

EXISTENCE FOR A CLASS OF NONLOCAL QUASIVARIATIONAL EVOLUTION PROBLEMS

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Abstract. We address the weak solvability of a class of evolution problems characterized by a nonlocal-in-time quasivariational character. In particular, this note is concerned with a suitably regular quasivariational functional dependence. The existence proof is achieved by means of the application of an infinite-dimensional version of KAKUTANI'S fixed-point theorem. An application of the abstract existence result to parabolic variational inequalities with Volterra-type nonlinear constraints is also provided.

1 Introduction

This short note is devoted to an abstract existence result for a class of nonlocal-in-time quasivariational inequalities of parabolic type. The prototype problem within this class is that of finding a (suitably weak) solution u of

$$\begin{cases} u \leq K(u) & \text{in } Q, \\ u_t - \operatorname{div} \alpha(\nabla u) + \beta(u) \leq g & \text{in } Q, \\ (u_t - \operatorname{div} \alpha(\nabla u) + \beta(u) - g)(u - K(u)) = 0 & \text{in } Q, \end{cases} \quad (1)$$

where

$$K(u)(x, t) = \gamma \left(\int_Q k(x, y, t, s) u(s, y) \, dy \, ds \right), \quad \forall (x, t) \in Q;$$

here $Q := \Omega \times [0, T]$ is a space-time cylindrical domain with a smooth bounded domain Ω in \mathbb{R}^d and a final reference time T , $\alpha \subset \mathbb{R}^d \times \mathbb{R}^d$ and $\beta \subset \mathbb{R} \times \mathbb{R}$ are maximal monotone graphs, $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is a given function, the kernel k is suitably smooth, and the datum g and boundary and initial data are prescribed.

In much greater generality, we will focus on the problem of proving the existence of a suitably weak solution to the abstract differential inclusion

$$u'(t) + \partial\phi_u(t, u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T), \quad u(0) = u_0. \quad (2)$$

Here, the abstract time evolution $t \mapsto u(t)$ takes place in a Hilbert space U and corresponds to the non-autonomous gradient flow in U^* (dual of U) driven by the functional ϕ which is assumed to be convex in its last variable. The prime denotes time-differentiation, the symbol ∂ stands for the usual subgradient in the sense of Convex Analysis taken with respect to the last variable, and the datum $t \mapsto f(t) \in U^*$ is given.

The key feature of this problem is the possible dependence of the functional ϕ_u on the function u as a whole trajectory. This dependence is intended to possibly model nonlocal-in-time effects such as memory etc. In this respect, this note is the continuation of a series of papers by the second author which is concerned with the treatment of nonlinear and nonlocal abstract evolution problems. Indeed, in [16] a doubly nonlinear nonlocal evolution equation in a Hilbert-space setting was discussed. The focus there was on existence and approximation issues. Then, in [17] the analysis was extended and complemented to the situation of a reflexive Banach-space framework. Finally, [18] deals exactly with (2) but under quite different monotonicity assumptions which will be commented below.

The main result of this note is the existence of suitably weak solutions to (2) (see Section 2). The strategy of the proof is fairly traditional. By firstly fixing a datum \bar{u} in the functional $\phi_{\bar{u}}(\cdot)$, we solve the corresponding variational problem. This is usually referred to as the *variational section* [2] of problem (2) and reads

$$u'(t) + \partial\phi_{\bar{u}}(t, u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T), \quad u(0) = u_0. \quad (3)$$

In case the dependence of ϕ on time is not regular, the latter problem may fail to admit strong solutions (see below). Hence we are forced to consider some suitable notion of weak solution of (3) which can be proved to exist, although non-uniquely. By defining

$\mathcal{S}(\bar{u})$ to be the set of all weak solutions to (3) (\mathcal{S} is generally referred to as the *variational selection* of the quasivariational problem (2)) the existence of a weak solution to (2), namely $u \in \mathcal{S}(u)$, will then follow by means of an application of an infinite-dimensional version of KAKUTANI'S fixed-point theorem. Finally, the abstract existence result will be applied to the concrete case of the evolution variational inequality with Volterra-type nonlinear constraints in (1) in Section 4.

Recently, an abstract inclusion of the form (2) has been investigated by the first author and his collaborators from a different point of view (see [11]).

2 Main result

We shall start by enlisting our assumptions.

(A1) Let U and V be real separable Hilbert spaces such that the injection $U \subset V$ is dense and continuous.

One denotes by U^* the dual of U and by $\langle \cdot, \cdot \rangle$ the duality pairing between U^* and U . Moreover, we identify $V \equiv V^*$ (dual) so that (U, V, U^*) form a classical Hilbert triplet. Note that the injection $U \subset V$ is not assumed to be compact. Finally, let $|\cdot|$ and $\|\cdot\|_E$ be the norm in V and in the generic Banach space E , respectively. We use the standard notation for vector-valued function spaces $L^2(0, T; U)$, $H^1(0, T; U)$ etc. [12], recall that

$$H^1(0, T; U^*) \cap L^2(0, T; U) \subset C([0, T]; V) \quad \text{continuously,}$$

and assume on data that

(A2) $u_0 \in U$, $f \in L^2(0, T; U^*)$.

Our structural assumptions on the potential ϕ read as follows.

(A3) Let $(\bar{u}, t, u) \in L^2(0, T; U) \times [0, T] \times U \mapsto \phi_{\bar{u}}(t, u) \in [0, +\infty]$ be such that for all $\bar{u} \in L^2(0, T; U)$ the function $(t, u) \mapsto \phi_{\bar{u}}(t, u)$ is a normal convex integrand.

We recall that, by letting $\mathcal{B}(U)$ be the Borel σ -algebra in U , \mathcal{L} be the σ -algebra of the Lebesgue measurable subsets of $(0, T)$, and $\mathcal{L} \otimes \mathcal{B}(U)$ the respective product σ -algebra, the functional $(t, u) \mapsto \phi_{\bar{u}}(t, u)$ is a normal convex integrand if it is $\mathcal{L} \otimes \mathcal{B}(U)$ -measurable and

$$u \mapsto \phi_{\bar{u}}(t, u) \quad \text{is convex and lower-semicontinuous for a.e. } t \in (0, T).$$

Moreover, for each $(\bar{u}, t, u) \in L^2(0, T; U) \times [0, T] \times U$, we recall that the subdifferential $\partial\phi_{\bar{u}}(t, u) \subset U^*$ with respect to u is classically defined by

$$v \in \partial\phi_{\bar{u}}(t, u) \iff \phi_{\bar{u}}(t, u) < \infty \quad \text{and} \quad \langle v, w - u \rangle \leq \phi_{\bar{u}}(t, w) - \phi_{\bar{u}}(t, u) \quad \forall w \in U.$$

For all fixed $\bar{u} \in L^2(0, T; U)$, let us introduce the convex and lower-semicontinuous functional on whole trajectories $\psi_{\bar{u}} : L^2(0, T; U) \rightarrow [0, +\infty]$ as

$$\psi_{\bar{u}}(u) = \begin{cases} \int_0^T \phi_{\bar{u}}(t, u(t)) dt & \text{if } t \mapsto \phi_{\bar{u}}(t, u(t)) \in L^1(0, T), \\ \infty & \text{otherwise.} \end{cases}$$

For the sake of later reference, let $\mathcal{D}(\bar{u})$ be the essential domain of $\psi_{\bar{u}}$, namely

$$\mathcal{D}(\bar{u}) = \{u \in L^2(0, T; U) : \psi_{\bar{u}}(u) < \infty\}.$$

From now on, the explicit dependence of the functional ϕ on t will be often omitted in order to shorten notations.

We shall be interested in the following quasivariational problem.

Problem Q To find $u \in \mathcal{D}(u)$ such that

$$\begin{aligned} \frac{1}{2}|v(0) - u_0|^2 + \int_0^T \langle v' - f, v - u \rangle + \int_0^T \phi_u(v) &\geq \int_0^T \phi_u(u) \\ \forall v \in \mathcal{D}(u) \cap H^1(0, T; U^*). \end{aligned}$$

The latter corresponds to a weak formulation of problem (2). Indeed, given a strong solution $u \in H^1(0, T; U^*) \cap L^2(0, T; U)$ of (2), we have that for any smooth test $v \in \mathcal{D}(u) \cap H^1(0, T; U^*)$

$$\langle u' - f, v - u \rangle + \phi_u(u) \leq \phi_u(v) \quad \text{a.e. in } (0, T) \quad \text{and} \quad u(0) = u_0.$$

Hence, by taking the integral in time on $[0, T]$, integrating by parts by means of the well-known formula for functions in $H^1(0, T; U^*) \cap L^2(0, T; U)$, and using $u(0) = u_0$ we deduce

$$\frac{1}{2}|v(0) - u_0|^2 + \int_0^T \langle v' - f, v - u \rangle + \int_0^T \phi_u(v) - \int_0^T \phi_u(u) \geq \frac{1}{2}|v(T) - u(T)|^2.$$

Therefore the formulation of Problem Q follows by simply neglecting the last (non-negative) contribution. This was exactly the starting point of MIGNOT & PUEL [13], later extended to the nonlocal case by the second author [18]. Note that the solution u of Problem Q needs not admit time-derivatives. Indeed, the notion of solution here considered is quite weak and, in particular, no uniqueness is expected (see Section 3).

A necessary assumption for the meaningfulness of Problem Q concerns the existence of smooth trajectories in $\mathcal{D}(u)$, that is

$$\mathcal{D}(u) \cap H^1(0, T; U^*) \neq \emptyset \quad \text{for some } u \in L^2(0, T; U).$$

In fact, if this was not the case, then all $u \in \mathcal{D}(u)$ would be solutions to Problem Q. Hence, we shall start by ruling out this degenerate possibility by requiring

$$(A4) \quad 0 = \phi_{\bar{u}}(t, 0) \quad \text{for a.e. } t \in (0, T), \quad \forall \bar{u} \in L^2(0, T; U).$$

Of course the latter assumption can be relaxed in many directions and is just chosen for simplicity. On the contrary note that the *viability problem* $u \in \mathcal{D}(u)$, which is certainly included into Problem Q, will not be directly assumed to admit a solution and will follow from our overall assumptions.

Let us now ask for some weak equi-coercivity of the functionals, namely

(A5) $\|u\|_{L^2(0,T;U)} \rightarrow \infty \Rightarrow \psi_{\bar{u}}(u)/\|u\|_{L^2(0,T;U)} \rightarrow \infty$ uniformly with respect to \bar{u} .

It should however be clear that other choices could be considered as well.

We now come to the *key assumptions* of this analysis. These regard the functional dependence of $\phi_{\bar{u}}$ on \bar{u} .

(A6) Letting $\bar{u}_n \rightarrow \bar{u}$ weakly in $L^2(0, T; U)$, we require that:

(A6.1) $\forall v \in \mathcal{D}(\bar{u}) \cap H^1(0, T; U^*)$ there exist $v_n \in \mathcal{D}(\bar{u}_n) \cap H^1(0, T; U^*)$ such that

$$\begin{aligned} v_n &\rightarrow v \text{ strongly in } H^1(0, T; U^*), \text{ weakly in } L^2(0, T; U), \\ v_n(0) &\rightarrow v(0) \text{ in } V, \text{ and } \psi_{\bar{u}_n}(v_n) \rightarrow \psi_{\bar{u}}(v), \end{aligned} \quad (4)$$

(A6.2) if $u_n \rightarrow u$ weakly in $L^2(0, T; U)$, then $\psi_{\bar{u}}(u) \leq \liminf_{n \rightarrow \infty} \psi_{\bar{u}_n}(u_n)$.

The latter are nothing but suitable Γ -convergence conditions. It is beyond the purposes of this short note to recall detailed definitions and results on Γ -convergence and the interested reader is referred to the original paper [8] as well as to the monographs [1, 7] for a through discussion. Let us just stress that, in the language of Γ -convergence, (A6.1) concerns the existence of a suitable *recovery sequence* for all smooth tests $v \in \mathcal{D}(\bar{u}) \cap H^1(0, T; U^*)$ whereas (A6.2) is simply the Γ -lim inf inequality for $(\bar{u}, u) \mapsto \psi_{\bar{u}}(u)$ with respect to the weak topology of the product $(L^2(0, T; U))^2$. Note that the convergences for v_n in (4) imply that $v_n \rightarrow v$ in $C([0, T]; V)$ as well. On the other hand, the latter uniform convergence will follow at once from $v_n \rightarrow v$ weakly in $H^1(0, T; U^*) \cap L^2(0, T; U)$ via Aubin-Lions Lemma [15, Cor. 4.1] whenever $U \subset V$ compactly. However, no compactness is directly assumed here and as our existence theory applies to the case $U \equiv U^*$ as well (see Section 4).

We shall mention that the latter condition (A6) is by far the most stringent of our analysis in the sense that it limits the application of the result to specific classes of suitably smooth functional dependencies $\bar{u} \mapsto \phi_{\bar{u}}$ (see the concrete examples in Section 4).

A completely different perspective is pursued in [13, 18] where Problem Q is addressed in a frame of an order structure on $L^2(0, T; U)$. In particular, MIGNOT & PUEL [13] discuss the local-in-time version of Problem Q (namely $\phi_{\bar{u}}(t, u) = \phi_{\bar{u}(t)}(t, u)$) by assuming additionally that, for any \bar{u}_1, \bar{u}_2 , almost every $t \in (0, T)$, and all $v_1 \in \partial\phi_{\bar{u}_1}(t, u_1)$, $v_2 \in \partial\phi_{\bar{u}_2}(t, u_2)$ one has

$$\bar{u}_1 \leq \bar{u}_2 \Rightarrow (v_1 - v_2, (u_1 - u_2)^+) \geq 0. \quad (5)$$

Here, we assume to be given a non-empty closed, and convex cone $P \subset H$ with $P \cap (-P) = \{0\}$ and define $\bar{u}_1 \leq \bar{u}_2$ iff $\bar{u}_2 - \bar{u}_1 \in P$. This is an order relation [14, Prop. 3.38, p. 95], P is interpreted as the cone of positive elements, u^+ corresponds to the projection of u on P , and one refers to (U, P) as a *Hilbert pseudo-lattice* [2]. One should notice that, in case the functional ϕ is independent of \bar{u} (namely, in the variational case), condition (5) is nothing but the classical *T-monotonicity* of the operator $\partial\phi$ originally introduced by BREZIS & STAMPACCHIA [3] and fully exploited in the framework of evolution problems by CALVERT [4, 5, 6]. In the current quasivariational setting, relation (5) turns out to be the natural extension of T-monotonicity and consists in an abstract monotonicity

condition *in the direction of the positive cone*. The analysis of the second author in the case of nonlocal-in-time quasivariational evolution problems in [18] builds up exactly on (5).

Our main existence result reads as follows.

Theorem 1 (Existence). *Assume (A1)-(A6) hold. Then Problem Q admits a solution.*

This result is proved in Section 3 below and some applications are eventually discussed in Section 4.

3 Proof of Theorem 1

As announced in the Introduction, the existence statement of Theorem 1 will follow by proving that the *variational selection* mapping \mathfrak{S} for Problem Q has a fixed point [2].

To this aim let us start by defining the *variational section* of Problem Q, namely

$$\begin{aligned} &\text{given } \bar{u} \in L^2(0, T; U), \text{ to find } u \in \mathcal{D}(\bar{u}) \text{ such that} \\ &\frac{1}{2}|v(0) - u_0|^2 + \int_0^T \langle v' - f, v - u \rangle + \int_0^T \phi_{\bar{u}}(v) \geq \int_0^T \phi_{\bar{u}}(u) \\ &\quad \forall v \in \mathcal{D}(\bar{u}) \cap H^1(0, T; U^*). \end{aligned} \tag{6}$$

This problem can be easily proved to admit a solution by means of standard Moreau-Yosida approximation techniques and the reader is referred to [18, Sec. 4.1] for details. We shall however remark that uniqueness for Problem (6) may fail due to the non-smoothness in time of the functional ϕ and the weakness of the solution notion. See [13, Exemple 1.2] for a scalar example.

We can hence define the variational selection mapping $\mathfrak{S} : L^2(0, T; U) \rightarrow 2^{L^2(0, T; U)}$ carrying the datum \bar{u} to the set $\mathfrak{S}(\bar{u})$ of all solutions to Problem (6). In order to prove that \mathfrak{S} has a fixed point, namely there exists $u \in L^2(0, T; U)$ such that $u \in \mathfrak{S}(u)$, one aims at exploiting the following generalized version of Kakutani's theorem due to GLICKSBERG (see also [2] for a proof).

Theorem 2 ([9]). *Let E be a locally convex Hausdorff topological vector space, K be a non-empty compact convex set in E , and $\mathcal{F} : K \rightarrow 2^K$ have non-empty, convex, and closed values. If \mathcal{F} is upper-semicontinuous, namely if*

$$\{e \in E : \mathcal{F}(e) \cap A \neq \emptyset\} \text{ is closed as long as } A \subset E \text{ is closed,} \tag{7}$$

then \mathcal{F} has at least one fixed point in K .

We apply Theorem 2 with the choice $E = L^2(0, T; U)$ endowed with the corresponding weak topology (which makes it into a locally convex Hausdorff topological vector space) and $\mathcal{F} = \mathfrak{S}$. We have already checked that \mathfrak{S} takes non-empty values. Moreover, given $u_1, u_2 \in \mathfrak{S}(\bar{u})$, $\theta \in [0, 1]$, and $u_\theta = \theta u_1 + (1 - \theta)u_2$, one has that $u_\theta \in \mathcal{D}(\bar{u})$, the latter

being a convex set, and

$$\begin{aligned} & \frac{1}{2}|v(0) - u_0|^2 + \int_0^T \langle v' - f, v - u_\theta \rangle + \int_0^T \phi_{\bar{u}}(v) \\ & \geq \int_0^T \theta \phi_{\bar{u}}(u_1) + \int_0^T (1 - \theta) \phi_{\bar{u}}(u_2) \geq \int_0^T \phi_{\bar{u}}(u_\theta) \\ & \quad \forall v \in \mathcal{D}(\bar{u}) \cap H^1(0, T; U^*). \end{aligned}$$

Namely $u_\theta \in \mathcal{S}(\bar{u})$ and \mathcal{S} is convex set-valued.

Finally, let $\mathcal{S}(\bar{u}) \ni u_n \rightarrow u$ weakly in $L^2(0, T; U)$. Then

$$\begin{aligned} & \frac{1}{2}|v(0) - u_0|^2 + \int_0^T \langle v' - f, v - u_n \rangle + \int_0^T \phi_{\bar{u}}(v) \geq \int_0^T \phi_{\bar{u}}(u_n) \\ & \quad \forall v \in \mathcal{D}(\bar{u}) \cap H^1(0, T; U^*). \end{aligned}$$

From (A4) it follows that $\limsup_{n \rightarrow \infty} \int_0^T \phi_{\bar{u}_n}(u_n) < \infty$, which shows $u \in \mathcal{D}(\bar{u})$. Moreover, by passing to the \liminf in the above inequality, one readily has that $u \in \mathcal{S}(\bar{u})$ as well. That is, \mathcal{S} has weakly closed values.

What we are left to prove is the upper-semicontinuity as (7). This is the point where our key assumption (A6) comes into play. Indeed, let $\bar{u}_n \rightarrow \bar{u}$ weakly in $L^2(0, T; U)$, any weakly closed set A be given and $u_n \in \mathcal{S}(\bar{u}_n) \cap A$ be such that $u_n \rightarrow u$ weakly in $L^2(0, T; U)$. We need to show that $u \in \mathcal{S}(\bar{u}) \cap A$. To this aim fix any $v \in \mathcal{D}(u) \cap H^1(0, T; U^*)$ and use (A6.1) in order to obtain the corresponding sequence $v_n \in \mathcal{D}(\bar{u}_n) \cap H^1(0, T; U^*)$ fulfilling (4). Next, we exploit the Γ - \liminf inequality of (A6.2) in order to have that

$$\begin{aligned} & \frac{1}{2}|v(0) - u_0|^2 + \int_0^T \langle v' - f, v - u \rangle + \int_0^T \phi_{\bar{u}}(v) \\ & \stackrel{(4)}{=} \liminf_{n \rightarrow \infty} \left(\frac{1}{2}|v_n(0) - u_0|^2 + \int_0^T \langle v'_n - f, v_n - u_n \rangle + \int_0^T \phi_{\bar{u}_n}(v_n) \right) \\ & \quad \geq \liminf_{n \rightarrow \infty} \int_0^T \phi_{\bar{u}_n}(u_n) \stackrel{(A6.2)}{\geq} \int_0^T \phi_{\bar{u}}(u), \end{aligned}$$

since $v_n(0) \rightarrow v(0)$ in V . That is $u \in \mathcal{S}(\bar{u})$ and the upper-semicontinuity of \mathcal{S} follows from the weak closeness of A .

4 Applications

We shall now give two concrete applications of Theorem 1 to the case of evolution inequalities with Volterra-type nonlinear constraints as (1). Let us start by fixing our assumptions:

$\Omega \subset \mathbb{R}^d$ is non-empty, open, bounded, and with Lipschitz boundary $\partial\Omega$; (8)

$T > 0$ is a final reference time, and $Q = \Omega \times (0, T)$; (9)

$\alpha \in \mathbb{R}^d \times \mathbb{R}^d$, $\hat{\alpha} \geq 0$ is the primitive of α (namely $\partial\hat{\alpha} = \alpha$) with $\hat{\alpha}(0) = 0$

and there exist constants $c_{1,\alpha}, c_{2,\alpha}, c_{3,\alpha} > 0$ such that

$$c_{1,\alpha}|\xi|^2 - c_{2,\alpha} \leq \hat{\alpha}(\xi) \leq c_{3,\alpha}(1 + |\xi|^2) \quad \forall \xi \in D(\hat{\alpha}); \quad (10)$$

$\beta \in \mathbb{R} \times \mathbb{R}$, $\hat{\beta}$ is the primitive of β with $\hat{\beta}(0) = 0$

and there exist constants $c_{1,\beta}, c_{2,\beta}, c_{3,\beta} > 0$ such that

$$c_{1,\beta}|r|^2 - c_{2,\beta} \leq \hat{\beta}(r) \leq c_{3,\beta}(1 + |r|^q) \quad \forall r \in D(\hat{\beta}) \quad (11)$$

where $q = 2^* = 2d/(d-2)$ if $d \geq 3$ and $1 \leq q < \infty$ otherwise; (12)

$g \in L^2(Q)$; (12)

$\gamma \in W^{2,\infty}(\mathbb{R})$, $\gamma(0) = 0$, $\gamma \geq 0$ on \mathbb{R} ; (13)

$k, k_t \in L^2(Q^2)$, $\nabla_x k \in (L^2(Q^2))^d$. (14)

Of course we shall complement problem (1) with initial condition

$$u_0 \in L^2(\Omega) \quad (15)$$

and homogeneous-Neumann boundary conditions (other conditions may be considered as well).

We can now variationally reformulate (1) along with the boundary and initial conditions in the form of the abstract problem (2) by letting

$$\begin{aligned} U &= H^1(\Omega), \quad V = L^2(\Omega), \quad \langle f(t), v \rangle = \int_{\Omega} g(x, t) v(x) dx \quad \forall v \in H^1(\Omega), \\ \phi_{\bar{u}}(t, v) &= \begin{cases} \int_{\Omega} \hat{\alpha}(\nabla v(x)) dx + \int_{\Omega} \hat{\beta}(v(x)) dx & \text{if } v(x) \leq K(\bar{u})(x, t) \text{ for a.e. } x \in \Omega, \\ \infty & \text{elsewhere in } H^1(\Omega), \end{cases} \end{aligned} \quad (16)$$

where

$$K(\bar{u})(x, t) = \gamma \left(\int_Q k(t, s, x, y) \bar{u}(y, s) dy ds \right) \quad \text{for a.e. } (x, t) \in Q.$$

For the sake of later reference we define, for all $\bar{u} \in L^2(Q)$, the non-empty, convex, and closed set

$$\mathcal{K}(\bar{u}) = \{v \in L^2(Q) : v(x, t) \leq K(\bar{u})(x, t) \text{ for a.e. } (x, t) \in Q\}.$$

Hence, we regard (1) as a nonlinear parabolic evolution problem where trajectories u are additionally constrained as $u \in \mathcal{K}(u)$. This motivates our reference to (1) as a *quasivariational* problem. It is however clear that this is in fact just a matter of terminology as we could equivalently interpret the evolution in (1) as constrained to the fixed set

$$\mathcal{K}' = \{u \in L^2(Q) : u \in \mathcal{K}(u)\}$$

(thus turning the problem into a *variational* one). We however refrain from adopting the latter perspective as it somehow prevents us from exploiting the structure of the problem. In particular, \mathcal{K}' fails to be convex (still being closed though). We however remark that, in case γ is linear \mathcal{K}' turns out to be a convex strict cone.

The application of our abstract existence Theorem 1 to the case of (1) is as follows.

Theorem 3 (Existence for parabolic inequalities with Volterra constraints).
Assume (8)-(15). Then, there exists $u \in \mathcal{K}(u) \cap L^2(0, T; H^1(\Omega))$ such that

$$\begin{aligned} \frac{1}{2} \int_{\Omega} (v(0) - u_0)^2 dx + \int_Q (v_t - g)(v - u) dx dt + \int_Q \left(\widehat{\alpha}(\nabla v) + \widehat{\beta}(v) \right) dx dt \\ \geq \int_Q \left(\widehat{\alpha}(\nabla u) + \widehat{\beta}(u) \right) dx dt \\ \forall v \in \mathcal{K}(u) \cap H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)). \end{aligned}$$

Proof. We are left to check that, along with positions (16), the assumptions (A4)-(A6) of Theorem 1 are fulfilled, conditions (A1)-(A3) being immediate. Let us mention that by (10) and (11)

$$\mathcal{D}(\bar{u}) = \mathcal{K}(\bar{u}) \cap L^2(0, T; H^1(\Omega)).$$

As for (A4), one simply checks that $0 \in \mathcal{K}(\bar{u})$ by recalling (13) for every $\bar{u} \in L^2(0, T; H^1(\Omega))$. The coercivity of ϕ is derived from (10)-(11); in fact, with $c_1 := \min\{c_{1,\alpha}, c_{1,\beta}\}$, $c_2 := \max\{c_{2,\alpha}, c_{2,\beta}\}$ and $c_3 := \max\{c_{3,\alpha}, c_{3,\beta}\}$ we have

$$\begin{aligned} c_1 \int_Q \{|\nabla u|^2 + |u|^2\} dx dt - c_2 |\Omega| T \leq \psi_{\bar{u}}(u) \\ \leq c_3 \int_Q \{|\nabla u|^2 + |u|^q\} dx dt + c_3 |\Omega| T, \end{aligned}$$

where $|\Omega|$ is the Lebesgue measure of Ω . Therefore, (A5) follows.

Let us now come to the check of the key assumptions (A6.1) and (A6.2). Due to the regularity of the kernel k in (14) and (13), we can prove that

$$\begin{aligned} K : L^2(0, T; H^1(\Omega)) \rightarrow H^1(0, T; L^2(\Omega)) \times L^2(0, T; H^1(\Omega)) =: W \\ \text{is weakly-strongly continuous, namely} \end{aligned}$$

$$u_n \rightarrow u \text{ weakly in } L^2(0, T; H^1(\Omega)) \implies K(u_n) \rightarrow K(u) \text{ strongly in } W.$$

In fact, owing to (14), K is a Hilbert-Schmidt operator [10, p. 64] such that $u \mapsto \partial_t(K(u))$ and $u \mapsto \nabla(K(u))$ are also Hilbert-Schmidt. In particular, they are compact in $L^2(Q)$ and from $L^2(Q)$ into $(L^2(Q))^d$, respectively. This implies that K is compact in W and the weak-strong continuity of K follows from the Lipschitz continuity of γ' (see (13)).

Now, let $\bar{u}_n \rightarrow \bar{u}$ weakly in $L^2(0, T; H^1(\Omega))$ and let

$$v \in \mathcal{D}(\bar{u}) \cap H^1(0, T; L^2(\Omega)) = \mathcal{K}(\bar{u}) \cap H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)).$$

We put

$$v_n = K(\bar{u}_n) - \text{proj}_P(K(\bar{u}_n) - v),$$

where $P \subset L^2(Q)$ is the cone of almost everywhere positive functions and proj_P is the projection onto P , taken with respect to the natural scalar product in W . In particular, we have that $K(\bar{u}_n) - v_n = \text{proj}_P(K(\bar{u}_n) - v)$ and hence almost everywhere non-negative. Namely $v_n \in \mathcal{K}(\bar{u}_n)$.

As $\bar{u}_n \rightarrow \bar{u}$ weakly in $L^2(0, T; H^1(\Omega))$, the convergence $K(\bar{u}_n) - v \rightarrow K(\bar{u}) - v$ holds strongly in W . Since proj_P is Lipschitz continuous (with constant 1), this entails that

$$\lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} \left(K(\bar{u}_n) - \text{proj}_P(K(\bar{u}_n) - v) \right) = K(\bar{u}) - \text{proj}_P(K(\bar{u}) - v) \stackrel{v \in \mathcal{K}(\bar{u})}{=} v,$$

the limit being strong in W . Moreover, by (10)-(11) and the Dominated Convergence Theorem [19, p. 1015]

$$\int_Q \left(\hat{\alpha}(\nabla v_n) + \hat{\beta}(v_n) \right) dx dt \rightarrow \int_Q \left(\hat{\alpha}(\nabla v) + \hat{\beta}(v) \right) dx dt.$$

Namely, we have provided a sequence v_n such that $v_n \rightarrow v$ strongly in W and $\psi_{\bar{u}_n}(v_n) \rightarrow \psi_{\bar{u}}(v)$ and (A6.1) follows.

Finally, assume that $u_n \rightarrow u$ weakly in $L^2(0, T; H^1(\Omega))$ with $u_n \in \mathcal{K}(\bar{u}_n)$ and $\{\psi_{\bar{u}_n}(u_n)\}$ is bounded. We have that

$$\int_Q (K(\bar{u}) - u) z dx dt = \lim_{n \rightarrow \infty} \int_Q (K(\bar{u}_n) - u_n) z dx dt \stackrel{u_n \in \mathcal{K}(\bar{u}_n)}{\geq} 0$$

for all smooth $z : Q \rightarrow [0, \infty)$. Hence $u \in \mathcal{K}(\bar{u})$ as well. Moreover, owing to (10)-(11), we have that

$$\int_Q \left(\hat{\alpha}(\nabla u) + \hat{\beta}(u) \right) dx dt \leq \liminf_{n \rightarrow \infty} \int_Q \left(\hat{\alpha}(\nabla u_n) + \hat{\beta}(u_n) \right) dx dt$$

and (A6.2) follows. \square

Let us remark that the compactness of the injection $H^1(\Omega) \subset L^2(\Omega)$ played no role in the above proof. In fact, our abstract result is suitable of describing some situation where $U \equiv U^*$ and hence no compactness is assumed. In particular, we shall now focus on the situation where $\alpha = 0$ in (1), namely the ODE case. In order to consider this situation, we have to reinforce the requirements on β by asking for a quadratic growth, namely

$$c_{1,\beta} r^2 - c_{2,\beta} \leq \hat{\beta}(r) \leq c_{3,\beta} (1 + r^2) \quad \forall r \in D(\hat{\beta}), \quad (17)$$

for some $c_{1,\beta}, c_{2,\beta}, c_{3,\beta} > 0$. On the other hand, the assumptions on the kernel k can be weakened as

$$k, k_t \in L^2(Q^2). \quad (18)$$

We have the following.

Theorem 4 (Existence for ODEs with Volterra constraints). *Assume (8)-(9), (12)-(13), (15), and (17)-(18). Then, there exists $u \in \mathcal{K}(u)$ such that*

$$\frac{1}{2} \int_{\Omega} (v(0) - u_0)^2 dx + \int_Q (v_t - g)(v - u) dx dt + \int_Q \hat{\beta}(v) dx dt \geq \int_Q \hat{\beta}(u) dx dt$$

$$\forall v \in \mathcal{K}(u) \cap H^1(0, T; L^2(\Omega)).$$

Sketch of the proof. One has simply to adapt the argument of Theorem 3 to the setting

$$U = V = L^2(\Omega), \quad \langle f(t), v \rangle = \int_{\Omega} g(x, t) v(x) dx \quad \forall v \in L^2(\Omega),$$

$$\phi_{\bar{u}}(t, v) = \begin{cases} \int_{\Omega} \widehat{\beta}(v(x)) dx & \text{if } v \in \mathcal{K}(\bar{u}) \\ \infty & \text{elsewhere in } L^2(\Omega). \end{cases}$$

Once again, along with these positions the check of (A1)-(A4) is straightforward. As for coercivity, one simply computes from (17)

$$c_{1,\beta} \|u\|_{L^2(Q)}^2 - c_{2,\beta} |\Omega| T \leq \int_Q \widehat{\beta}(u) dx dt \leq \psi_{\bar{u}}(u)$$

so that (A5) holds. The proof of (A6) follows along the same lines of Theorem 3, the only difference being that the operator K will now be defined on $L^2(Q)$ and turns out to be weakly-strongly continuous from $L^2(Q)$ to $H^1(0, T; L^2(\Omega))$ (no space-regularity for k with respect to x is here required, see (18)). Moreover, the projection proj_P exploited in the construction of the recovery sequence v_n is to be taken with respect to the topology of $H^1(0, T; L^2(\Omega))$. \square

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