Nonlocal quasivariational evolution problems*

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Abstract

This note addresses a class of abstract quasivariational evolution equations taking into account nonlocality with respect to time. We present an existence result for suitably weak solutions to such problems, which extend previous contributions. The existence argument relies on some order technique and exploits a fixed point result for multivalued applications in ordered spaces. Moreover, we discuss the application of our results to classes of ODE and parabolic PDE problems.

Key words: quasivariational inequalities, nonlocality, order techniques, fixed point

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1 Introduction

The present analysis is concerned with a class of quasivariational evolution problems taking into account nonlocal in time dynamics. In particular, assume we are given a separable Hilbert space H and a reference time $T > 0$. We shall be concerned with the evolution of $u : (0,T) \to H$ governed by the relations

$$u'(t) + \partial\varphi(u, t, u(t)) \ni 0 \quad \text{in } H \text{ for } t \in (0,T), \quad u(0) = u_0, \tag{1.1}$$

where the prime denotes the derivative with respect to time and u_0 is a datum. In the latter relation φ is a suitable functional which is convex in its last occurrence. The symbol ∂ stands for the usual subgradient in the sense of Convex Analysis taken with respect to the last variable (see below). As one shall see, we directly allow for some time dependence in φ since this is crucial with respect to applications.

The *key feature* of this analysis is the possible *functional dependence* of φ on u, namely its first occurrence. Indeed, we have in mind to investigate a global dependence of φ on the function u as a whole in order to possibly take into account nonlocal in time effects such as memory etc. In particular, this

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paper is a continuation of a series of papers that has been concerned with the treatment of nonlinear and nonlocal abstract evolution problems. Indeed, in [38] a doubly nonlinear nonlocal evolution equation in a hilbertian setting was discussed. The focus there was on existence and approximation issues. Then, in [39] the analysis was extended and complemented to the situation of a reflexive Banach space framework.

Before going on, let us briefly motivate our interest in the abstract problem (1.1) by pointing out some examples of well-known evolution problems that may be included in our generalized formulation.

Example 1.1 (Moreau's sweeping process). Let $K : [0,T] \to 2^H$ (i.e. the parts of H) be such that, for all $t \in [0,T]$, the set $K(t)$ is non-empty, convex, and closed. Moreover, we denote by $I_{K(t)} : H \to [0,+\infty]$ the *indicator function* of $K(t)$, namely

$$I_{K(t)}(u) = 0 \quad \text{if } u \in K(t) \quad \text{and} \quad I_{K(t)}(u) = +\infty \quad \text{otherwise.}$$

Hence, the choice $\varphi(u,t,u(t)) := I_{K(t)}(u(t))$ in (1.1) corresponds to the well-known *Moreau's sweeping process* [25, 33, 34], i.e.

$$u'(t) + \partial I_{K(t)}(u(t)) \ni 0 \quad \text{for } t \in (0,T), \quad u(0) = u_0,$$

for some $u_0 \in K(0)$, where $\partial I_{K(t)}(u(t))$ denotes the *normal cone* to $K(t)$ at $u(t)$. The latter problem may arise in connection to a variety of applications related to non-smooth mechanics, convex optimization, mathematical economics among others [27, 32]. Moreover, it formally includes as a special case the *evolution variational inequality*

$$v(t) \in K', \quad \big(v'(t) - f(t), v(t) - w\big) \le 0 \quad \forall w \in K', \quad t \in (0,T), \quad v(0) = u_0.$$

Here $(\cdot,\cdot)$ denotes the scalar product in H, $K' \subset H$ is non-empty, convex, and closed, and $f \in L^1(0,T;H)$, by means of the positions

$$u(t) := v(t) - \int_0^t f(s)\,ds, \qquad K(t) := K' - \int_0^t f(s)\,ds \qquad t \in (0,T),$$

with obvious notations.

Example 1.2 (Quasivariational sweeping process). Let now $K : [0,T] \times H \to 2^H$ be such that, for all $(t,u) \in [0,T] \times H$, the set $K(t,u)$ is non-empty, convex, and closed. The choice $\varphi(u,t,u(t)) := I_{K(t,u(t))}(u(t))$ turns out to be a generalization of the above mentioned Moreau's sweeping process to the case of *state-dependent* and moving convex sets, i.e.

$$u'(t) + \partial I_{K(t,u(t))}(u(t)) \ni 0 \quad \text{for } t \in (0,T), \quad u(0) = u_0, \tag{1.2}$$

for some $u_0 \in K(0,u_0)$. The latter quasivariational problem arises in connection with the treatment of quasi-statical evolution problems with friction, micro-mechanical damage models (see [24] and the references therein), and the evolution of shape memory alloys [2, 3]. Moreover, the latter extended version of Moreau's sweeping process formally includes also the case of *quasivariational evolution inequalities*

$$v(t) \in K'(v(t)), \quad \big(v'(t) - f(t), v(t) - w\big) \le 0$$
$$\forall w \in K'(v(t)), \ t \in (0,T), \ \ v(0) = u_0,$$

where now $K' : H \to 2^H$ has non-empty, convex, and closed values, by means of the choices, for $t \in (0,T)$,

$$u(t) := v(t) - \int_0^t f(s)\, ds, \quad K(t, u(t)) := K'\left(u(t) + \int_0^t f(s)\, ds\right) - \int_0^t f(s)\, ds.$$

Example 1.3 (Gradient flow). Assume we are given a convex, proper, and lower semicontinuous functional $\psi : H \to [-\infty, +\infty]$. Then, the gradient flow problem for ψ starting from the initial state $u_0 \in H$ with $\psi(u_0) < +\infty$ can be reduced to (1.1) through the position $\varphi(u, t, u(t)) := \psi(u(t))$. Time-dependent problems may also be considered. Let, for instance, $\Lambda : D(\Lambda) \to H$ be linear, positive, and symmetric where $D(\Lambda)$ stands for the domain of Λ. Hence, the problem

$$u' + \Lambda u = f \quad \text{in} \quad (0,T), \quad u(0) = u_0,$$

may be reformulated as problem (1.1) with the choice

$$\varphi(u, t, u(t)) := \frac{1}{2}(\Lambda u(t), u(t)) - (f(t), u(t)).$$

Example 1.4 (Parabolic variational inequalities). Assume that the Hilbert space V is continuously embedded into H and $a : V \times V \to \mathbb{R}$ is a bilinear, continuous, and symmetric form. Moreover, let $K \subset V$ be non-empty, convex, and closed, $u_0 \in K$, and $f : [0,T] \to H$. Then the parabolic variational inequality

$$u(0) = u_0, \qquad u(t) \in K \qquad t \in (0,T),$$
$$\big(u'(t) - f(t), u(t) - v\big) + a\big(u(t), u(t) - v\big) \le 0$$
$$\forall v \in K, \qquad t \in (0,T),$$

may be included in (1.1) by letting, for all $\bar{u} \in L^2(0,T;H)$, $u \in V$, and $t \in (0,T)$, $\varphi(\bar{u}, t, u) := \frac{1}{2}a(u,u) + I_K(u) - (f(t), u)$. Time dependencies in K or a may also be considered (see below).

Example 1.5 (Parabolic quasivariational inequalities). Under the above notations and assumptions, let $K : H \to 2^V$ have non-empty, convex, and closed values. Hence the following parabolic quasivariational inequality

$$u(0) = u_0, \qquad u(t) \in K(u(t)) \qquad t \in (0,T),$$
$$\big(u'(t) - f(t), u(t) - v\big) + a\big(u(t), u(t) - v\big) \le 0 \quad \forall v \in K(u(t)), \qquad t \in (0,T), \tag{1.3}$$

may be rewritten as (1.1) with the choice, for all $\bar{u} \in L^2(0,T;H)$, $u \in V$, and $t \in (0,T)$, $\varphi(\bar{u}, t, u) := \frac{1}{2}a(u,u) + I_{K(\bar{u}(t))}(u) - (f(t), u)$.

Example 1.6 (Parabolic evolution problems). More generally, with the same notations as above, let $\psi : H \times H \to [0, +\infty]$ be such that $\psi(u, \cdot)$ is proper, convex, and lower semicontinuous for all $u \in H$. We will denote its effective domain by $D(u) := \{v \in V : \psi(u, v) < +\infty\}$. Hence, the problem

$$\begin{gathered} u(0) = u_0, \qquad u(t) \in D(u(t)) \quad \text{for} \ \ t \in (0, T), \\ \big(u'(t) - f(t), u(t) - v\big) + a\big(u(t), u(t) - v\big) + \psi(u(t), u(t)) \leq \psi(u(t), v) \\ \forall v \in D(u(t)), \qquad t \in (0, T), \end{gathered}$$

may be reduced to (1.1) with the choice $\varphi(\bar{u}, t, u) := \frac{1}{2} a(u, u) + \psi(\bar{u}(t), u) - (f(t), u)$.

Of course the assumptions on data in the above examples are chosen just in order to justify notations. In particular we stress that one has clearly to impose suitable restrictions for the aim of obtaining some existence result. We will briefly discuss in the forthcoming Section 6 some possible application to the above problems, among others.

The main focus of this paper is that of providing an existence result for a suitably weak version of (1.1). The key assumption of our analysis will clearly concern the functional dependence of φ on u and shall be regarded as of *monotonicity type* (see (A3) below). Our interest in quasivariational problems with ordering properties is clearly motivated by applications since monotonicity stems as a common feature in many modeling situations. A concrete example of a nonlocal material model where monotonicity comes naturally into play is discussed to some extent in [40] where some generalized kinetic hardening model in associative elastoplasticity [18] is addressed.

We shall briefly sketch the lines of our existence argument and refer the reader to the forthcoming analysis for the details. We will firstly check for some (weak) solvability of the so-called *variational section* [4] of problem (1.1)

$$u'(t) + \partial\varphi(\bar{u}, t, u(t)) \ni 0 \quad \text{in } H \ \text{ for } t \in (0, T), \quad u(0) = u_0, \tag{1.4}$$

where $\bar{u}$ is a given datum. Even in the simplest case of Example 1.1 (where indeed φ is independent of $\bar{u}$), it is clear that, in case the dependence of φ on t is not regular, the latter problem (1.4) may fail to have strong solutions [30]. Hence we are forced to consider some suitable notion of weak solution to (1.4) (see Subsection 3.2) which is proved to exist although uniqueness may fail [30, Exemple 1.2]. We denote by $\mathcal{S}(\bar{u})$ the set of weak solutions to (1.4) and recall that the set-valued mapping $\mathcal{S}$ is generally referred to as the *variational selection* of the quasivariational problem (1.1). By introducing an *order structure* on the solution set, we claim that our key assumption of φ entails the validity of an abstract *comparison tool* among weak solutions. In particular, the comparison principle asserts that, whenever we refer to ordered data $\bar{u}_1$ and $\bar{u}_2$, the corresponding solution sets $\mathcal{S}(\bar{u}_1)$ and $\mathcal{S}(\bar{u}_2)$ show some ordering property as well (see below). Finally, we shall present a suitable *fixed point device* for multivalued applications in ordered sets that entails, in particular, the existence of a fixed point for the variational selection $\mathcal{S}$. The latter fixed point is nothing but a generalized solution to problem (1.1).

Our order approach to (1.1) will provide new existence results for suitable nonlocal in time versions of the above mentioned problems among others (see Section 6 below). On the other hand, the order technique here developed may supply some novel existence results for local in time problems as well. This is for instance the case for Example 1.2. In fact, the present approach to quasivariational sweeping processes is quite different from the usual one since most of the contributions to the subject focus on different possible regularity requirements for the set-valued map $(t,u) \mapsto K(t,u)$ [10, 24, 28] rather than on monotonicity issues. In particular, the latter map is usually asked to the Lipschitz continuous with respect to the Hausdorff distance, with a Lipschitz dependence constant related to the dependence on the state u which is *strictly less* than 1. This restriction is motivated from the following counterexample to global strong solvability [25]. Let $H = \mathbb{R}$ and consider the function $\psi : \mathbb{R} \to \mathbb{R}$ defined as $\psi(v) := \max\{-1, \min\{2v - 1/2, 1\}\}$. We shall find a solution $w : [0,T] \to \mathbb{R}$ to the problem

$$w'(t) + \partial I_{C(w(t))}(w(t)) \ni 1\,, \quad w(0) = 0, \tag{1.5}$$

where we set $C(v) := [\psi(v), +\infty)$ for $v \in \mathbb{R}$. Letting $u(t) = w(t) - t$ and defining $K(t,u) = C(u+t) - t$ we check that (1.5) is indeed of the form of (1.2) and the dependence of K on the Hausdorff distance d_H (for all nonempty sets $A, B \subset H$ we let $d_H(A,B) := \max\{\sup_{a\in A} \inf_{b\in B} |a-b|, \sup_{b\in B} \inf_{a\in A} |a-b|\}$) is

$$d_H(K(t,u), K(s,v)) \;=\; |\psi(u+t) - \psi(v+s) - t + s| \;\leq\; |t-s| + 2|u-v|$$

for all $t, s \in [0,T]$ and $u, v \in \mathbb{R}$. Hence, the Lipschitz dependence of K with respect to u shows a factor 2. It is straightforward to check that (1.5) has a unique solution $w(t) = t$ on $[0, 1/2)$ (note that $\psi(1/2) = 1/2$). On the other hand, there is no absolutely continuous solution for $t > 1/2$. Indeed, one has $\partial I_{C(w)}(w) \subset (-\infty, 0]$, hence $w' \geq 1$ almost everywhere. On the other hand $w(t)$ cannot enter the region $\{1/2 < w < 1\}$, since in such a region we have $w < \psi(w)$, hence $w \notin C(w)$. Of course, by suitably tailoring the choice of ψ we could prevent the problem to have even local strong solutions. On the contrary, we will show our weak notion of solution to (1.2) to be well-suited for the the above described critical situation. In particular, we are in the position of proving the existence of a global (weak) solution to relation (1.5) (see Section 6).

Before moving on, we shall remark that the ordering techniques exploited here have interesting analogies with the theory of the solvability of equations by the well-known Perron's method. Indeed, as one shall see, our existence result will rely both on the above mentioned comparison result and on the existence of a pair of ordered sub and supersolutions. One has to mention that the idea of exploiting ordering arguments in connection to quasivariational problems is quite classical [6, 7]. In particular, it was successfully applied to elliptic quasivariational inequalities by Tartar [41] and then extended to some class of evolution quasivariational inequalities by Mignot & Puel [29, 30]. Namely, in [30] the authors address the situation of Example 1.5 in the special case where K models a *unilateral constraint from above* [30, eq. (2.4)-(2.6)] and fulfills a monotonicity condition (see the forthcoming $(A3)_3$ below). In [30], an existence result is provided for a suitable weak formulation of (1.3). Here, the same notion of weak solution is considered and we extend the above referred existence result to more general problems, possibly including general constraints, functions that are not indicators of convex sets, and nonlocality with respect to time. It should be remarked, however, that [30] contains a finer description of the variational selection mapping and of the structure of the solution set.

As for possible applications of our abstract results, we shall give some examples of integrodifferential problems that turn out to admit a weak solution according to our analysis. In particular, we give some detail on the applicability of our abstract results to quasivariational sweeping processes and some parabolic quasivariational inequalities with unilateral constraints depending from the unknown solution in a space-time nonlocal fashion. Although, the formulation of (1.1) is suited as an abstract version of a parabolic PDE problem governed by *symmetric* second order differential operators, let us explicitly stress that we are in the position of considering the situation of *non-symmetric* operators as well. In particular, letting $V \subset H$ continuously and $A : V \to V^*$ (i.e., the dual of V) be linear and continuous (possibly non-symmetric), our existence technique is applied in Section 5 to the problem

$$u'(t) + Au(t) + \partial\varphi(u, t, u(t)) \ni 0 \quad \text{in } V^* \text{ for } t \in (0,T), \quad u(0) = u_0, \tag{1.6}$$

for some suitable initial datum $u_0 \in V$. Finally, we discuss the possibility of including in our framework also some differential problems presenting globally nonlocal nonlinearities of order-preserving type.

Our work is organized as follows. In Section 2 we discuss some preliminary material on orders in Hilbert spaces. In particular, we present our fixed point device for multivalued applications. Then, in Section 3 we introduce our assumptions and state the main results. Moreover, we present some comments on the monotonicity requirements on φ. Section 4 is then devoted to proofs while Section 5 focuses on problem (1.6). Finally, Section 6 contains some applications.

2 Preliminaries

Let us start by setting some notation and presenting some preliminary material. This introductory discussion mainly follows Baiocchi & Capelo [4]. We shall also refer the reader to the work of Calvert [12, 13, 14, 15] for some additional material and results on abstract evolution problems in ordered spaces.

2.1 Orders

Let $(E, \leq)$ denote a non-empty ordered set and $F \subset E$. We recall that $f \in F$ is a *maximal (minimal) element* of F iff, for all $f' \in F$, $f \leq f'$ ($f' \leq f$, respectively) implies $f = f'$. Then, f is the *maximum (minimum)* of F iff $f' \leq f$ ($f \leq f'$, respectively) for all $f' \in F$. Moreover, $e \in E$ is an *upper bound (lower bound)* of F iff $f \leq e$ ($e \leq f$, respectively) for all $f \in F$ and $e \in E$ is the *supremum* or *least upper bound* (*infimum* or *greatest lower bound*) iff e is the minimum (maximum) of the set of upper bounds (lower bounds, respectively) of F. Moreover, we say that F is a *chain* if it is totally ordered and that F is an *interval* iff there exist $e_*, e^* \in E$ such that $F \equiv \{e \in E : e_* \leq e \leq e^*\}$. In the latter case we use the notation $F = [e_*, e^*]$. The set $(E, \leq)$ is said to be *s-inductive (i-inductive)* iff every chain of E is bounded above (below, respectively) and $(E, \leq)$ is said to be *completely s-inductive (completely i-inductive)* iff every chain of E has a supremum (infimum, respectively). Finally $(E, \leq)$ is said to be *inductive (completely inductive)* iff it is both s-inductive and i-inductive (completely s-inductive and completely i-inductive, respectively). The well-known Zorn lemma reads then as follows.

Lemma 2.1. *Let $(E, \leq)$ be s-inductive. Then E has a maximal element.*

Our fixed point tool will result from a suitable extension to set-valued applications in Hilbert ordered spaces of the following lemma.

Lemma 2.2. *Let $(E, \leq)$ be an ordered set and $I := [u_*, u^*] \subset E$ be completely s-inductive. Suppose that $S : (I, \leq) \to (I, \leq)$ is non-decreasing. Then, the set $\{u \in I \ : \ u = S(u)\}$ is non-empty and has a minimum.*

The latter result was announced by Kolodner [23] and turns out to be the main tool in the analysis of [30, 41]. Its proof is to be found, for instance, in [4, Thm. 9.26, p. 223].

2.2 Orders in Hilbert spaces

Assume we are given a Hilbert space H and a non-empty, closed, and convex cone C and define $u \leq v$ iff $v - u \in C$. The latter is an order relation [36, Prop. 3.38, p. 95] and we shall interpret C as the cone of positive elements. By defining the *polar cone* as $C^* := \{u \in H \ : \ (u, v) \leq 0 \ \ \forall v \in C\}$, we possibly obtain, for all $u \in H$, the (orthogonal) decomposition [31] $u = u_1 + u_2$ where $u_1 \in C$, $u_2 \in C^*$ and $(u_1, u_2) = 0$. Indeed the latter elements u_1 and u_2 are exactly the corresponding projections. Owing to these considerations we will use the notation $u_1 = u^+ = \pi_C(u)$ and $u_2 = -u^- = \pi_{C^*}(u)$ (here π stands for the projection). These notations are particularly well motivated in the special case of a *self-polar cone* $C^* = -C$. In the latter case one indeed has $u^- = \pi_C(-u)$. Moreover, we will use the following notation

$$u \vee v := v + (u - v)^+, \quad u \wedge v := u - (u - v)^+.$$

In the particular case of a self-polar cone one of course has that $u \vee v = u + (v-u)^+$ and $u \wedge v = v - (v-u)^+$ as well while this is not true in general. Let us stress that the symbols $\wedge$ and $\vee$ are chosen just for the sake of notational simplicity. Indeed, we are not claiming that one is able to find, for all $u, v \in H$, the element $\inf\{u,v\}$ or $\sup\{u,v\}$ although, whenever they exist, they coincide with $u \wedge v$ and $u \vee v$, respectively. Although all of our analysis can be formulated in the case of a closed convex cone C such that $-C \subset C^*$, we shall restrict ourselves from the very beginning to the situation of self-polar cones $-C = C^*$ instead. Let us explicitly observe that self-polar cones have vertex at the origin. Moreover, we will term the datum (H, C) of a separable Hilbert space and a non-empty self-polar cone as a *Hilbert pseudo-lattice* (see [4, Sec. 19.5, p. 399]). Let us collect here for the reader's convenience some examples in this direction.

Example 2.3 (Orthant). Our first example of a Hilbert pseudo-lattice is $H = \mathbb{R}^n$ $(n \in \mathbb{N})$ and $C := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_i \geq 0 \ \text{for} \ i = 1, \ldots, n\}$, i.e., the n-dimensional non-negative *orthant*. This actually turns out to be a lattice.

Example 2.4 (Non-negative functions). Let (Ω, μ) be a measure space with positive measure μ and denote by $L^2(\Omega, \mu)$ the Hilbert space of all square $\mu-$integrable functions on Ω endowed with the standard inner product. By letting $C := \{u \in L^2(\Omega, \mu) \ : \ u \geq 0 \ \ \mu\text{−a.e. in} \ \Omega\}$ we check that (H, C) is a Hilbert lattice. In particular, $u \leq v$ iff $u(x) \leq v(x)$ for $\mu-$almost every $x \in \Omega$.

Example 2.5 (Positive semi-definite matrices). Let H be the space of symmetric $n \times n$ real matrices endowed with the standard contraction product $(A, B) := \text{tr}(AB)$ for all $A, B \in H$, where tr stands for the trace. We define C as the set of positive semidefinite matrices. Again it is a standard matter to check that (H, C) is a Hilbert pseudo-lattice [19, Cor. 7.5.4, p. 459]. Of course $A \leq B$ iff $B - A$ is positive semidefinite.

Example 2.6 (Second order cone). Given the space H, let us consider the convex cone in $\mathbb{R} \times H$ defined by $C := \{(t, u) \in \mathbb{R} \times H \ : \ t \geq |u|\}$. One easily checks that $(\mathbb{R} \times H, C)$ is a Hilbert pseudo-lattice and that $(t, u) \geq (s, v)$ iff $t - s \geq |u - v|$. In particular, it is easy to check that indeed $(\mathbb{R} \times H, C)$ fails to be a lattice.

Example 2.7 (Infinite dimensional orthant). Assume we are given (H_i, C_i) Hilbert (pseudo-)lattices with $i = 1, \ldots, n$, $n \in \mathbb{N}$. Then $(H_1 \times \cdots \times H_n, C_1 \times \cdots \times C_n)$ is a Hilbert (pseudo-)lattice. Example 2.3 follows with the choice $H = \mathbb{R}$.

Example 2.8 (Conic combination). Let u_n denote a countable orthonormal basis for the separable Hilbert space H. We denote by C the range of the mapping $u \mapsto \sum_{n \in \mathbb{N}} (u, u_n)^+ u_n$. Namely, C is the set of linear combinations of u_n with non-negative coefficients. It is straightforward to check that (H, C) is a Hilbert pseudo-lattice.

Whenever (H, C) is a Hilbert pseudo-lattice, one readily checks that the same holds for $(L^2(0,T;H), C')$ with $C' := \{u \in L^2(0,T;H) \, : \, u \in C \text{ a.e. in } (0,T)\}$. Namely, the space $L^2(0,T;H)$ is endowed with the order $\leq'$ defined, for all $u, v \in L^2(0,T;H)$, as $u \leq' v$ iff $v \leq v$ a.e in $(0,T)$. For the sake of notational simplicity we will use the same symbol $\leq$ for the two orders in H and in $L^2(0,T;H)$ throughout the remainder of the paper.

2.3 A relation on functionals

We now follow for instance [4, 17] and define on the set of convex, proper, and lower semicontinuous functions ψ on a Hilbert pseudo-lattice (H, C) the relation $\prec$ as

$$\psi_1 \prec \psi_2 \quad \text{iff} \quad \forall u_1, u_2 \in H \quad \psi_1(u_1 \wedge u_2) + \psi_2(u_1 \vee u_2) \leq \psi_1(u_1) + \psi_2(u_2).$$

Then, by introducing on the set of non-empty, convex, and closed sets of H the relation $\prec\prec$ as

$$K_1 \prec\prec K_2 \quad \text{iff} \quad \Big(k_1 \in K_1, \ k_2 \in K_2 \ \Rightarrow \ k_1 \wedge k_2 \in K_1, \ k_1 \vee k_2 \in K_2\Big),$$

with obvious notations, we observe that whenever $\psi_1 \prec \psi_2$, one has that $D(\psi_1) \prec\prec D(\psi_2)$, where D stands for the effective domain. In particular, by restricting ourselves to indicator functions $\psi_1 = I_{K_1}, \psi_2 = I_{K_2}$, we readily deduce that $I_{K_1} \prec I_{K_2}$ iff $K_1 \prec\prec K_2$. In particular, relation $\prec\prec$ turns out to be an order on the non-empty closed intervals of H. On the other hand, relation $\prec$ is not an order on the set convex, proper, and lower semicontinuous functions, as it may be plainly checked.

Given a convex, proper, and lower semicontinuous function $\psi : H \to [0, +\infty]$ we can define a possibly multivalued map $\partial\psi : H \to 2^H$ as

$$v \in \partial\psi(u) \quad \text{iff} \quad u \in D(\psi) \ \text{ and } \ (v, w - u) \leq \psi(w) - \psi(u) \quad \forall w \in D(\psi).$$

The latter is referred to as the ***subgradient*** of ψ and is a maximal monotone operator. The reader shall refer to [8] for a detailed discussion. Assume now we are given $\psi_1, \psi_2 : H \to [0, +\infty]$ convex, proper, and lower semicontinuous such that $\psi_1 \prec \psi_2$. It is a standard matter to exploit the definition of subgradient and deduce that, for all $v_1 \in \partial\psi_1(u_1)$, $v_2 \in \partial\psi_2(u_2)$, one has

$$(v_i, u_i - w_i) \geq \psi_i(u_i) - \psi_i(w_i) \quad \forall w_i \in D(\psi_i) \qquad i = 1, 2.$$

Since we have $D(\psi_1) \prec\prec D(\psi_2)$, we may now choose $w_1 := u_1 \wedge u_2$, $w_2 := u_1 \vee u_2$ above, take the sum in the corresponding inequalities and deduce that

$$\big(v_1 - v_2, (u_1 - u_2)^+\big) \geq 0. \tag{2.1}$$

Whenever $\psi_1 = \psi_2 = \psi$, the latter property is nothing but the *T-monotonicity* of $\partial\psi$, originally introduced by Brezis & Stampacchia [9].

2.4 Fixed point lemma

Let us now come to our fixed point device, namely Lemma 2.9. The latter is an extension to the case of set-valued mappings of the former Lemma 2.2. Of course the current literature on fixed point results for multivalued applications is quite rich. Nevertheless, let us stress that we could not find a reference for the forthcoming Lemma 2.9. Hence, we aim to provide here a direct proof together with some comments.

Let us now introduce some notations. Namely, letting F, G denote non-empty subsets of H, we define the relation $\leq^*$ as $F \leq^* G$ iff, for all $f \in F$ there exists $g \in G$ such that $f \leq g$. Of course $F \prec G$ implies that $F \leq^* G$ while the opposite implication does not hold. For the sake of notational simplicity, in the following we write, for instance, $f \leq^* F$ instead of $\{f\} \leq^* F$, etc. We are in the position of proving the following lemma.

Lemma 2.9. *Let (H, C) be a Hilbert pseudo-lattice and $I := [u_*, u^*] \subset H$. Assume that $S : (I, \leq) \to (2^I, \leq^*)$ is non-decreasing and has non-empty and weakly compact values. Then, there exists $u \in I$ such that $u \in S(u)$.*

Proof. Let $U := \{v \in I \,:\, v \leq^* S(v)\}$. We will prove that: (i) U is non-empty, (ii) U with the induced order is completely s-inductive, (iii) U has a maximal element u, (iv) u is a fixed point for S (namely $u \in S(u)$).
Proof of (i): since $S(u_*) \subset I$, we readily check that $u_* \in U$.
Proof of (ii): let $L = \{\lambda_\alpha\}_{\alpha \in A}$ be a chain in U, where $(A, <)$ is a totally ordered set of indices. Owing to [4, Thm. 19.12, p. 399], the interval I turns out to be completely s-inductive. Hence, $\lambda = \sup_{\alpha \in A} \lambda_\alpha \in I$ exists and λ_α converges to λ as α increases. Of course, any subsequence $\{\lambda_n\}_{n \in \mathbb{N}}$ of L is converging to the same limit. Since $\lambda_n \leq \lambda$ and $\lambda_n \in U$, we have that $\lambda_n \leq^* S(\lambda_n) \leq^* S(\lambda)$. Namely, there exist $s_n \in S(\lambda)$ such that $\lambda_n \leq s_n$. Being $S(\lambda)$ weakly compact, one can extract a (not relabeled subsequence) such that s_n weakly converges to $s \in S(\lambda)$. Then, we have that, for all $c \in C$, $(s - \lambda, c) = \lim_{n \to +\infty}(s_n - \lambda_n, c) \geq 0$. Finally $\lambda \leq s \in S(\lambda)$ which amounts to say that $\lambda \in U$.
Proof of (iii): one applies Lemma 2.1.
Proof of (iv): the maximal element u belongs to U, thus there exists $v \in S(u)$ such that $u \leq v$. Hence $S(u) \leq^* S(v)$ and, in particular $v \leq^* S(v)$. Finally, $v \in U$ and, since u is maximal, one has that $u = v \in S(u)$. □

A few comments on the latter lemma are in order. First of all, one observes that, since any non-decreasing function $S : I \to I$ may be regarded as a non-decreasing multivalued application $S : (I, \leq) \to (2^I, \leq^*)$ with non-empty and weakly compact values, Lemma 2.9 actually extends the existence result of the former Lemma 2.2. On the other hand, nothing can be said in general on the existence of a minimum for the set of fixed points of the application S in the framework of Lemma 2.9. Indeed, let us consider $I := [0, 1]$ endowed with the usual order and the map $S_1(0) := \{1\}$ and $S_1(u) := \{u, 1\}$ for all $u \in (0, 1]$. We readily check that $\{u \in I : u \in S_1(u)\} \equiv (0, 1]$ (see Figure 1).

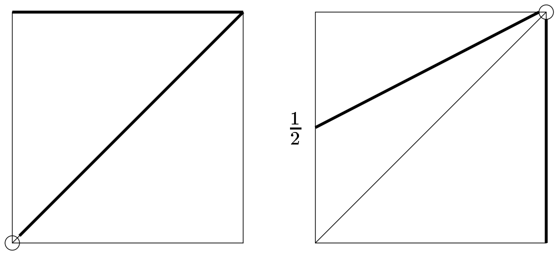

Figure 1: Examples of non-decreasing functions

Moreover, it is clear that the weak compactness of the values of the mapping S is not necessary in order to have fixed points. Nevertheless, we cannot remove this assumption from the statement of Lemma 2.9 as it is shown by the counterexample $I := [0,1]$ and $S_2(u) := (u+1)/2$ for all $u \in [0,1)$, $S_2(1) := [0,1)$ (see Figure 1).

3 Main results

We shall collect in this section the statement our main existence result for problem (1.1).

3.1 Data

Let us start from the assumptions on data.

(A0) Let (H, C) be a Hilbert pseudo-lattice and $u_0 \in H$. We will denote by $(\cdot,\cdot)$ the scalar product in H and by $|\cdot|$ the corresponding norm.

(A1) Let $\varphi : L^2(0,T;H)\times[0,T]\times H \to [0,+\infty]$ such that $\forall(\bar{u},t) \in L^2(0,T;H)\times [0,T]$ the function $\varphi(\bar{u},t,\cdot)$ is convex, proper, and lower semicontinuous and, $\forall \bar{u}, u \in L^2(0,T;H)$, the function $t \mapsto \varphi(\bar{u},t,u(t))$ is measurable.

Let us stress from the very beginning that the assumption on the non-negativity of φ is just motivated for the sake of simplicity and could be weakened, for instance by allowing linear perturbations (see Examples 1.3-1.6 above). This is particularly important within applications and the reader is referred to Section 6 and especially Subsection 6.1 for some further comment. Of course the measurability requirement of (A1) is fulfilled whenever $\varphi(\bar{u},\cdot)$ is a *normal integrand* for all $\bar{u} \in L^2(0,T;H)$ (see, e.g., [36, Sec. 14.D, p. 660]), namely, letting $\mathcal{B}(H)$ be the Borel $\sigma-$algebra in H and $\mathcal{L}$ be the $\sigma-$algebra of Lebesgue measurable subsets of $(0,T)$, the function $\varphi(\bar{u},\cdot)$ is $\mathcal{L}\otimes\mathcal{B}(H)-$measurable and $u \mapsto \varphi(\bar{u},t,u)$ is lower semicontinuous for almost every $t \in (0,T)$.

Before going on we shall detail the notion of subgradient in this context. In particular, let us fix $(\bar{u}, t) \in L^2(0,T;H) \times [0,T]$ and denote by $\partial\varphi(\bar{u}, t, \cdot) : H \to 2^H$ the multivalued map

$$\partial\varphi(\bar{u}, t, u) := \big\{ v \in H \ : \ (v, w-u) \leq \varphi(\bar{u}, t, w) - \varphi(\bar{u}, t, u) \ \ \forall v \in D(\varphi(\bar{u}, t, \cdot)) \big\},$$

which is non-empty for all $u \in D(\partial\varphi(\bar{u}, t, \cdot)) \subset D(\varphi(\bar{u}, t, \cdot))$. As for the time-dependent setting a few comments are in order. Indeed, for all fixed $\bar{u} \in L^2(0,T;H)$, let us introduce the function $\Phi(\bar{u}, \cdot) : L^2(0,T;H) \to [0, +\infty]$ as

$$\Phi(\bar{u}, u) := \begin{cases} \displaystyle\int_0^T \varphi(\bar{u}, t, u(t))\, dt & \text{if } u \in L^2(0,T;H) \text{ and } \varphi(\bar{u}, \cdot, u(\cdot)) \in L^1(0,T) \\ +\infty & \text{if } u \in L^2(0,T;H) \text{ and } \varphi(\bar{u}, \cdot, u(\cdot)) \notin L^1(0,T). \end{cases}$$

Owing to (A1), it is straightforward to check that Φ is convex and lower semicontinuous. Hence the subdifferential $\partial\Phi(\bar{u}, \cdot) : L^2(0,T;H) \to 2^{L^2(0,T;H)}$ is also well-defined and turns out to be non-empty for all $u \in D(\partial\Phi(\bar{u}, \cdot))$. For the sake of later reference we shall let

$$\mathcal{D}(\bar{u}) := D(\Phi(\bar{u}, \cdot)).$$

Moreover, one readily obtains that $v \in \partial\Phi(\bar{u}, u)$ if and only if $v \in L^2(0,T;H)$ and $v(t) \in \partial\varphi(\bar{u}, t, u(t))$ for almost every $t \in (0,T)$. We shall ask for the following.

(A2) $\forall \bar{u} \in L^2(0,T;H)$ the set $\mathcal{D}(\bar{u}) \cap H^1(0,T;H)$ is non-empty,

Let us stress that the latter assumption entail some regularity for the time dependence of φ. On the other hand, we observe that some condition in the direction of (A2) was already considered in the framework of [30]. Moreover, as it will become clear in the sequel (see Subsection 3.2) (A2) will turn out to be a minimal requirement in order the problem not to degenerate into merely viability issues. Let us note that (A2) implies $\Phi(\bar{u}, \cdot)$ to be proper for all $\bar{u} \in L^2(0,T;H)$.

In the remainder of the paper and for the sake of notational simplicity we will often omit to indicate explicitly time dependencies, unless needed.

Let us now come to the *main assumption* of this analysis. We will ask for

(A3) for any $\bar{u}_1, \bar{u}_2$, almost every $t \in (0,T)$, and all $v_1 \in \partial\varphi(\bar{u}_1, t, u_1)$, $v_2 \in \partial\varphi(\bar{u}_2, t, u_2)$ one has

$$\bar{u}_1 \leq \bar{u}_2 \quad \Rightarrow \quad \big(v_1 - v_2, (u_1 - u_2)^+\big) \geq 0. \tag{3.2}$$

In the variational and time-independent case, i.e. φ independent of both $\bar{u}$ and time, (3.2) corresponds to the *T-monotonicity* of the operator $\partial\varphi$ originally introduced by Brezis & Stampacchia [9] and fully exploited in the framework of evolution problems by Calvert [12, 13, 14]. In the current quasivariational setting, assumption (A3) turns out to be the natural extension of T-monotonicity and consists is an abstract monotonicity condition *in the direction of the positive cone*. Before going on we shall clarify this statement by presenting some situations where (A3) is fulfilled. To this aim, we shall look back to the former contributions [30] and [41].

Example 3.1. Let us start by analyzing the monotonicity condition exploited by Tartar [41, eq. (9)] (the same notion was firstly discussed by Duvaut & Lions [17] in order to establish some comparison result for evolution variational inequalities). In [41] the monotonicity assumption reads as (the corresponding time-independent version of) the following

$(\mathrm{A3})_2$ for any $\bar{u}_1, \bar{u}_2 \in L^2(0,T;H)$ one has that

$$\bar{u}_1 \le \bar{u}_2 \quad \Rightarrow \quad \varphi(\overline{u}_1, t, \cdot) \prec \varphi(\overline{u}_2, t, \cdot) \qquad \text{for a.e. } t \in (0,T).$$

As we have already observed in Subsection 2.3, assumption $(\mathrm{A3})_2$ implies (A3) (indeed it is equivalent, see the forthcoming Lemma 4.1).

Example 3.2. We shall now turn to the monotonicity condition of [30]. In particular, we reduce ourselves to the case of *unilateral constraints from above* and introduce a non-decreasing function $M : H \to H$ and, for all $\bar{u} \in L^2(0,T;H)$, the convex set $K(\bar{u}) := \{v \in L^2(0,T;H) : v \le M(\bar{u}) \text{ a.e. in } (0,T)\}$. The requirement of [30, eq. (2.4)-(2.6)] is equivalent to the following

$(\mathrm{A3})_3$ $\forall \bar{u}_1, \bar{u}_2 \in L^2(0,T;H)$ one has that

$$\bar{u}_1 \le \bar{u}_2 \;\Rightarrow\; K(\bar{u}_1) \subset K(\overline{u}_2).$$

Once again it is easy to check (see also [30, eq. (2.5)]) that this is equivalent, whenever restricted to the special geometry of unilateral constraints from above, to the former (A3).

Example 3.3. We shall give an example of functionals fulfilling (A3) without being indicators of closed convex sets. To this aim, let us firstly follow Tartar [41], introduce a bounded open set $\Omega \subset \mathbb{R}^n$, let $H = L^2(\Omega)$, and consider, for $\bar{u} \in L^2(0,T;H)$, $u \in H$, the choice $\varphi(\bar{u}, t, u) := \psi(\bar{u}(t), u)$ for almost every $t \in (0,T)$, where $\psi : H \times H \to [0, +\infty]$ is defined by $\psi(\bar{u}, u) := \int_\Omega \zeta(\bar{u}(x), u(x))\, dx$, and ζ is non-negative, smooth, and convex in u. It is straightforward to check that, whenever $\partial^2\zeta/\partial\bar{u}\partial u \le 0$, the resulting φ fulfills $(\mathrm{A3})_2$. See Tartar [41] for some comment on vector valued functions ζ.

As regards the time-dependent case, we could consider functionals defined almost everywhere by $\varphi(\bar{u}, t, u) := \iint_{\Omega\times(0,t)} \bar{\zeta}(s, \bar{u}(x,s), u(x))\, dx\, ds$, where $\bar{\zeta}$ is, for instance, non-negative, smooth, convex in u, and satisfies $\partial^2\bar{\zeta}/\partial\bar{u}\partial u \le 0$ almost everywhere in time. The constructed φ still fulfills $(\mathrm{A3})_2$.

3.2 Results

We are now in the position of stating our weak formulation of the quasivariational problem (1.1). In particular, we shall be concerned with the following.

Problem Q. To find $u \in \mathcal{D}(u)$ such that

$$\frac{1}{2}|v(0) - u_0|^2 + \int_0^T \Big((v', v - u) + \varphi(u, v) \Big) \geq \int_0^T \varphi(u, u)$$
$$\forall v \in \mathcal{D}(u) \cap H^1(0, T; H).$$

The latter formulation arises from the variational inequality (1.1) whenever a suitably time-regular test function v is chosen. The reader should notice that, since we are assuming (A2), the set of regular test functions $\mathcal{D}(u) \cap H^1(0, T; H)$ is non-empty for all $u \in L^2(0, T; H)$. Stated differently, one observes that the formulation of Problem Q clearly requires some assumption in the direction of (A2). In particular, in case $\mathcal{D}(u) \cap H^1(0, T; H)$ is empty, the inequality is always satisfied and Problem Q reduces to the *viability problem* $u \in \mathcal{D}(u)$. On the other hand, we shall mention that we are not explicitly requiring in our assumptions that there exists some function u such that $u \in \mathcal{D}(u)$. This is of course needed in order to solve Problem Q and will turn out to be a by-product of the specific assumptions of Theorem 3.4 below instead.

Following the general theory, in order to solve the above quasivariational Problem Q we shall be concerned with its *variational section*, namely the following variational counterpart.

Problem V. Given $\bar{u} \in L^2(0, T; H)$, to find $u \in \mathcal{D}(\bar{u})$ such that

$$\frac{1}{2}|v(0) - u_0|^2 + \int_0^T \Big((v', v - u) + \varphi(\bar{u}, v) \Big) \geq \int_0^T \varphi(\bar{u}, u)$$
$$\forall v \in \mathcal{D}(\bar{u}) \cap H^1(0, T; H).$$

Indeed, we first prove that the variational Problem V and admits at least a solution for all data $\bar{u} \in L^2(0, T; H)$. Hence, we will define the *variational selection* mapping $\mathcal{S} : L^2(0, T; H) \to 2^{L^2(0,T;H)}$ carrying the datum $\bar{u}$ into the solution of Problem V. Finally, we will prove that $\mathcal{S}$ possesses a fixed point by means of an application of Lemma 2.9.

We will call $\bar{u} \in L^2(0, T; H)$ a *subsolution* of Problem Q if $\bar{u} \leq u$ for some $u \in \mathcal{S}(\bar{u})$. Analogously $\bar{u} \in L^2(0, T; H)$ is called *supersolution* of Problem Q if $u \leq \bar{u}$ for all $u \in \mathcal{S}(\bar{u})$. Our main results read as follows.

Theorem 3.4. *Assume* (A0)-(A3) *and that there exist a subsolution* u_* *and a supersolution* u^* *to Problem Q with* $u_* \leq u^*$*. Then, the set of solutions* u *to Problem Q such that* $u_* \leq u \leq u^*$ *is non-empty.*

The latter existence results are proved by means of the above described technique in Section 4 below. The above theorems leave open the question whether a subsolution u_* and a supersolution u^* such that $u_* \leq u^*$ exist. Indeed this does not follow from the data and we must explicitly require it. Instead of presenting some abstract conditions for the existence of such sub and supersolutions, we prefer to refer the reader to the forthcoming Section 6 for some examples of concrete constructions, mainly based on maximum principles. As commented above, the existence of suitable sub and supersolutions entails in particular the possibility of solving the nested viability problem $u \in \mathcal{D}(u)$. This fact turns out to be evident within applications where the concrete construction of u_* and u^* is often subject to the solution of the latter viability problem.

Let us now briefly comment of the asymmetry of the above definitions of sub and supersolutions. Indeed this asymmetry is due to the fact that we have chosen to present our result for the s-inductive situation (see Lemmas 2.1-2.2 and 2.9). It is however noteworthy to observe that the same existence results may be obtained in the i-inductive case as well, by suitably modifying the statements. We preferred to stick to the s-inductive situation for the sake of clarity.

As far as uniqueness is concerned, let us stress that nothing can be said in general for Problem Q. Indeed, also Problem V fails to have a unique solution as it is shown in [30, Exemple 1.2]. We shall remark that Mignot & Puel [30] are able to prove a full well-posedness result for *maximum solutions* to Problem Q (namely, functions $\bar{u}$ such that $u \leq \bar{u}$ for all solutions u to Problem Q) under proper additional assumptions on data inspired by Laetsch [26]. This is however the effect of the special unilateral structure of the constraints in [30] and cannot be recovered in the present abstract setting. On the other hand, some previous uniqueness results for abstract elliptic quasivariational inequalities are already outlined in [41] and detailed in [4, Thm 11.7, p. 247]. Unfortunately, an application of the above referred result in our functional setting seems not obvious.

4 Proofs

This section is devoted to the proof of Theorem 3.4. In particular, Subsection 4.1 is concerned with the solvability of Problem V, i.e. the definition of the above introduced solution mapping $\mathcal{S}$. Then, the existence of a fixed point for $\mathcal{S}$ and thus the proof of Theorem 3.4 is obtained in Subsection 4.2.

4.1 Problem V

Let us fix the datum $\bar{u} \in L^2(0,T;H)$ and omit from the very beginning and throughout the remainder of this subsection to indicate explicitly the dependence of φ on $\bar{u}$, for the sake of notational simplicity. Moreover, we denote by $v_{\bar{u}}$ a fixed element in $\mathcal{D}(\bar{u}) \cap H^1(0,T;H)$ (which is non-empty according to (A2)).

In order to prove the existence of a solution to Problem V we will focus on a suitable approximation of φ. In particular, let us fix $\varepsilon > 0$ and regularize φ by means of its Yosida approximation φ_ε defined, for all $t \in [0,T]$ and $u \in H$, as

$$\varphi_\varepsilon(t,u) := \inf_{v \in H} \left(\frac{|v-u|^2}{2\varepsilon} + \varphi(t,v) \right) \tag{4.1}$$

As it is well-known [5, 8], the above regularization enjoys some interesting properties. Here we limit ourselves to point out those features that are exploited in our analysis leaving indeed the reader to the above cited monographes for some further discussion. In particular, we shall make use of the following

$$\forall (t,u) \in [0,T]\times H \quad \varphi_\varepsilon(t,u) \le \varphi(t,u), \tag{4.2}$$

$$\forall t \in [0,T], \forall u_1, u_2 \in H \quad |\partial\varphi_\varepsilon(t,u_1) - \partial\varphi_\varepsilon(t,u_2)| \le |u_1-u_2|/\varepsilon, \tag{4.3}$$

$$\forall u \in L^2(0,T;H) \quad t \mapsto \varphi_\varepsilon(t,u(t)) \text{ and } t \mapsto \partial\varphi_\varepsilon(t,u(t)) \text{ are measurable,} \tag{4.4}$$

$$\text{for any sequence } u_\varepsilon \text{ weakly converging to } u \text{ in } H \text{ as } \varepsilon\to 0 \text{ one has that } \varphi(t,u) \le \liminf_{\varepsilon\to 0} \varphi_\varepsilon(t,u_\varepsilon) \quad \text{for } t\in[0,T], \tag{4.5}$$

$$\forall (t,u) \in [0,T]\times H \qquad \lim_{\varepsilon\to 0} \varphi_\varepsilon(t,u) = \varphi(t,u). \tag{4.6}$$

Namely, owing to (4.1), we readily deduce (4.2) and the fact that φ_ε is convex for all $t \in [0,T]$. Hence the symbol ∂ in (4.3) is justified and we simply refer to [8, Prop. 2.6.i, p. 28] for a proof. The measurability in (4.4) is a consequence of (A1) and (4.1). Finally, (4.5) follows from the Mosco convergence of $\varphi_\varepsilon(t,\cdot)$ to $\varphi(t,\cdot)$ for $t\in[0,T]$ while (4.6) is ensured for instance by [5, Thm. 2.2, p. 57]. The reader is referred to Attouch [1] for some details in this direction.

Let us consider the following.

PROBLEM V_ε. To find $u_\varepsilon \in H^1(0,T;H)$ such that $u(0)=u_0$ and

$$u_\varepsilon'(t) + \partial\varphi_\varepsilon(t,u_\varepsilon(t)) = 0 \quad \text{for a.e. } t\in(0,T), \tag{4.7}$$

Making use of (4.3)-(4.4), it is quite standard to check that Problem V_ε admits a unique solution.Let us now establish some estimates on u_ε, independently of the approximation parameter ε. First of all we observe that one has

$$(u_\varepsilon', u_\varepsilon - v) + \varphi_\varepsilon(u_\varepsilon) \le \varphi_\varepsilon(v) \quad \text{a.e. in } (0,T), \quad \forall v\in L^2(0,T;H). \tag{4.8}$$

Let us now choose $v = v_{\bar u}$ in the latter inequality and integrate on $(0,t)$ for $t\in(0,T)$ in order to get that

$$\begin{aligned} &\frac{1}{2}|(u_\varepsilon - v_{\bar u})(t)|^2 + \int_0^t \varphi_\varepsilon(u_\varepsilon) \\ \le\ &\frac{1}{2}|u_0 - v_{\bar u}(0)|^2 + \int_0^t \varphi_\varepsilon(v_{\bar u}) - \int_0^t (v_{\bar u}', u_\varepsilon - v_{\bar u}). \end{aligned}$$

Hence, also using (4.2), we obtain that

$$|u_\varepsilon(t)| + \int_0^t \varphi_\varepsilon(u_\varepsilon) \ \le\ C \qquad \text{a.e. in } (0,T),$$

where C depends on $|v_{\bar u}(0)|$, $\int_0^T |v_{\bar u}'|^2$, $\int_0^T \varphi(\bar u, v_{\bar u})$, and $|u_0|$ but not on ε.

We are now in the position of finding a function $u \in L^\infty(0,T;H)$ and a (not relabeled) subsequence $\varepsilon \to 0$ such that

$$u_\varepsilon \to u \quad \text{weakly star in } \ L^\infty(0,T;H) \ \text{ as } \ \varepsilon \to 0. \tag{4.9}$$

Let us now fix $v \in \mathcal{D}(\bar{u}) \cap H^1(0,T;H)$ and integrate (4.8) on $(0,T)$, getting

$$\int_0^T (u'_\varepsilon, v - u_\varepsilon) + \int_0^T \big(\varphi_\varepsilon(v) - \varphi_\varepsilon(u_\varepsilon)\big) \geq 0.$$

In particular, we readily check that

$$\frac{1}{2}|v(0) - u_0|^2 + \int_0^T \Big((v', v - u_\varepsilon) + \varphi_\varepsilon(v)\Big) \ \geq \ \frac{1}{2}|(u_\varepsilon - v)(t)|^2 + \int_0^T \varphi_\varepsilon(u_\varepsilon).$$

Next, we exploit (4.2), (4.5), (4.9), pass to the $\liminf$ as $\varepsilon \to 0$ in both sides of the latter inequality, and get that

$$\frac{1}{2}|v(0) - u_0|^2 + \int_0^T \Big((v', v - u) + \varphi(v)\Big) \ \geq \ \int_0^T \varphi(u).$$

Namely, u solves Problem V.

4.2 Problem Q

Let us now turn to the proof of Theorem 3.4. For any given $\bar{u} \in L^2(0,T;H)$ let us introduce the set $S(\bar{u})$ defined as that of *weak-star limits* in $L^\infty(0,T;H)$ of subsequences of solutions to Problem V_ε as $\varepsilon \to 0$. The argument developed in the latter subsection entails in particular that

$$\forall \bar{u} \in L^2(0,T;H) \qquad \emptyset \neq S(\bar{u}) \subset \mathcal{S}(\bar{u}).$$

We shall turn our attention to the map S instead of $\mathcal{S}$. First of all, let us check that the mapping S has indeed some monotonicity property in $[u_*, u^*]$ where u_* and u^* are exactly the sub and supersolution to Problem Q whose existence is assumed in Theorem 3.4. To this aim we shall exploit some further property of the Yosida approximation φ_ε of φ and provide some tools in the direction of [8, Prop. 4.7, p. 134] and [35]. In particular, we shall reproduce in the current quasivariational nonlocal setting some equivalences that were originally reported in [22, Thm. 2.1] (see also [20, 21]).

Lemma 4.1. *Under assumptions* (A0)-(A2)*, the following are equivalent*

- *(i)* φ *fulfills* (A3)*,*
- *(ii)* φ *fulfills* $(\mathrm{A3})_2$*,*
- *(iii)* φ_ε *fulfills* (A3) *for all* $\varepsilon > 0$*,*
- *(iv)* φ_ε *fulfills* $(\mathrm{A3})_2$ *for all* $\varepsilon > 0$*.*

Proof. We shall prove $(ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$. We simply argue as in Subsection 2.3.

$(i) \Rightarrow (iii)$. Let us fix $\bar{u}_1, \bar{u}_2 \in L^2(0,T;H)$ such that $\bar{u}_1 \le \bar{u}_2$ and any $t \in (0,T)$ such that (3.2) holds. From the general theory on the Yosida approximation (see for instance [8]), we readily get that, for all $u_1, u_2 \in H$ and $v_1^\varepsilon = \partial\varphi_\varepsilon(\bar{u}_1, t, u_1)$, $v_2^\varepsilon = \partial\varphi_\varepsilon(\bar{u}_2, t, u_2)$, one has that $v_i^\varepsilon = (u_i - J_i^\varepsilon u_i)/\varepsilon$ for $i = 1, 2$, where $J_i^\varepsilon u_i$ is the unique solution to

$$J_i^\varepsilon u_i + \varepsilon \partial\varphi(\bar{u}_i, t, J_i^\varepsilon u_i) \ni u_i \qquad i = 1, 2.$$

Note in particular that $v_i^\varepsilon \in \partial\varphi(\bar{u}_i, t, J_i^\varepsilon u_i)$ for $i = 1, 2$. Hence, we easily compute that

$$\begin{aligned} &\big(v_1^\varepsilon - v_2^\varepsilon, (u_1 - u_2)^+\big) \\ &= \big(v_1^\varepsilon - v_2^\varepsilon, (J_1^\varepsilon u_1 - J_2^\varepsilon u_2)^+\big) + \big(v_1^\varepsilon - v_2^\varepsilon, (u_1 - u_2)^+ - (J_1^\varepsilon u_1 - J_2^\varepsilon u_2)^+\big) \\ &\ge \frac{1}{\varepsilon}\big((u_1 - u_2) - (J_1^\varepsilon u_1 - J_2^\varepsilon u_2), (u_1 - u_2)^+ - (J_1^\varepsilon u_1 - J_2^\varepsilon u_2)^+\big) \end{aligned}$$

where we exploited (A3). Finally, since $(\cdot)^+$ is monotone, the conclusion holds.

$(iii) \Rightarrow (iv)$. Let us reason by contradiction, consider again $\bar{u}_1, \bar{u}_2 \in L^2(0,T;H)$ such that $\bar{u}_1 \le \bar{u}_2$, and assume that $(A2)_2$ fails on a non-negligible set $E \subset (0,T)$. Hence, letting $t \in E$ be fixed, one finds $u_1, u_2 \in H$ such that

$$\varphi_\varepsilon(\bar{u}_1, t, u_1 \wedge u_2) + \varphi_\varepsilon(\bar{u}_2, t, u_1 \vee u_2) > \varphi_\varepsilon(\bar{u}_1, t, u_1) + \varphi_\varepsilon(\bar{u}_2, t, u_2). \tag{4.10}$$

Let us define, for $\tau \in [0,1]$, the following quantities

$$\begin{aligned} &p := (u_1 - u_2)^+, \quad n := (u_1 - u_2)^-, \\ &p_1(\tau) := u_1 + (\tau - 1)p, \quad p_2(\tau) := u_2 + \tau p, \\ &g_i(\tau) := (\partial\varphi_\varepsilon(\bar{u}_i, t, p_i(\tau)), p) \quad \text{for} \quad i = 1, 2. \end{aligned}$$

We clearly have that

$$\begin{aligned} \varphi_\varepsilon(\bar{u}_1, t, u_1) - \varphi_\varepsilon(\bar{u}_1, t, u_1 \wedge u_2) &= \int_0^1 g_1(s)\, ds, \\ \varphi_\varepsilon(\bar{u}_2, t, u_1 \vee u_2) - \varphi_\varepsilon(\bar{u}_2, t, u_2) &= \int_0^1 g_2(s)\, ds. \end{aligned}$$

Hence, by exploiting (4.10), we claim that there exist $\delta > 0$ and $\bar{\tau} \in (\delta, 1)$ such that $g_1(\bar{\tau}) < g_2(\bar{\tau} - \delta)$. Let us now call $q_1 := p_1(\bar{\tau})$ and $q_2 := p_2(\bar{\tau} - \delta)$ and compute

$$(q_1 - q_2)^+ = \big((u_1 - u_2) + (\delta - 1)p\big)^+ = (\delta p - n)^+ = \delta p,$$

(recall that C has vertex at the origin). Finally, one readily checks that

$$\big(\partial\varphi_\varepsilon(\bar{u}_1, t, q_1) - \partial\varphi_\varepsilon(\bar{u}_2, t, q_2), (q_1 - q_2)^+\big) = \delta\big(g_1(\bar{\tau}) - g_2(\bar{\tau} - \delta)\big) < 0.$$

Namely, also (A3) fails in E, a contradiction.

$(iv) \Rightarrow (ii)$. We simply exploit the convergence (4.6) and pass to the limit as $\varepsilon \to 0$. □

Let us fix $\bar{u}_1, \bar{u}_2 \in L^2(0,T;H)$ and denote by $u_{1\varepsilon}$ and $u_{2\varepsilon}$ the solutions to Problem V_ε with data $\bar{u}_1$ and $\bar{u}_2$, respectively. By taking the difference in the respective equations (4.7), testing on $(u_{1\varepsilon} - u_{2\varepsilon})^+$ and integrating on $(0,t)$ for $t \in (0,T)$, we get that

$$\frac{1}{2}|(u_{1\varepsilon} - u_{2\varepsilon})^+(t)|^2 + \int_0^t \left(v_{1\varepsilon} - v_{2\varepsilon}, (u_{1\varepsilon} - u_{2\varepsilon})^+\right) = 0,$$

where $v_{i\varepsilon} := \partial\varphi_\varepsilon(\bar{u}_i, u_{i\varepsilon})$ for almost every time and $i = 1,2$. Finally, it is a standard matter to apply Lemma 4.1 and deduce that

$$\bar{u}_1 \le \bar{u}_2 \quad \Rightarrow \quad u_{1\varepsilon} \le u_{2\varepsilon} \ \text{ for all } \ \varepsilon > 0. \tag{4.11}$$

Unfortunately, moving from the latter position we cannot infer that $\bar{u}_1 \le \bar{u}_2$ implies $u_1 \le u_2$ for all $u_i \in S(\bar{u}_i)$, $i = 1,2$, since the extracted subsequences converging to u_1 and u_2 need not have the same indices. Nevertheless, by successively extracting subsequences, we are readily in the position of claiming that $\bar{u}_1 \le \bar{u}_2$ implies $S(u_1) \le^* S(u_2)$. In particular, we have that, for all $\bar{u} \in [u_*, u^*] =: I$, one has that $S(\bar{u}) \subset I$ as well. On the other hand, owing for instance to the metrizability of the weak topology of $L^2(0,T;H)$ on bounded sets, we readily check that $S(\bar{u})$ is weakly compact. We are now in the position of applying Lemma 2.9 with $E = L^2(0,T;H)$ and deduce that the set $\{u \in I : u \in S(u)\}$ is non-empty, whence Theorem 3.4 is completely proved.

By carefully reconsidering the latter proof one readily checks that the existence of sub and supersolutions to Problem Q assumed in the statement of Theorem 3.4 may be substantially weakened. Indeed, one needs just the existence of points u_*, u^* in $L^2(0,T;H)$ such that, for all $\bar{u} \in [u_*, u^*]$, one has that $S(\bar{u}) \subset [u_*, u^*]$. This is especially interesting with respect to applications where it is in general useful to exploit the approximation properties of the points in the image of S (see the forthcoming Section 6). According to these considerations we stress that we actually proved the following stronger existence result.

Theorem 4.2. *Assume* (A0)-(A3) *and that there exist* $u_*, u^* \in L^2(0,T;H)$ *such that* $S([u_*, u^*]) \subset [u_*, u^*]$*. Then, the set of solutions* u *to Problem Q such that* $u_* \le u \le u^*$ *is non-empty.*

5 Existence for problem (1.6)

We shall now turn to the analysis of problem (1.6). To this aim let us start by stating our assumptions

(B0) Let (H,C) be a separable Hilbert pseudo-lattice with norm $|\cdot|$ and V be a Hilbert space with norm $\|\cdot\|$, $V \subset H$ continuously and densely such that, for all $v \in V$, one has that $v^+ \in V$ and $\|v^+\| \le C_0\|v\|$ for some $C_0 > 0$. We will denote by $(\cdot,\cdot)$ both the scalar product in H and the duality pairing between V^* and V. Finally, let $u_0 \in H$.

(B1) Let $a : (0,T) \times V \times V \to \mathbb{R}$ be such that

$\forall (u,v) \in V \times V$ the function $t \mapsto a(t,u,v)$ is measurable,
for a.e. $t \in (0,T)$ the form $a(t,\cdot,\cdot)$ is bilinear and fulfills
$\exists C_1 \geq 0 \ : \ \forall (u,v) \in V \times V, \ \ |a(\cdot,u,v)| \leq C_1 \|u\| \, \|v\|$ a.e. in $(0,T)$,
$\exists C_2 > 0, C_3 \geq 0$ such that
$\forall v \in V, \ \ a(\cdot,v,v) \geq C_2 \|v\|^2 - C_3 |v|^2$ a.e. in $(0,T)$,
$\forall v \in V, \ \ a(\cdot,v^+,v^-) \leq 0$ a.e. in $(0,T)$.

The latter assumptions were already considered in [30]. The reader is referred to the above mentioned paper or the forthcoming Section 6 for some concrete examples of spaces and forms fulfilling the above requirements. As for the function φ we will moreover consider (A1) along with the structural monotonicity condition (A3). On the other hand, we will replace (A2) with the following requirement which closely reflects the regularity framework of the problem.

(B2) $\forall \bar{u} \in L^2(0,T;H)$ the set $\mathcal{D}(\bar{u}) \cap H^1(0,T;V^*) \cap L^2(0,T;V)$ is non-empty.

Once again, the latter entails some time-regularity for the function φ and is strongly motivated by our problem formulation (see Subsection 3.2). With these notation at hand, we are in the position of stating the following.

Problem Q$'$. To find $u \in \mathcal{D}(u) \cap L^2(0,T;V)$ such that

$$\frac{1}{2}|v(0)-u_0|^2 + \int_0^T \Big((v', v-u) + a(u, v-u) + \varphi(u,v) \Big) \geq \int_0^T \varphi(u,u)$$
$$\forall v \in \mathcal{D}(u) \cap H^1(0,T;V^*) \cap L^2(0,T;V).$$

Again, exactly as in Section 3.2, we shall start from considering the corresponding variational selection of Problem Q$'$.

Problem V$'$. Given $\bar{u} \in L^2(0,T;H)$, to find $u \in \mathcal{D}(\bar{u}) \cap L^2(0,T;V)$ such that

$$\frac{1}{2}|v(0)-u_0|^2 + \int_0^T \Big((v', v-u) + a(u, v-u) + \varphi(\bar{u},v) \Big) \geq \int_0^T \varphi(\bar{u},u)$$
$$\forall v \in \mathcal{D}(\bar{u}) \cap H^1(0,T;V^*) \cap L^2(0,T;V).$$

The latter may be proved to admit (possibly many) solutions by the same approximation and passage to the limit argument of Section 4. Indeed, one can replace a and φ by

$$\bar{a}(u,v) := a(u,v) + C_3(u,v), \qquad \bar{\varphi}(\bar{u},u) := \varphi(\bar{u},u) - C_3(\bar{u},v)$$
$$\forall \bar{u} \in L^2(0,T;H), \ u,v \in H \ \text{ a.e. in } \ (0,T),$$

where C_3 is exactly the constant in (B1). Indeed, one has that $\bar{a}$ is coercive and $\bar{\varphi}$ fulfills (A1) as well. In particular, the mappings $\mathcal{S}'$, $S' : L^2(0,T;H) \to 2^{L^2(0,T;H)}$ corresponding to *solutions* to Problem Q$'$ for a given datum and *weak-star limits in* $L^\infty(0,T;H)$ of the approximating processes are well-defined. Hence, suitable notions of sub and supersolutions to Problem Q$'$ may now be introduced in the same way as in Section 3.2 and the following theorem holds.

Theorem 5.1. *Assume* (B0)-(B1), (A1), (B2), (A3) *and that there exist a subsolution* u_* *and a supersolution* u^* *to Problem* Q' *with* $u_* \leq u^*$. *Then, the set of solutions* u *to Problem* Q' *such that* $u_* \leq u \leq u^*$ *is non-empty.*

We shall now observe that, whenever φ fulfills (A3), the same holds true for $\bar{\varphi}$. Namely also in this situation, we easily deduce again that relation (4.11) is fulfilled. Finally, the existence of a fixed point in $[u_*, u^*]$ for $\mathcal{S}'$ can be obtained by arguing as in Subsection 4.2. Again, we shall remark that we are actually in the position of proving the following stronger result.

Theorem 5.2. *Assume* (B0)-(B1), (A1), (B2), (A3) *and that there exist* $u_*, u^* \in L^2(0,T;H)$ *such that* $S'([u_*, u^*]) \subset [u_*, u^*]$. *Then, the set of solutions* u *to Problem* Q' *such that* $u_* \leq u \leq u^*$ *is non-empty.*

6 Applications

We aim to give here some concrete application of the latter abstract construction. In particular, we shall present some example of ODE and PDE problems that can be addressed in the above framework and make precise some assumptions on the data of Examples 1.2-1.6 that allow to apply our abstract machinery. Let us remark that the following examples are chosen merely to suggest a variety of problems that can be resolved by the present method and that they are not intended to be the best possible in any sense.

6.1 ODE problems with nonlocal constraints

Let $\Omega \subset \mathbb{R}^n$ be an open and non-empty set and define $H := L^2(\Omega)$, $C := \{v \in H \ : \ v \geq 0 \ \text{ a.e. in } \Omega\}$, and $Q := \Omega \times (0,T)$, so that (A0) is plainly fulfilled. Moreover, let $M : L^2(0,T;H) \to L^2(0,T;H)$ be everywhere defined and non-decreasing. Some examples in this direction are given by the functions $(M_1 u)(x,t) := m(x,t,u(x,t))$ or $(M_2 u)(x,t) := \int_{\Omega\times(0,t)} m(x,s,u(x,s))\,dx\,ds$ for almost every $t \in (0,T)$, where $m : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, non-decreasing with respect to the second variable and bounded (for simplicity). Another class of examples stems from supremum constructions, the reader is referred to the classical examples discussed in [7]. Moreover, one can consider, for $h \in L^1((0,T)^2)$ with $h \geq 0$, the function $M_3 u(t) := \int_0^t h(t,s)u(s)\,ds$, possibly being a convolution term of the type $h(t,s) = j(t-s)$ for some $j \in L^1(0,T)$, etc. Let us stress that, taking into account our concept of weak solution, no *causality* on M has to be required. Namely, the value $(Mu)(t)$ for some $u \in L^2(0,T;H)$ and $t \in (0,T)$, may depend also on the values of u on (t,T) (i.e. the future). Of course, whenever referred to reasonable models, the causality of M is a natural requirement.

We now define $K(u) := \{v \in L^2(0,T;H) \ : \ v \leq M(u) \ \text{ a.e. in } \ Q\}$, and fix $u_0 \in H$ and $f \in L^2(0,T;H)$. Finally, for all $u \in L^2(0,T;H)$ and almost every $t \in (0,T)$, let $N(u,t) := \{v \in H \ : \ v \leq (Mu)(t) \ \text{ a.e. in } \ \Omega\}$,

$\psi(u,t,v) := I_{N(u,t)}(v)$, and $\varphi(u,t,v) := \psi(u,t,v) - \int_\Omega f(t)\, v\, dx$. We claim that, under the above choices, problem (1.1) reduces to

$$u'(t) + \partial I_{N(u,t)}(u(t)) \ni f(t) \quad \text{for} \quad t \in (0,T), \quad u(0) = u_0,$$

modeling indeed the situation of an ODE (nonlocally) constrained by above. We stress that the latter potential φ, which is indeed a very natural choice within applications, may attain negative values as well. This is for the moment in contradiction with (A1) where φ is supposed to be non-negative on its effective domain. The reader is however asked to check that our abstract results still hold in the case of a linear perturbation of a non-negative potential. Assumption (A2) obviously depends upon the current choice of M. For instance, it follows immediately in the above examples M_1 and M_2 since, for all $u \in L^2(0,T;H)$, the functions $M_1 u$ and $M_2 u$ are bounded (hence a suitable constant realizes (A2)). As for M_3 one could simply consider the situation $j \in W^{1,1}(0,T)$ and check that $t \mapsto \int_0^t j(t-s)u(s)\, ds \in H^1(0,T;H)$ itself. Finally, for all $\bar{u}_1,\ \bar{u}_2 \in L^2(0,T;H)$ with $\bar{u}_1 \leq \bar{u}_2$, we have that $M(\bar{u}_1) \leq M(\bar{u}_2) \ \Rightarrow\ K(\bar{u}_1) \subset K(\bar{u}_2)$, and we are in the situation of $(A3)_3$. Our aim is to exploit Theorem 5.1 and deduce that, there exists at least one function $u \in L^2(0,T;H)$ such that

$$u \leq M(u) \quad \text{a.e. in } Q \quad \text{and} \quad \frac{1}{2}|v(0) - u_0|^2 + \int_Q (v' - f)(v - u)\, dx\, dt \geq 0$$
$$\forall v \in H^1(0,T;H) \ \text{ such that } \ v \leq M(u) \ \text{ a.e. in } \ Q. \qquad (6.1)$$

One still needs to provide a sub and a supersolution u_*, u^* to the corresponding variational section (see Section 1). A straightforward choice for u^* is $u^*(t) := u_0 + \int_0^t f^+(s) ds$ for $t \in (0,T)$. Indeed, consider a sequence u_ε of solutions to the corresponding regularized problem V_ε that weakly-star converges to a point $u \in S(u^*)$. Hence we readily check that

$$(u_\varepsilon - u^*)' + \partial\psi_\varepsilon(u^*, u_\varepsilon) = -f^- \quad \text{a.e. in } \ Q.$$

By multiplying the latter inequality by $(u_\varepsilon - u^*)^+$ and taking the integral on $\Omega \times (0,t)$ for some $t \in (0,T)$, we deduce that

$$\frac{1}{2}|(u_\varepsilon - u^*)^+(t)|^2 + \int_{\Omega\times(0,t)} \left(\partial\psi_\varepsilon(u^*, u_\varepsilon) - \partial\psi_\varepsilon(u^*, u^*), (u_\varepsilon - u^*)^+\right) \leq 0,$$

where we also exploited the fact that $u_\varepsilon(0) = u^*(0) = u_0$, and $\partial\psi_\varepsilon(u^*, u^*) \geq 0$ almost everywhere in Q. Hence, it is a standard matter to make use of the T-monotonicity of $\partial\psi_\varepsilon(u^*, \cdot)$ and deduce that $u \leq u^*$.

Some quite similar argument ensures that ant function $u_* \in H^1(0,T;H)$ such that $u_*(0) \leq 0$, $u' \leq f$, and $u_* \leq M u_*$ is a suitable subsolution to (6.1). In order the latter set of relations to admit a solution one could consider some further assumption relating u_0, f, and M. We prefer instead to tackle the (simplified) situation where $0 \leq u_0, f$ and $M0 \geq 0$ (the latter follows from the above examples by choosing $m(\cdot, 0) \geq 0$, for instance). In this case $u_* := 0$ turns out to be a suitable subsolution (recall that, $u_* = 0 \leq u^*$). Namely, since we have that $S([u_*, u^*]) \subset [u_*, u^*]$, we are in the framework of Theorem 4.2.

6.2 PDE problems with nonlocal constraints

We present here an extension of the former results of [30] to the nonlocal case in the framework of Theorem 5.2. Let $\Omega \subset \mathbb{R}^n$ be non-empty, open, and with a regular boundary. Let $H = L^2(\Omega)$, and V be a closed subset of $H^1(\Omega)$ containing $H^1_0(\Omega)$. We shall consider the bilinear form

$$a(t,u,v) :=$$

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,t) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \sum_{i=1}^{n} \int_\Omega b_i(x,t) \frac{\partial u}{\partial x_i}\, v\, dx + \int_\Omega c(x,t)\, u\, v\, dx,$$

for all $u, v \in V$, $t \in (0,T)$, where $a_{ij},\ b_i, c \in L^\infty(Q)$ and fulfill

$$\exists C_2 > 0 \ : \ \forall \xi \in \mathbb{R}^n \quad \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ \geq \ C_2 \sum_{i=1}^{n} \xi_i^2 \quad \text{a.e. in } Q.$$

Let now M, K, and φ be as in Subsection 6.1 and fix $u_0 \in H$ and $f \in L^2(0,T;H)$ such that $0 \leq u_0,\ f$ (again for simplicity) and the time-dependent operator $A : V \to V^*$ defined by $(Au,v) = a(t,u,v)$ for all $v, w \in V$ and almost everywhere in $(0,T)$ (the time dependence is systematically dropped in the notation for the sake of clarity). Along with these choices, it is easy to check that (B0)-(B1), (A1), (B2), and (A3) are fulfilled. Let us now consider the problem of finding solutions $u \in L^2(0,T;V)$ to

$$u \leq M(u) \quad \text{a.e. in } Q, \tag{6.2}$$

$$\frac{1}{2}|v(0)-u_0|^2 + \int_Q (v'-f)(v-u)\, dx\, dt + \int_0^T a(u, v-u)\, dt \geq 0$$
$$\forall v \in H^1(0,T;H) \cap L^2(0,T;V) \text{ such that } v \leq M(u) \text{ a.e. in } Q. \tag{6.3}$$

A suitable choice for the supersolution is $u^* \in H^1(0,T;H) \cap L^2(0,T;V)$ such that

$$(u^*)' + Au^* = f^+ \quad \text{in } V^*, \quad \text{a.e. in } (0,T), \quad u^*(0) = u_0, \tag{6.4}$$

and one can prove as above that $u \leq u^*$ for all $u \in S'(u^*)$ (we recall that $S'(\bar{u})$ stands for the set of weak limits of solutions to Problem V'_ε corresponding to the datum $\bar{u} \in L^2(0,T;H)$). As for the subsolution u_* one readily checks that the choice $u_* = 0$ is still admissible since $u^* \geq 0$. Hence, since $S'([u_*,u^*]) \subset [u_*,u^*]$, we are in the framework of Theorem 5.2, and there exists a solution to (6.2)-(6.3).

The latter extends the result of [30] to the nonlocal in time case. However, we would be in the position of considering bilateral constraints as well as general functional nonlinearities φ. Again referring to [30] for the details, we might extend our applications and consider some class of *nonlocal boundary constraints*. Moreover, we could turn to systems of inequalities by referring to the product space $H = (L^2(\Omega))^m$ provided a suitable pseudo-lattice structure (see, e.g., Example 2.7).

6.3 Quasivariational sweeping processes

Our next aim is to apply the results of Subsection 3.2 to the situation of problem (1.2). Assume we are given a Hilbert pseudo-lattice (H, C) and a function $K : [0, T] \times H \to 2^H$ with non-empty, convex, and closed values. We will ask that, for all $\bar{u} \in L^2(0, T; H)$, there exists a selection $u(t) \in K(t, \bar{u}(t))$ almost everywhere with $u \in H^1(0, T; H)$ so that (A2) is fulfilled. Moreover, for almost every t, we require, for all $u_1, u_2 \in H$ with $u_1 \leq u_2$, that $K(t, u_1) \prec\prec K(t, u_2)$. Hence (A3) follows and we are in the position of applying Theorems 3.4 and 4.2 to (1.2).

Let us again stress that our approach to problem (1.2) is quite different from the current literature [10, 24, 28] on quasivariational sweeping processes. In the above mentioned papers the map $(t, u) \mapsto K(t, u)$ is asked to be Lipschitz continuous (with respect to the Hausdorff topology) with Lipschitz constant related to the dependence on u which is strictly less than 1. Moreover, some additional compactness [24] or smoothness and non-empty interior conditions [10] are assumed. Here instead K is allowed to be non-smooth and have non-compact values. This is particularly well suited with respect to infinite dimensional applications (see Subsections 6.1-6.2 above). On the other hand K is asked to be suitably non-decreasing in u with respect to relation $\prec\prec$. This entails the possibility of (weakly) solving (1.2) for choices of K that could not be handled by previous contributions. A first example of this fact is the counterexample to strong solvability of (1.5). The latter fits indeed into our framework by setting $u(t) = w(t) - t$ and $K(t, u) = C(u + t) - t$. In particular, can apply Theorem 4.2 along with the natural choices $u_* := 0 \leq 1 =: u^*$ and find suitable solutions to (1.5) in the sense Problem Q. We shall however observe that the latter solutions show some additional unphysical features (non-uniqueness, for instance). More recently, again along the lines of Theorem 4.2, we have succeeded in extending the above referred weak existence result to a stronger $BV-$type functional setting where indeed solutions (still not unique) are better behaved [11]. Let us mention that Theorems 3.4 and 4.2 allow us to consider the situation of nonlocal in time state dependent sweeping processes as well. The reader is referred to [40] for some detail in this direction. Finally, we refer to [37] and [16] for some further application of order techniques to quasivariational sweeping processes.

6.4 Parabolic quasivariational inequalities

We shall state some precise assumption in order that Examples 1.5-1.6 can be handled in the framework of Section 5. To this aim, we let a be coercive (or better $H-$coercive), $u_0 \in H$, $f \in L^2(0, T; V^*)$, and we ask K and ψ to be such that, for all $u_1, u_2 \in H$ with $u_1 \leq u_2$, one has $K(u_1) \prec\prec K(u_2)$ and $\psi(u_1, \cdot) \prec \psi(u_2, \cdot)$. Hence, letting (H, C) be a Hilbert pseudo-lattice, assumptions (B0)-(B1), (A1), and (A3) will be clearly fulfilled. The measurability requirement of (A2) and the existence of suitable sub and supersolutions (here not addressed) will then suffice in order to apply Theorems 5.1 and 5.2 and possibly deduce the existence of a weak solution to the quasivariational evolution problems of Examples 1.5-1.6. As before, this technique is not restricted to the above mentioned local in time problems and could be extended as well to some nonlocal analogue (see also Subsections 6.1-6.2).

6.5 Equations with non-decreasing nonlinearities

We are entitled to provide the weak solvability in the sense of Problems Q and Q′ of some differential problems including nonlocal non-decreasing nonlinearities. In particular, let us assume (A0), $f \in L^2(0,T;H)$, and $u_0 \in H$. Moreover, let $M : L^2(0,T;H) \to L^2(0,T;H)$ be an everywhere defined, non-decreasing mapping (for instance any of the nonlocal operators introduced in Subsection 6.1). Let us consider the problem of finding $u \in L^2(0,T;H)$ such that

$$\frac{1}{2}|v(0)-u_0|^2 + \int_0^T (v' - Mu - f, v - u) \geq 0 \qquad \forall v \in H^1(0,T;H). \tag{6.5}$$

The latter stands for a suitable weak formulation of the equation $u' = Mu + f$. Problem (6.5) may be rewritten in our abstract setting by letting, for all $u \in L^2(0,T;H)$, $v \in H$, and almost every $t \in (0,T)$, the function φ be defined by $\varphi(u,t,v) = -\big((Mu)(t), v\big) - \big(f(t), v\big)$. In particular, it is straightforward to check that (A1) as well as (A3) (recall that M is non-decreasing) hold true.

We shall briefly comment the possibility of providing suitable sub and supersolutions to (6.5). To this aim, some restriction on the choice of M (related to u_0 and f, see above) has to be introduced. For the sake of simplicity (other choices are possible) we limit ourselves to consider the (simplified) situation of Subsections 6.1-6.2 namely $0 \leq u_0, f$ and $0 \leq M0$. Moreover we ask for $Mu \leq M^*$ for all $u \in L^2(0,T;H)$ and some $M^* \in L^2(0,T;H)$ (the latter is a rather mild restriction with respect to concrete situations, see Subsections 6.1-6.2). Hence, one readily checks that $u_* = 0$ is a subsolution to (6.5). On the other hand, a suitable supersolution u^* to (6.5) is provided by $u^*(t) = u_0 + \int_0^t (M^* + f)$ for all $t \in [0,T]$. Indeed, since $(u^*)' = M^* + f \geq Mu^* + f$ almost everywhere in $(0,T)$, the fact that $\mathcal{S}(u^*) \leq u^*$ (note that here $\mathcal{S}$ and S coincide and reduce to a point) follows by standard comparison arguments. The regularity assumption (A2) is not fulfilled in the present situation since $\partial\varphi(u,t,v) = -(Mu)(t) - f(t) \in L^2(0,T;H)$. On the other hand, the reader may check that (A2) is actually not needed in the current setting since we immediately solve the variational section problem without exploiting the Yosida approximation. Finally, one readily checks that $u_* = 0 \leq u^*$ and the existence of a solution to (6.5) follows along the lines of Theorem 3.4.

Let us now briefly discuss a corresponding PDE analogue to (6.5). To this aim, we shall assume (B0)-(B1) and consider the problem of finding $u \in L^2(0,T;V)$ such that

$$\frac{1}{2}|v(0)-u_0|^2 + \int_0^T (v' - Mu - f, v - u) + \int_0^T a(v, v-u) \geq 0$$
$$\forall v \in H^1(0,T;H) \cap L^2(0,T;V). \tag{6.6}$$

The latter stands for a suitable weak formulation of $u' + Au = Mu + f$ where A is defined from a as in Subsection 6.2. Again, whenever M is non-decreasing, the latter fits into the framework of (A1)-(A3). Moreover, again referring to the simple case $0 \leq u_0, f$ and $0 \leq M0$, the choice $u_* = 0$ and u^* solving $(u^*)' + Au^* = M^* + f$ almost everywhere in $(0,T)$, $u^*(0) = u_0$, still provide suitable sub and supersolutions to (6.6) with $u_* \leq u^*$. Finally, our existence theory applies to (6.6). Let us stress that the latter technique may be adapted to a variety of different situations including, for instance, nonlocal constraints.

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