take the sum, integrate on $(0, t)$ for $t \in (0, T]$, and exploit (2.7) and (3.4) in order to have that $\|(\xi - \xi_A)(t)\|_{L^2(\Omega)}^2 = 0$ for every $t \in [0, T]$. \square

The latter lemma is crucial since it reduces the *quasivariational* problem (2.7)-(2.8) to a variational one which is considerably simpler. It is however necessary to check that indeed in the current framework, the function ξ_A solves the quasivariational problem. This is the aim of the following result.

Lemma 3.6. *Under the assumptions (B1)-(B3), the solution $\xi_A \in H^1(0, T; L^2(\Omega))$ of (3.4)-(3.5) solves (2.7)-(2.8) as well.*

Proof. First of all, owing to (3.7) we readily obtain that $\xi_A \in K(\varepsilon, \xi_A)$ almost everywhere in Q . Let us now fix $t \in [0, T]$ such that (3.5) holds. It is a standard matter to check that, for almost every $x \in D := \{y \in \Omega : a_*(y, t) \neq a^*(y, t)\}$ one has that

$$\dot{\xi}_A(x, t) \begin{cases} \in (-\infty, 0] & \text{if } \xi_A(x, t) = a^*(x, t), \\ = 0 & \text{if } a_*(x, t) < \xi_A(x, t) < a^*(x, t), \\ \in [0, +\infty) & \text{if } \xi_A(x, t) = a^*(x, t). \end{cases} \quad (3.8)$$

We prove the first claim, the proof for the others being analogous. Assume $x \in D$ be a Lebesgue point for $y \mapsto \dot{\xi}_A(y, t)(\xi_A(y, t) - a_*(y, t)) \in L^1(\Omega)$. Then, owing to (3.5), it is straightforward to choose $z \in A(t)$ in order to get that

$$\int_{B_\rho(x)} \dot{\xi}_A(y, t)(\xi_A(y, t) - a_*(y, t)) dy = \int_{\Omega} \dot{\xi}_A(y, t)(\xi_A(y, t) - z(y)) dy \leq 0,$$

where $B_\rho(x) := \{y \in \Omega : |x - y| < \rho\}$. By letting ρ go to zero we obtain that

$$\dot{\xi}_A(x, t)(\xi_A(x, t) - a_*(x, t)) \leq 0$$

and the first of (3.8) follows from the fact that $\xi_A(x, t) = a^*(x, t)$.

We now fix $w \in K(\varepsilon, \xi_A(t))$ almost everywhere in Ω and define

$$\bar{w}(\cdot) := a_*(\cdot, t) \vee (w(\cdot, t) \wedge a^*(\cdot, t)).$$

It is straightforward to check that indeed $\bar{w} \in A(t)$. Moreover, we readily have that

$$a_*(x, t) = a^*(x, t)$$

for almost every $x \in \Omega \setminus D$. Hence, in particular, $w = \bar{w}$ almost everywhere in $\Omega \setminus D$. Now, we aim to prove that indeed

$$\int_{\Omega} \dot{\xi}_A(y, t)(\xi_A(y, t) - w(y)) dy \leq 0. \quad (3.9)$$

To this aim we simply exploit (3.4) and (3.8) and compute

$$\begin{aligned} & \int_{\Omega} \dot{\xi}_A(y, t)(\xi_A(y, t) - w(y)) dy \\ &= \int_{\Omega} \dot{\xi}_A(y, t)(\xi_A(y, t) - \bar{w}(y)) dy + \int_{\Omega} \dot{\xi}_A(y, t)(\bar{w}(y) - w(y)) dy \\ &\leq \int_{\{\xi_A = a^*\}} \dot{\xi}_A(y, t)(\bar{w}(y) - w(y)) dy + \int_{\{\xi_A = a_*\}} \dot{\xi}_A(y, t)(\bar{w}(y) - w(y)) dy. \end{aligned} \quad (3.10)$$

It is now straightforward to check that $\bar{w} - w \geq 0$ (≤ 0 , respectively) almost everywhere on $\{\xi_A = a^*\}$ ($\{\xi_A = a_*\}$ respectively). Then, taking again into account (3.8), we deduce that the right hand side of (3.10) is non-positive and (3.9) follows. \square

Owing to these considerations, we are allowed to omit the proofs of Lemmas 3.1 and 3.2 since they can be readily recovered by exploiting Lemma 3.5 and the well-known results on (3.4)-(3.5) (see, e.g., [19]).

On the other hand, let us briefly survey the current literature on (2.7)-(2.8). Indeed, we remark that the contributions on (2.7)-(2.8) usually exploit the regularity of the set-valued map $(\xi, t) \mapsto K(\varepsilon(t), \xi)$. In particular, the latter map is generally assumed to be Lipschitz continuous with respect to the Hausdorff metric with a Lipschitz constant w.r.t. ξ to be strictly lower than 1. This restriction is indeed motivated by the existence of some counterexample to strong solvability for Lipschitz constant w.r.t. ξ greater or equal 1 (see [5, 12]). Here instead we investigate a possibly just continuous but monotone framework, still obtaining the existence of strong solutions. In particular, no Lipschitz regularity with respect to ξ is assumed. The reader is referred to [17] for a related abstract analysis of the problem.

3.3 Approximation

We shall briefly collect here some results of the variable time-step discretization of (3.4)-(3.5). To this aim we start by introducing the partition

$$\mathcal{P} := \{0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T\},$$

with variable time-step $\tau_i := t_i - t_{i-1}$ and let $\tau := \max_{1 \leq i \leq N} \tau_i$ denote the diameter of the partition \mathcal{P} . No constraints are imposed on the possible choice of the time-steps.

In the forthcoming analysis the following notation will be extensively used: being $\{u_i\}_{i=0}^N$ a vector, we denote by $u_{\mathcal{P}}$ and $\bar{u}_{\mathcal{P}}$ two functions of the time interval $[0, T]$ which interpolate the values of the vector $\{u_i\}$ piecewise linearly and backward constantly on the partition \mathcal{P} , respectively. Namely

$$\begin{aligned} u_{\mathcal{P}}(0) &:= u_0, & u_{\mathcal{P}}(t) &:= \alpha_i(t)u_i + (1 - \alpha_i(t))u_{i-1}, \\ \bar{u}_{\mathcal{P}}(0) &:= u_0, & \bar{u}_{\mathcal{P}}(t) &:= u_i, \quad \text{for } t \in (t_{i-1}, t_i], \quad i = 1, \dots, N \end{aligned}$$

where

$$\alpha_i(t) := (t - t_{i-1})/\tau_i \quad \text{for } t \in (t_{i-1}, t_i], \quad i = 1, \dots, N.$$

Given a vector $\{u_i\}_{i=0}^N$, we shall define a second vector $\{\delta u_i\}_{i=1}^N$ as $\delta u_i := (u_i - u_{i-1})/\tau_i$ (namely a discrete derivative). Finally, we indicate with $\varepsilon_{\mathcal{P}}$ and $\xi_{\mathcal{P}}^0$ two suitable approximations of ε and ξ^0 fulfilling the compatibility condition (see (B3))

$$\xi_{\mathcal{P}}^0 \in K(\varepsilon_0, \xi_{\mathcal{P}}^0) \quad \text{a.e. in } \Omega. \quad (3.11)$$

The latter inclusion is, for instance, ensured by (B3) whenever we choose $\varepsilon_0 := \varepsilon(0)$ and $\xi_{\mathcal{P}}^0 := \xi^0$. Of course we stress that our discretization applies to a wider class of possible choices. Let us now consider, for all $i = 1, \dots, N$, the solutions to the nonlinear equations

$$a_i^* = k^*(\varepsilon_i, a_i^*), \quad a_{*i} = k_*(\varepsilon_i, a_{*i}),$$

and the set

$$A_i := [a_{*i}, a_i^*].$$

We are interested in the study of the following discrete scheme

$$\xi_0 = \xi_{\mathcal{P}}^0 \quad \text{a.e. in } \Omega, \quad (3.12)$$

$$\frac{\xi_i - \xi_{i-1}}{\tau_i} + \partial I_{A_i}(\xi_i) \ni 0 \quad \text{a.e. in } \Omega, \quad \text{for } i = 1, \dots, N. \quad (3.13)$$

Let us state the following results.

Lemma 3.7 (Stability, convergence). *The scheme (3.12)-(3.13) has a unique solution $\{\xi\}_{i=0}^N \in (L^2(\Omega))^{N+1}$. Moreover, it fulfills the stability estimate*

$$|\dot{\xi}_{\mathcal{P}}| \leq C_K |\dot{\varepsilon}_{\mathcal{P}}| \quad \text{a.e. in } Q,$$

for all partitions \mathcal{P} . Whenever $\varepsilon_{\mathcal{P}} \rightarrow \varepsilon$ weakly in $H^1(0, T; L^2(\Omega))$ and $\xi_{\mathcal{P}}^0 \rightarrow \xi^0$ weakly in $L^2(\Omega)$, then $\xi_{\mathcal{P}} \rightarrow \xi$ weakly in $H^1(0, T; L^2(\Omega))$ and ξ solves (2.7)-(2.8).

Lemma 3.8 (Error control). *Let (ε, ξ^0) fulfill (B2)-(B3), $(\varepsilon_{\mathcal{P}}, \xi_{\mathcal{P}}^0)$ be suitable approximations, and $\xi, \xi_{\mathcal{P}}$ be the solutions to (2.7)-(2.8), (3.12)-(3.13), respectively. Then we have that, for any $[t_1, t_2] \subset [0, T]$,*

$$\max_{[t_1, t_2]} |\xi - \xi_{\mathcal{P}}| \leq \max \left\{ |\xi^0 - \xi_{\mathcal{P}}^0|, C_K \max_{[t_1, t_2]} |\varepsilon - \bar{\varepsilon}_{\mathcal{P}}| \right\} \quad \text{a.e. in } \Omega. \quad (3.14)$$

The proofs of the the above results rely of course in an application of Lemma 3.5 and former results [14, 19]. The estimate (3.14) turns out to be interesting from the point of view of applications. Indeed, let us stress that no constraint on the possible choice of time-steps is introduced throughout the analysis. In particular, the time-steps could be tailored according to

some experimental or numerical consideration, such as adaptivity. Moreover, the estimate (3.14) also shows some kind of optimality. Consider indeed the situation $T = 1$, $k^*(\varepsilon) = k_*(\varepsilon) := \varepsilon$, $\varepsilon(t) := \min\{1, nt\}$, $n \in \mathbb{N}$. So we have that $C_K = 1$ and, choosing $\mathcal{P} := \{0, 1\}$, $\xi^0 = \xi_{\mathcal{P}}^0 \equiv 0$, and $\varepsilon_i := \varepsilon(t_i)$, one easily checks that, almost everywhere in Ω ,

$$\begin{aligned} \max_{t \in [0,1]} |(\xi - \xi_{\mathcal{P}})(t)| &= \max_{t \in [0,1]} |(\varepsilon - \varepsilon_{\mathcal{P}})(t)| = 1 - \frac{1}{n}, \\ \max_{t \in [0,1]} |(\varepsilon - \bar{\varepsilon}_{\mathcal{P}})(t)| &= 1. \end{aligned} \quad (3.15)$$

One can also reformulate the latter example as $k^* \equiv 1$, $k_*(\varepsilon) := \min\{1, \max\{0, \varepsilon\}\}$, $\varepsilon(t) := t$, and $\xi^0 \equiv 0$ so that we have $C_K = 1$. By choosing $\mathcal{P} := \{0, T\}$, $\xi_{\mathcal{P}}^0 \equiv 0$, and $\varepsilon_i := \varepsilon(t_i)$ one easily finds almost everywhere in Ω that

$$\lim_{T \rightarrow +\infty} \max_{[0, T]} |\xi - \xi_{\mathcal{P}}| = 1 = \lim_{T \rightarrow +\infty} \max_{[0, T]} |\varepsilon - \bar{\varepsilon}_{\mathcal{P}}|.$$

Hence the error estimate (3.14) turns out to be sharp in the above asymptotic sense.

We shall now turn the discussion to the particular case of assumptions (B4). The interest for this special situation is twofold. First of all, looking back to the modeling discussion of Section 2, it turns out that (B4) is fulfilled in our concrete case. On the other hand, in the special monotone case of (B4) we are indeed in the position of proving a stronger error control result. Indeed, one has the following.

Lemma 3.9 (Exact scheme). *Let (B4) and the assumptions of Lemma 3.8 hold. Moreover, let $\xi_{\mathcal{P}}^0 = \xi^0$ and ε be piecewise monotone on \mathcal{P} in the following sense*

$$\begin{aligned} \forall i = 1, \dots, N, \text{ one has that } \varepsilon(t) \lesssim \varepsilon(s) \text{ or } \varepsilon(s) \lesssim \varepsilon(t) \\ \text{a.e. in } \Omega, \forall t, s \in (t_{i-1}, t_i] \text{ with } t < s. \end{aligned} \quad (3.16)$$

Then the scheme (3.12)-(3.13) is exact on \mathcal{P} in the sense that $(\xi - \xi_{\mathcal{P}})(t_) = 0$ almost everywhere in Ω , for all $t_* \in \mathcal{P}$.*

Proof. We simply sketch the argument by considering the discretized problem for two partitions \mathcal{P}, \mathcal{Q} with $\mathcal{P} \subset \mathcal{Q}$. In the above monotone case we readily have that $\xi_{\mathcal{P}} = \xi_{\mathcal{Q}}$ on the partition \mathcal{P} and the assertion follows from the convergence of the algorithm as the diameter of the partition \mathcal{Q} goes to 0. \square

We stress that the monotonicity assumption (B4) is crucial in order to ensure the validity of the sharp result of Lemma 3.9. Indeed, consider the case $\varepsilon(t) := t$, $k^*(\varepsilon) := |\varepsilon - 1|$, $k_* \equiv 0$, $\xi^0 := 1$, where of course (B4) fails to hold. Then, letting $T = 2$, $\mathcal{P} = \{0, 2\}$, $\varepsilon_i := \varepsilon(t_i)$, $\xi_{\mathcal{P}}^0 := 1$ one has that

$$\xi(t) = (1 - t)^+ \quad \text{and} \quad \xi_{\mathcal{P}}(t) \equiv 1 \quad \text{on } [0, 2].$$

In particular $\xi(2) \neq \xi_{\mathcal{P}}(2)$ and the assertion of Lemma 3.9 does not hold.

Before closing this section, let us briefly sketch a comparison argument for discrete solutions that, together with the above proved convergence of the algorithm, will in particular entail the proof of Lemma 3.4.

Lemma 3.10 (Comparison). *Assume (B1) and (B4), let (ε_1, ξ_1^0) and (ε_2, ξ_2^0) fulfill (B2)-(B3), $(\varepsilon_{1,\mathcal{P}}, \xi_{1,\mathcal{P}}^0)$ and $(\varepsilon_{2,\mathcal{P}}, \xi_{2,\mathcal{P}}^0)$ be suitable approximations and $\xi_{1,\mathcal{P}}$ and $\xi_{2,\mathcal{P}}$ be the respective discrete solutions. Moreover, let $\varepsilon_{1,\mathcal{P}} \lesssim \varepsilon_{2,\mathcal{P}}$ almost everywhere in Q and $\xi_{1,\mathcal{P}}^0 \leq \xi_{2,\mathcal{P}}^0$ almost everywhere in Ω . Then $\xi_{1,\mathcal{P}} \leq \xi_{2,\mathcal{P}}$ almost everywhere in Q .*

Proof. It is straightforward to follow the same argument of relations (3.2)-(3.3) and check that, taking into account the above assumptions, deduce that

$$a_{*,1,\mathcal{P}} \leq a_{*,2,\mathcal{P}} \text{ and } a_{1,\mathcal{P}}^* \leq a_{2,\mathcal{P}}^* \text{ almost everywhere in } Q,$$

(with obvious notations). Hence, the assertion follows from $\xi_{1,\mathcal{P}}^0 \leq \xi_{2,\mathcal{P}}^0$ almost everywhere in Ω and (3.12)-(3.13). \square

4 PDE problems

We shall now turn to the study of the full PDE problem consisting in the coupling of the constitutive relation (2.4) the evolution of the internal parameter (2.6), and the *quasi-stationary* momentum equation

$$\sigma_x + b = 0 \text{ in } Q, \quad (4.1)$$

where $b \in L^1(Q)$ is a given load. Due to the one-dimensional nature of the problem, the choice of the boundary conditions

$$u(0, \cdot) = 0, \quad \sigma(1, \cdot) = j(\cdot) \in L^1(0, T),$$

for some traction j , gives rise to a *traction driven problem* which can be easily solved. Indeed, one has just to compute $\sigma(x, t) = j(t) + \int_x^1 b(r, t) dr$ almost everywhere. Then, the evolution of ξ is completely determined through (2.6). Finally (2.4) and the initial condition for ξ allow us to recover the displacement u .

On the other hand, as soon as one is interested in the *displacement driven problem*, namely

$$u(0, \cdot) = 0, \quad u(1, \cdot) = v(\cdot), \quad (4.2)$$

for some given $v : [0, T] \rightarrow (0, +\infty)$, the system does not decouple anymore and its analysis, which is indeed considerably more complex, will be the argument of the forthcoming subsections.

4.1 Results

Arguing as above, as soon as $b(\cdot, t) \in L^1(\Omega)$ for almost every $t \in (0, T)$, we let $g(x, t) := \int_x^1 b(r, t) dr$ and deduce that

$$\sigma(x, t) = s(t) + g(x, t) \text{ a.e. in } Q, \quad (4.3)$$

where s is a real valued function. For the sake of notational convenience and simplicity we will fix $K(\sigma) := [\lambda_\ell(F(\sigma)), \lambda_u(F(\sigma))]$ and $L = 1$. Hence, also owing to (4.2)-(4.3), we may take the integral of (2.4) on $(0, 1)$ and rewrite the full PDE problem (2.4), (2.6), and (4.1) as

$$s(t) + \int_0^1 g(r, t) dr + \kappa \int_0^1 \xi(r, t) \eta(r, t) dr = v(t) \text{ for } t \in [0, T], \quad (4.4)$$

$$\eta(x, t) \in \partial F(s(t) + g(x, t)) \text{ for } (x, t) \in Q, \quad (4.5)$$

$$\dot{\xi}(x, t) + \partial I_{K(s(t)+g(x,t))}(\xi(x, t)) \ni 0 \text{ for } (x, t) \in Q, \quad (4.6)$$

$$\xi(x, 0) = \xi^0(x) \text{ for } x \in \Omega, \quad (4.7)$$

As regard the data g and ξ^0 , we shall make the following assumptions:

$$(C1) \quad g \in W^{2,1}(0, T; L^1(\Omega)) \cap W^{1,1}(0, T; L^\infty(\Omega)), \quad v \in H^1(0, T),$$

$$(C2) \quad \xi^0(\cdot) \in K(s^0 + g(\cdot, 0)) \text{ almost everywhere in } \Omega, \text{ where } s^0 \in \mathbb{R} \text{ fulfills } s^0 + \int_0^1 g(0) + \kappa \int_0^1 \xi^0 \eta^0 = v(0), \text{ for some } \eta^0(\cdot) \in \partial F(s^0 + g(\cdot, 0)) \text{ almost everywhere in } \Omega.$$

Some comments on the latter requirements are in order. Indeed, as a by-product of (C1) and the results of Subsection 4.2, one readily checks that, for all $\xi^0 \in L^\infty(\Omega)$ with $\xi^0 \geq 0$ almost everywhere, there exists a unique $s^0 \in \mathbb{R}$ and $\eta^0 \in L^\infty(\Omega)$ such that the equation and inclusion in (C2) hold. Hence, the requirement of (C2) is actually reduced to the validity of the compatibility condition $\xi^0 \in K(s^0 + g(0))$ which is nothing but the equivalent in this setting of (B3).

We are in the position of proving the following.

Theorem 4.1. *Under the assumptions (A1)-(A2) and (C1)-(C2) there exists*

$$(s, \eta, \xi) \in H^1(0, T) \times L^\infty(Q) \times H^1(0, T; L^2(\Omega))$$

fulfilling (4.4)-(4.7) almost everywhere.

The proof of the latter result will be established by introducing an effective approximation procedure in the forthcoming two sections. Moreover, we are in the situation of proving the following *local Hölder continuous dependence* result.

Lemma 4.2. *Assume (g_1, v_1, ξ_1^0) and (g_2, v_2, ξ_2^0) are two sets of data fulfilling (C1)-(C2). Then, the respective solutions (s_1, η_1, ξ_1) and (s_2, η_2, ξ_2) to problem (4.4)-(4.7) fulfill*

$$\begin{aligned} \|s_1 - s_2\|_{C[0, T]} + \|\xi_1 - \xi_2\|_{C([0, T]; L^p(\Omega))} &\leq \|\xi_1^0 - \xi_2^0\|_{L^p(\Omega)} \\ &+ C_p \left(\|g_1 - g_2\|_{C([0, T]; L^p(\Omega))} + \|g_1 - g_2\|_{L^\infty(0, T; L^1(\Omega))} \right. \\ &\quad \left. + \|g_1 - g_2\|_{L^\infty(0, T; L^1(\Omega))}^{1/2} + \|v_1 - v_2\|_{C[0, T]} \right), \end{aligned} \quad (4.8)$$

for $p \in [1, +\infty]$ and a suitable positive constant C_p depending on $p, \kappa, F, \lambda_\ell, \lambda_u$, and $\|g_i\|_{L^\infty(Q)}$ for $i = 1, 2$.

4.2 Stationary problem

As a first step in the direction of proving Theorem 4.1 we shall be concerned with some stationary problem. Indeed, we assume

$$(C3) \quad \bar{\xi} \in L^\infty(\Omega), \quad g(x, t) = \bar{g}(x) \in L^\infty(\Omega), \quad \bar{v} > 0,$$

replace g with \bar{g} in relations (4.4)-(4.5), and study the stationary system

$$s + \int_0^1 \bar{g} + \kappa \int_0^1 \xi \eta \ni \bar{v}, \quad (4.9)$$

$$\eta \in \partial F(s + \bar{g}) \text{ in } \Omega, \quad (4.10)$$

$$\xi + \partial I_{K(s+\bar{g})}(\xi) \ni \bar{\xi} \text{ in } \Omega. \quad (4.11)$$

As one will see, the latter is nothing but the step-problem for some time-discretization of (4.4)-(4.7). This subsection is devoted to the proof of the following result.

Lemma 4.3. *Under the assumptions (A1)-(A2) and (C3) there exists $(s, \eta, \xi) \in \mathbb{R} \times L^\infty(\Omega) \times L^\infty(\Omega)$ fulfilling (4.9)-(4.11) almost everywhere. Moreover, s and ξ are uniquely determined and one has that*

$$-\sup \bar{g} \leq s \leq -\inf \bar{g}. \quad (4.12)$$

Proof. We start by suitably regularizing the possibly multivalued graph ∂F by means of a parameter dependent monotone Lipschitz continuous function ∂F_ε , $\varepsilon \in (0, 1)$, such that $0 \in \partial F_\varepsilon(0)$, and the following *graph convergence* holds

$$\begin{aligned} \forall x, y \in \mathbb{R} \text{ with } y \in \partial F(x) \text{ there exist } x_\varepsilon, y_\varepsilon \in \mathbb{R} \text{ such that} \\ y_\varepsilon \in \partial F_\varepsilon(x_\varepsilon) \text{ and } (x_\varepsilon, y_\varepsilon) \rightarrow (x, y). \end{aligned}$$

Indeed, owing to (A1), there are actually various possibilities of performing such an approximation. In particular, let us choose the so-called *Yosida* approximation [4]

$$\partial F_\varepsilon(x) := (x - (id + \varepsilon \partial F)^{-1}(x))/\varepsilon,$$

where id is the identity in \mathbb{R} . In this case, ∂F_ε turns out to be Lipschitz continuous of constant ε^{-1} . By replacing ∂F by ∂F_ε in (4.10) and observing that (4.11) is actually equivalent to

$$\xi = \pi_{K(s+\bar{g})}(\bar{\xi}),$$

where $\pi_{K(s+\bar{g})}$ denotes the projection on the nonempty, convex, and closed set $K(s + \bar{g})$, our *stationary and regularized problem* reduces to that of determining $s_\varepsilon \in \mathbb{R}$ such that

$$s_\varepsilon + \int_0^1 \bar{g} + \kappa \int_0^t \pi_{K(s_\varepsilon+\bar{g})}(\bar{\xi}) \partial F_\varepsilon(s_\varepsilon + \bar{g}) = \bar{v}. \quad (4.13)$$

We term the left hand side of (4.13) as $V_\varepsilon(s_\varepsilon)$. Hence it is clear that V_ε is continuous since all the functions under the integral sign are Lipschitz continuous with respect to s_ε . Moreover, one easily check that, if $s_* < -\sup \bar{g}$, then $\partial F_\varepsilon(s_* + \bar{g}) \leq 0$ and also $s_* + \int_0^1 \bar{g} < 0$, hence $V_\varepsilon(s_*) < \bar{v}$. On the other hand, whenever $s^* > -\inf \bar{g}$, then $V_\varepsilon(s^*) > \bar{v}$. Thus, we readily have at least one point s_ε such that (4.13) is fulfilled. As a by-product, we have also deduced the bounds

$$-\sup \bar{g} \leq s_\varepsilon \leq -\inf \bar{g}, \quad (4.14)$$

which are of course independent of ε . Starting from the latter bounds and (A1)-(A2) one easily deduces that

$$\eta_\varepsilon := \partial F_\varepsilon(s_\varepsilon + \bar{g}) \text{ and } \xi_\varepsilon := \pi_{K(s_\varepsilon+\bar{g})}(\bar{\xi}),$$

are uniformly bounded in $L^\infty(\Omega)$. Let us now pass to the limit as $\varepsilon \rightarrow 0$. Owing to the above bounds and possibly extracting some (not relabeled) subsequence, one has that there exist $(s, \eta, \xi) \in \mathbb{R} \times L^\infty(\Omega) \times L^\infty(\Omega)$ such that

$$s_\varepsilon \rightarrow s, \quad (4.15)$$

$$\eta_\varepsilon \rightarrow \eta \text{ weakly star in } L^\infty(\Omega), \quad (4.16)$$

$$\xi_\varepsilon \rightarrow \xi \text{ weakly star in } L^\infty(\Omega). \quad (4.17)$$

Moreover, since $s_\varepsilon + \bar{g}$ converges uniformly to $s + \bar{g}$, one readily checks that the convergence of ξ_ε is uniform as well. In fact, owing to (A2), it is straightforward to bound $|\xi_\varepsilon - \xi_\delta|$ for two parameters $\varepsilon, \delta \in (0, 1)$ with (a multiple of) the quantity $|F(s_\varepsilon + \bar{g}) - F(s_\delta + \bar{g})|$, which actually goes uniformly to zero due to the local Lipschitz continuity of F (see (A1)) and (4.14). Hence, in particular

$$\int_0^1 \xi_\varepsilon \eta_\varepsilon \rightarrow \int_0^1 \xi \eta,$$

and we can pass to the limit in (4.13) and get (4.9). On the other hand, (4.16) ensures that

$$\int_0^1 \eta_\varepsilon(s_\varepsilon + \bar{g}) \rightarrow \int_0^1 \eta(s + \bar{g}),$$

hence (4.10) is a standard consequence of the above graph convergence. Finally, we shall check for (4.11). To this aim, let us choose any $w \in K(s + \bar{g})$ almost everywhere in Ω and define $w_\varepsilon := \pi_{K(s_\varepsilon + \bar{g})}(w)$. It is straightforward to check that $w_\varepsilon \rightarrow w$ uniformly. On the other hand we readily have that

$$(\xi_\varepsilon - \bar{\xi})(\xi_\varepsilon - w_\varepsilon) \leq 0 \quad \text{a.e. in } \Omega.$$

Hence the latter inequality passes to the limit and yields

$$(\xi - \bar{\xi})(\xi - w) \leq 0 \quad \text{a.e. in } \Omega.$$

We conclude for (4.11) by observing that indeed $\xi \in K(s + \bar{g})$.

In order to complete the proof, we shall now check the uniqueness of such solution. Hence, assume we are given two solutions (s_1, η_1, ξ_1) and (s_2, η_2, ξ_2) in the sense of Lemma 4.3 such that, without loss of generality, $s_1 > s_2$. Then, we take the difference of equation (4.9) written for (s_1, η_1, ξ_1) and the same equation written for (s_2, η_2, ξ_2) , and multiply by $s_1 - s_2$ obtaining

$$(s_1 - s_2)^2 + \kappa \int_0^1 (\xi_1 \eta_1 - \xi_2 \eta_2)(\sigma_1 - \sigma_2) = 0,$$

where of course $\sigma_i := s_i + \bar{g}$, $i = 1, 2$ and $\sigma_1 > \sigma_2$. We shall check that the integrand $I := (\xi_1 \eta_1 - \xi_2 \eta_2)(\sigma_1 - \sigma_2)$ is almost everywhere non-negative on Ω . Indeed, we distinguish three cases

- i) $0 \leq \sigma_2 < \sigma_1$. Then we have that $\eta_2 \leq \eta_1$, $0 \leq \eta_1$, and $\xi_2 \leq \xi_1$ almost everywhere. Hence $\xi_2 \eta_2 \leq \xi_2 \eta_1 \leq \xi_1 \eta_1$ and $I \geq 0$ almost everywhere.
- ii) $\sigma_2 < 0 < \sigma_1$. In this case we readily check that $\eta_2 \leq 0 \leq \eta_1$. Hence, of course, $\xi_1 \eta_1 - \xi_2 \eta_2 \geq 0$ and $I \geq 0$ almost everywhere.
- iii) $\sigma_2 < \sigma_1 \leq 0$. We reason as in case i). We have that $\eta_2 \leq \eta_1$, $\eta_2 \leq 0$, and $\xi_2 \geq \xi_1$ almost everywhere. Hence $\xi_2 \eta_2 \leq \xi_1 \eta_2 \leq \xi_1 \eta_1$ and $I \geq 0$ almost everywhere.

Finally, we have proved that $s_1 = s_2$, a contradiction. Since ξ is uniquely determined by the evolution of $K(s + \bar{g})$, we deduce that $\xi_1 = \xi_2$ as well. As regards η_1, η_2 we just claim that η turns out to be uniquely determined where $\partial F(\sigma)$ is a point and nothing can be said in general where $\partial F(\sigma)$ is multivalued. We stress that this is exactly the case of physical interest. Indeed, the straightforward choices $F(\sigma) = |\sigma|$ and λ_ℓ, λ_u as in Figure 1 entail that ∂F is multivalued in 0. On the other hand, η is uniquely determined in the situation of the relaxed problem (4.13). Moreover, the above proof ensures that actually V_ε turns out to be strongly monotone of constant 1. Namely, at each level ε , the point s_ε is unique. \square

Remark 4.4. We stress that the latter argument may be suitably adapted in order to prove that

$$L\|\dot{\sigma}\|_{L^2(Q)} \leq \|\dot{\varepsilon}\|_{L^2(Q)}.$$

The proof of the latter fact is here omitted. Indeed this argument is very close to that of the forthcoming proof of Lemma 4.3. On the other hand, we stress that the above claimed bound justifies the computation of Subsection 2.4. In particular, relation (2.11) turns out to hold for all $t_1, t_2 \in [0, T]$ with $t_1 \leq t_2$.

4.3 Evolutionary problem

We are now ready to present the proof of Theorem 4.1 and Lemma 4.2. Let us refer at once to the notations of Subsection 3.3 and deal with the implicit time discretization scheme

$$s_i + \int_0^1 g_i + \kappa \int_0^1 \xi_i \eta_i = v_i \quad \text{for } i = 1, \dots, N, \quad (4.18)$$

$$\eta_i \in \partial F(s_i + g_i) \quad \text{a.e. in } \Omega, \quad \text{for } i = 1, \dots, N, \quad (4.19)$$

$$\frac{\xi_i - \xi_{i-1}}{\tau_i} + \partial I_{K(s_i + g_i)}(\xi_i) \ni 0 \quad \text{a.e. in } \Omega, \quad \text{for } i = 1, \dots, N, \quad (4.20)$$

$$\xi_0 = \xi^0 \quad \text{a.e. in } \Omega. \quad (4.21)$$

In the latter relations g_i and v_i stand for suitable approximations of g and v fulfilling

$$\sum_{i=2}^N \tau_i \|\delta(\delta g)_i\|_{L^1(\Omega)} \quad \text{and} \quad \sum_{i=1}^N \tau_i \|\delta g_i\|_{L^2(\Omega)}^2$$

are uniformly bounded independently of \mathcal{P} , (4.22)

$$\bar{g}_{\mathcal{P}} \rightarrow g \quad \text{uniformly as the diameter } \tau \rightarrow 0, \quad (4.23)$$

$$\sum_{i=1}^N \tau_i |\delta v_i|^2 \quad \text{is uniformly bounded independently of } \mathcal{P}, \quad (4.24)$$

$$\bar{v}_{\mathcal{P}} \rightarrow v \quad \text{uniformly as the diameter } \tau \rightarrow 0. \quad (4.25)$$

Owing to (C1), the latter requirements may be easily achieved with the choices $g_i := g(0) + \int_0^{t_i} \dot{g}(s) ds$ and $v_i := v(0) + \int_0^{t_i} \dot{v}(s) ds$ which are of course well-defined. In particular, standard interpolation results ensure that $W^{2,1}(0, T; L^2(\Omega)) \cap W^{1,1}(0, T; L^\infty(\Omega))$ is continuously embedded in $H^1(0, T; H^1(\Omega))$.

Under the above assumptions we simply exploit Lemma 4.3 and obtain by induction some vectors $\{s_i\}_{i=1}^N \in \mathbb{R}^N$, $\{\eta_i\}_{i=1}^N \in (L^\infty(\Omega))^N$, and $\{\xi_i\}_{i=0}^N \in (L^\infty(\Omega))^{N+1}$ fulfilling (4.18)-(4.21). In particular, again referring to Subsection 3.3 for notations, we may rewrite the system in the compact form

$$\bar{s}_{\mathcal{P}} + \int_0^1 \bar{g}_{\mathcal{P}} + \kappa \int_0^1 \bar{\xi}_{\mathcal{P}} \bar{\eta}_{\mathcal{P}} = \bar{v}_{\mathcal{P}} \quad \text{a.e. in } (0, T), \quad (4.26)$$

$$\bar{\eta}_{\mathcal{P}} \in \partial F(\bar{s}_{\mathcal{P}} + \bar{g}_{\mathcal{P}}) \quad \text{a.e. in } Q, \quad (4.27)$$

$$\dot{\bar{\xi}}_{\mathcal{P}} + \partial I_{K(\bar{s}_{\mathcal{P}} + \bar{g}_{\mathcal{P}})}(\bar{\xi}_{\mathcal{P}}) \ni 0 \quad \text{a.e. in } Q, \quad (4.28)$$

$$\bar{\xi}_{\mathcal{P}}(0) = \xi^0 \quad \text{a.e. in } Q. \quad (4.29)$$

By also exploiting (4.12) and (4.22), one readily gets that $\|\bar{s}_{\mathcal{P}}\|_{L^\infty(0,T)}$, $\|\bar{\eta}_{\mathcal{P}}\|_{L^\infty(Q)}$, and $\|\bar{\xi}_{\mathcal{P}}\|_{L^\infty(Q)}$ are bounded independently of \mathcal{P} . We now deduce some bound on time variations, we take the difference between (4.18) written at level i and the same relation at level $i-1$ and multiply the resulting equation by $(s_i - s_{i-1})/\tau_i^2$, and have that

$$|\delta s_i|^2 + \left(\int_0^1 \delta g_i \right) \delta s_i + \kappa \int_0^1 \delta(\xi_i \eta_i) \delta s_i = \delta v_i \delta s_i, \quad (4.30)$$

which is actually valid for $i=1$ as well (see (C2)). The last integral in (4.30) shall be handled as follows

$$\kappa \int_0^1 \delta(\xi_i \eta_i) \delta s_i = \kappa \int_0^1 \delta(\xi_i \eta_i) \delta \sigma_i - \kappa \int_0^1 \delta(\xi_i \eta_i) \delta g_i \geq -\kappa \int_0^1 \delta(\xi_i \eta_i) \delta g_i,$$

where, of course, $\sigma_i := s_i + g_i$, for $i=1,2$, and the inequality is motivated by the fact that $\delta(\xi_i \eta_i) \delta \sigma_i \geq 0$ almost everywhere, exactly as argued in the Subsection 4.2. Hence, multiplying (4.30) by τ_i and taking the sum for $i=1, \dots, m$, one may perform a discrete integration by parts and obtain that

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^m \tau_i |\delta s_i|^2 &\leq \sum_{i=1}^m \tau_i \left(\left(\int_0^1 |\delta g_i| \right)^2 + |\delta v_i|^2 \right) \\ &+ \kappa \int_0^1 \xi_m \eta_m \delta g_m - \kappa \int_0^1 \xi_0 \eta_0 \delta g_1 - \kappa \sum_{i=1}^{m-1} \tau_{i+1} \int_0^1 \xi_i \eta_i \delta(\delta g)_{i+1}. \end{aligned}$$

Namely, we may exploit (4.22) and (4.24) in order to deduce that $\|s_{\mathcal{P}}\|_{H^1(0,T)}$ is bounded independently of \mathcal{P} . The latter bound and again (4.22) entail by the same considerations as above that

$$\|\xi_{\mathcal{P}}\|_{H^1(0,T;L^2(\Omega))} \text{ is bounded independently of } \mathcal{P} \quad (4.31)$$

as well. Owing to the above stated a priori bounds and by letting the diameter τ of partition \mathcal{P} go to zero, we may find (possibly taking not relabeled subsequences and thanks to the compact injection $H^1(0,T) \subset C[0,T]$) a triplet (s, η, ξ) such that

$$s_{\mathcal{P}} \rightarrow s \text{ uniformly in } [0, T], \quad (4.32)$$

$$\bar{\eta}_{\mathcal{P}} \rightarrow \eta \text{ weakly star in } L^\infty(Q), \quad (4.33)$$

$$\xi_{\mathcal{P}} \rightarrow \xi \text{ weakly star in } L^\infty(\Omega) \cap H^1(0, T; L^2(\Omega)). \quad (4.34)$$

Let now \mathcal{Q} be a second partition of the interval $[0, T]$ and denote by

$$\mathcal{P} \cup \mathcal{Q} := \{0 = r_0 < r_1 < \dots < r_{M-1} < r_M = T\}$$

for some $M \in \mathbb{N}$. We claim that

$$\max_{1 \leq j \leq i} \|\xi_{\mathcal{P}}(r_j) - \xi_{\mathcal{Q}}(r_j)\|_{L^\infty(\Omega)} \leq c_K \max_{1 \leq j \leq i} \|(s_{\mathcal{P}} + g_{\mathcal{P}})(r_j) - (s_{\mathcal{Q}} + g_{\mathcal{Q}})(r_j)\|_{L^\infty(\Omega)},$$

where c_K bounds the Lipschitz continuity constants of $\lambda_\ell \circ F$ and $\lambda_u \circ K$. In order to prove the latter relation it suffices to reproduce in the current framework the argument of [19, Lemma III.2.1, p. 66]. In particular, we stress that no compactness in space is needed and (4.23) and (4.31) ensure that

$$\xi_{\mathcal{P}}, \bar{\xi}_{\mathcal{P}} \rightarrow \xi \text{ strongly in } L^\infty(Q). \quad (4.35)$$

In particular, (s, η, ξ) fulfills the regularity requirements of Theorem 4.1 and the initial conditions (4.7) almost everywhere. Since (4.23) together with (4.32) imply that

$$\bar{s}_{\mathcal{P}} + \bar{g}_{\mathcal{P}} \rightarrow s + g \quad \text{uniformly in } Q, \quad (4.36)$$

and we readily observe that

$$\int_0^1 \bar{\eta}_{\mathcal{P}}(\bar{s}_{\mathcal{P}} + \bar{g}_{\mathcal{P}}) \rightarrow \int_0^1 \eta(s + g),$$

one can exploit the properties of maximal monotone graphs [4, Prop. 2.5, p. 27] in order to deduce (4.5) almost everywhere. Moreover, thanks to (4.33) and (4.35), we obtain that

$$\int_0^1 \bar{\xi}_{\mathcal{P}} \bar{\eta}_{\mathcal{P}} \rightarrow \int_0^1 \xi \eta \quad \text{a.e. in } (0, T),$$

thus (4.4) holds almost everywhere. As for (4.6), we simply refer to (4.28) in order to claim that

$$\dot{\xi}_{\mathcal{P}}(\bar{\xi}_{\mathcal{P}} - \bar{w}_{\mathcal{P}}) \leq 0 \quad \text{a.e. in } Q, \quad \forall \bar{w}_{\mathcal{P}} \in K(\bar{s}_{\mathcal{P}} + \bar{g}_{\mathcal{P}}) \quad \text{a.e. in } Q.$$

Let us now fix $w \in K(s + g)$ almost everywhere and define $\bar{w}_{\mathcal{P}} := \pi_{K(\bar{s}_{\mathcal{P}} + \bar{g}_{\mathcal{P}})}(w)$. As above, owing to (4.36), we obtain that $\bar{w}_{\mathcal{P}}$ converges uniformly to w in Q . In particular, the convergence (4.34) ensures that

$$0 \geq \int_Q \dot{\xi}_{\mathcal{P}}(\bar{\xi}_{\mathcal{P}} - \bar{w}_{\mathcal{P}}) \rightarrow \int_Q \dot{\xi}(\xi - w),$$

and the almost everywhere validity of (4.6) follows.

We now turn to the proof of Lemma 4.2. In order to simplify notations, henceforth C stands for a positive constant, possibly depending on $p, \kappa, F, \lambda_u, \lambda_\ell$, and $\|g_i\|_{L^\infty(\Omega)}$, $i = 1, 2$, and possibly varying from line to line. First of all we exploit (4.12) an recall that

$$\|s_i\|_{C[0, T]} \leq C. \quad (4.37)$$

We now take the difference between equation (4.4) written for (s_1, η_1, ξ_1) and the same equation for (s_2, η_2, ξ_2) and multiply it by $s_1 - s_2$ obtaining

$$\begin{aligned} (s_1 - s_2)^2 + \left(\int_0^1 g_1 - g_2 \right) (s_1 - s_2) + \kappa \int_0^1 (\xi_1 \eta_1 - \xi_2 \eta_2) (\sigma_1 - \sigma_2) \\ = \kappa \int_0^1 (\xi_1 \eta_1 - \xi_2 \eta_2) (g_1 - g_2) + (v_1 - v_2) (s_1 - s_2), \end{aligned}$$

where $\sigma_i := s_i + g_i$, for $i = 1, 2$. Thus it is a standard matter to reason as above and, also owing to (A1)-(A2), obtain that

$$\frac{1}{2} |s_1 - s_2|^2 \leq \left(\int_0^1 g_1 - g_2 \right)^2 + (v_1 - v_2)^2 + C \int_0^1 |g_1 - g_2|. \quad (4.38)$$

On the other hand, by exploiting (4.6)-(4.7) and recalling, for instance [19, Lemma III.2.1, p.66], it is easy to check that

$$\begin{aligned} \|\xi_1 - \xi_2\|_{C([0, T]; L^p(\Omega))} \leq \|\xi_1^0 - \xi_2^0\|_{L^p(\Omega)} \\ + C (\|s_1 - s_2\|_{C[0, T]} + \|g_1 - g_2\|_{C([0, T]; L^p(\Omega))}). \end{aligned} \quad (4.39)$$

Whence the assertion of Lemma 4.2 follows. Our continuous dependence result entails in particular that the components s and ξ of the solution to problem (4.4)-(4.7) are uniquely determined. As before, η may not be unique in many interesting situations.

4.4 Space approximation

We provide now additional discussion in the direction of the effective approximation of the evolutionary PDE problem (4.4)-(4.7). Indeed, the previous section is devoted to its *fully implicit* time-discretization with variable time-steps. Of course we are also entitled to consider some space discretization as well. To this aim, let us give here some easy example in this direction. Indeed, let us consider the space partition

$$\mathcal{T} := \{0 = x_0 < x_1 < \dots < x_{M-1} < x_M = 1\},$$

and denote by $h_i := x_i - x_{i-1}$ the space step and $h := \max_{1 \leq i \leq M} h_i$ the diameter. Then, we approximate $L^1(\Omega)$ by means of the finite element space of piecewise constant on \mathcal{T} polynomials and define the projector

$$\Pi_{\mathcal{T}}(w)(x) := \frac{1}{h_i} \int_{x_{i-1}}^{x_i} w(r) dr \quad \text{for } x \in (x_{i-1}, x_i], \quad i = 1, \dots, M, \quad \forall w \in L^1(\Omega).$$

It is then straightforward to check that the above requirements are fulfilled. Moreover, one also has that the classical estimate

$$\|w - \Pi_{\mathcal{T}}(w)\|_{L^1(\Omega)} \leq h \text{Var}_{[0,1]} w \quad \forall w \in BV(\overline{\Omega}),$$

where $BV(\overline{\Omega})$ denotes the space of functions $w : [0, 1] \rightarrow \mathbb{R}$ of bounded variation $\text{Var}_{[0,1]} w$. Then, looking back to the previous subsection, we shall make precise this discussion in the following positions

$$\begin{aligned} \gamma_{ij} &:= \frac{1}{h_j} \int_{x_{j-1}}^{x_j} \left(g(x, 0) + \int_0^{t_i} \dot{g}(x, t) dt \right) dx, \\ \bar{g}_{\mathcal{P}\mathcal{T}}(x, t) &:= g_{ij} \quad \text{for } (x, t) \in (x_{j-1}, x_j] \times (t_{i-1}, t_i], \end{aligned}$$

for $i = 1, \dots, N$, $j = 1, \dots, M$. Then, we readily state the following estimate.

Lemma 4.5. *Under the assumptions (A1)-(A2) and (C1)-(C2), let (s, η, ξ) denote an almost everywhere solution to problem (4.4)-(4.7) whose existence is stated in Theorem 4.1 and $(\bar{s}_{\mathcal{P}\mathcal{T}}, \bar{\eta}_{\mathcal{P}\mathcal{T}}, \bar{\xi}_{\mathcal{P}\mathcal{T}})$ a solution to (4.26)-(4.29) where $\bar{g}_{\mathcal{P}}$, ξ^0 are replaced by $\xi_{\mathcal{T}}^0 := \Pi_{\mathcal{T}}(\xi^0)$ and $\bar{g}_{\mathcal{P}\mathcal{T}}$ defined as above. Then, we have that*

$$\begin{aligned} \|s - \bar{s}_{\mathcal{P}\mathcal{T}}\|_{L^\infty(0,T)} + \|\xi - \bar{\xi}_{\mathcal{P}\mathcal{T}}\|_{L^\infty(0,T;L^p(\Omega))} &\leq \|\xi^0 - \xi_{\mathcal{T}}^0\|_{L^p(\Omega)} \\ C_p \left(\|g - \bar{g}_{\mathcal{P}\mathcal{T}}\|_{L^\infty(0,T;L^p(\Omega))} + \|g - \bar{g}_{\mathcal{P}\mathcal{T}}\|_{L^\infty(0,T;L^1(\Omega))} \right. \\ &\quad \left. + \|g - \bar{g}_{\mathcal{P}\mathcal{T}}\|_{L^\infty(0,T;L^1(\Omega))}^{1/2} + \|v - \bar{v}_{\mathcal{P}}\|_{L^\infty(0,T)} \right), \end{aligned}$$

where C_p is the same constant of (4.8). Moreover, if in addition to (C1)-(C2) we also have that

$$(g, v, \xi^0) \in L^\infty(0, T; BV(\overline{\Omega})) \times W^{1,\infty}(0, T) \times BV(\overline{\Omega}),$$

then

$$\|s_1 - \bar{s}_{\mathcal{P}\mathcal{T}}\|_{L^\infty(0,T)} + \|\xi - \bar{\xi}_{\mathcal{P}\mathcal{T}}\|_{L^\infty(0,T;L^1(\Omega))} \leq C(h^{1/2} + \tau),$$

for h small enough, where $C > 0$ depends just on g, v, ξ^0 , and C_p .

We shall omit the proof of the latter *a priori bound* on the total discretization error since it suffices to adapt the argument of Lemma 4.2. Nevertheless, we stress that the above error control imposes no constraints on the possible choice of the partitions and it is optimal with respect to the order of convergence in time since we approximated the time derivative with a first order difference. Finally, we could of course strengthen our assumptions on data and present some alternative error estimates in L^p norms.

Before closing this section we remark that of course some other choice for the space approximation may be considered and we restricted here ourselves to the finite element space P_0 of piecewise constant on \mathcal{T} polynomials case just for the sake of simplicity and a variety of other choices may be considered. Moreover, we stress that in the physically relevant situation $F(\sigma) = |\sigma|$ and λ_u, λ_t as in Figure 1, no regularization of ∂F is needed in order to solve (4.13). Indeed, it is clear that $\xi = 0$ where ∂F is not regular.

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