EXISTENCE AND NONEXISTENCE FOR THE FULL THERMOMECHANICAL SOUZA-AURICCHIO MODEL OF SHAPE MEMORY WIRES

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Abstract. We provide an existence theory for the full thermomechanical quasi-static evolution of a shape memory wire described by the Souza-Auricchio constitutive model. The analysis requires some mild restriction on the choice of the thermomechanical coupling term in the expression on the free energy of the material. This restriction slightly deviates from the original Souza-Auricchio modeling frame, still being compatible with real situations. We additionally show that, by imposing no such restriction, the original Souza-Auricchio model is ill-posed.

Key words. Shape Memory Alloys, Thermomechanics, Existence result, Blowup in finite time

AMS subject classifications. 74N30, 74C05, 35K55

1. Introduction. Shape memory materials are metallic alloys (but also polymers and ceramics) exhibiting an amazing thermomechanical behavior: severely deformed specimens regain their original shape after a thermal cycle. This is the so-called *shape memory effect*. Moreover, at some higher temperatures regimes, the same materials are *superelastic*, namely, they recover large deformations during mechanical loading-unloading cycles [20].

These features are not present (at least to a comparable extent) in materials traditionally used in Engineering and are at the basis of a variety of innovative applications to aeronautical, structural, and earthquake technologies [15, 16]. The most successful application field of shape memory alloys (SMAs) is probably that of biomedical devices. Nowadays, SMAs are successfully used in orthodontics (archwires), orthopedics (bone anchors, intramedullary fixations, bone staples), medical instruments and minimal invasive surgery/technology (endoguidewires, grippers, cutters, vena cava filters), and drug delivery systems. The most emerging biomedical applications of SMAs are stents for intra-vascular or extra-vascular scaffolding (cardiovascular stenting, bronchial biliary, aortic aneurysm, carotid stenosis).

The relevance of these applications recently trimmed an intense and always increasing interest in the description of the thermomechanical behavior of SMAs and a whole menagerie of models has been proposed by addressing different alloys (NiTi, CuAlNi, Ni_2MnGa, and many others) at different scales (atomistic, microscopic with micro-structures, mesoscopic with volume fractions, macroscopic) and emphasizing different principles (minimization of stored energy vs. maximization of dissipation, phenomenology vs. rational crystallography and Thermodynamics) and different structures (single crystals vs. polycrystalline aggregates, possibly including intragranular interaction). Correspondingly, the Engineering literature on SMA modeling is vast. By limiting ourselves to macroscopic-phenomenological models we can refer, without claim of completeness, to [3, 18, 19, 21, 22, 26, 27, 28, 36, 38, 39, 42, 43]. The referred models have of course ambitions for different ranges of applicability (from lab single-crystal experiments to commercially exploitable tools) and different abilities to fit particular experiments and to explain microstructures, stress/strain relations, or hysteresis. The mutual relations between models at different scales are to a large extent still to be investigated [40].

We shall focus here on a phenomenological, internal-variable-type model model for polycrystalline materials which is capable of describing both the shape memory and the superelastic effect. The model has been originally advanced in the small-strain regime by Souza, Mamiya, & Zouain [41] and then combined with finite elements by Auricchio & Petrini [6, 7, 8]. We hence resort in referring to it as the *Souza-Auricchio* model in the following.

The interest in this model is motivated by its robustness with respect to parameters and discretizations despite its *simplicity*: in the three-dimensional situation, the constitutive behavior of the specimen is determined by the knowledge of just 8 material parameters, see (2.4) (note that linearized thermo-plasticity with linear hardening already requires 5 material parameters). These parameters are directly available for they can be easily fitted from experimental data (see Section 2 below). The Souza-Auricchio model has been analyzed from the viewpoint of existence and approximation of solutions of the three-dimensional quasi-static evolution problem in [4]. Later on, convergence rates for space-time discretization of the problem were obtained in [32, 33]. Extensions of the original model to the case of permanent inelastic effects have been provided in

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[5, 9, 10] and further improvements in the direction of the description of more realistic non-symmetric behaviors and transformation-dependent material parameters are reported in [11].

All the above-mentioned contributions on the Souza-Auricchio model focus on the isothermal situation, namely the temperature of the specimen is assumed to be fixed, uniform, and known. In particular, the cited literature is basically concerned with the description of the super-elastic effect only, for no shape-memory behavior occurs without temperature changes.

Some first results in the direction of including temperature changes in the Souza-Auricchio model has been obtained by Mielke, Paoli, & Petrov [30, 31]. In these papers, the temperature of the body is assumed to be changing in time, being however *given a-priori*. This simplification is justified if the body is relatively thin in at least one direction *and* the mechanical evolution is so slow that the produced heat may be assumed to be (almost) instantaneously dissipated (almost) in the environment. However, simple cyclic loading-unloading experiments on *wires* reveal that the heat production due to the dissipative phase-transformation is not at all negligible [37] and the temperature rise of the specimen with respect to the surrounding environment can indeed be relevant. This applicative observation and the interest in a complete material theory are the leading motivations for our analysis in the full thermomechanically coupled evolution of SMA specimens under the Souza-Auricchio model.

The main issue of this paper is hence that of addressing a full thermomechanical quasi-static evolution problem, letting indeed the temperature of the body to be one of the unknown of the system. Our main result is an existence theory for the resulting system of nonlinear PDEs deriving from the conservation of energy and momentum (quasi-static) and the constitutive equations of the material. The analysis deeply relies on the specific form of the free energy for the Souza-Auricchio model and is obtained by space-discretization and passage to the limit. Statements and proofs are given in Sections 3-5.

A crucial by-product of our investigation is the *constructive* proof of the fact that the original formulation of the Souza-Auricchio model *necessarily requires* some (minor) modification in order to comply with Thermodynamics, this modification being indeed compatible with real experimental data. In particular, we show that the model is ill-posed if the dependence of the latent heat on temperature is not smooth enough, *or* if the hardening constant is too small, *or* if the dissipation is too large. In these cases, we explicitly construct solutions which fail to exist for all times. Details in this direction will be given in Subsection 4.1.

Before closing this introduction let us mention that, besides the Souza-Auricchio framework, at least two other models for SMAs have proved to permit some analytic discussion in the fully coupled thermomechanical situation. These are the Frémond [19] and the Falk, Falk & Konopka models [17, 18]. With respect to these models, the Souza-Auricchio model provides some clear advantages as it directly features a rate-independent evolution of internal variables with respect to stress and temperature and it is capable of reproducing the crucial phenomenon of *martensitic reorientation* in the three-dimensional situation. It is beyond our purposes even to attempt a complete literature review on mathematical results for SMA-related systems of PDEs. The reader is however referred to the contributions [1, 2, 12, 13, 14, 23, 34, 35] and the related references for a comprehensive collection of results.

2. The model. We shall start by briefly recalling the Souza-Auricchio model together with some notation. The interested reader is referred to the original papers [6, 4, 41] for some extra detail.

The SMA body is modeled within the frame of Generalized Standard Materials (see Maugin [29]) and the small-strain approximation by additively decomposing the linearized deformation $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})) = (u_{i,j} + u_{j,i})/2$, ($\boldsymbol{u} : \Omega \to \mathbb{R}^3$ being the displacement from a fixed reference configuration $\Omega \subset \mathbb{R}^3$) into an elastic part $\boldsymbol{\varepsilon}^{el} \in \mathbb{R}^{3\times 3}$ and an inelastic (or transformation) part $\boldsymbol{\varepsilon}^{tr} \in \mathbb{R}^{3\times 3}$ as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{el} + \boldsymbol{\varepsilon}^{tr}. \tag{2.1}$$

At the microscopic level the super-elastic effect is interpreted as the result of a structural phase transition between different configurations of the material lattice, namely the *non-oriented phase* (austenite and twinned martensite) and the *oriented phase* (detwinned martensite). In particular, the *tensorial* internal variable $\boldsymbol{\varepsilon}^{tr}$ is assumed to be descriptive of the mechanical effect of the detwinning observed in the material.

The free energy of the body is prescribed as

$$\psi(\theta, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{tr}) = c\theta(1 - \log\theta) + \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{tr}) : \mathbb{C}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{tr}) + \frac{1}{2}\boldsymbol{\varepsilon}^{tr} : \mathbb{H}\boldsymbol{\varepsilon}^{tr} + f(\theta)|\boldsymbol{\varepsilon}^{tr}| + I_{\varepsilon_L}(\boldsymbol{\varepsilon}^{tr}). \tag{2.2}$$

The first term in the expression of ψ is the purely caloric part and c stands for the specific heat of the material. The second and third terms are classical in linearized elastoplasticity with linear hardening. In particular $\mathbb{C}$ is

FIG. 2.1. *Parameter fitting on two thermal test at different tension levels.*

the elasticity tensor (isotropy assumed)

$$\mathbb{C} := \frac{E}{1+\nu}(\mathbb{I}_4 + \mathbb{I}_2 \otimes \mathbb{I}_2) + \frac{E}{1-2\nu}\mathbb{I}_2 \otimes \mathbb{I}_2$$

where E is the Young modulus, ν is the Poisson ratio, and $\mathbb{I}_4$ and $\mathbb{I}_2$ denote the identity 4 and 2-tensor, respectively. Instead, $\mathbb{H} := E_h \mathbb{I}_4$ for $E_h > 0$ is the hardening tensor.

The last two terms in the expression of ψ are the distinguishing traits of the Souza-Auricchio model. The function f is convex, nonnegative, nondecreasing, and suitably smooth. Note that the original choice of the Souza-Auricchio model for the function f is

$$f_{\mathrm{SA}}(\theta) = b(\theta - \theta_M)^+ \tag{2.3}$$

where θ_M is the critical temperature for the austenite-martensite transition at zero stress and $b > 0$. Note however that the choice $f = f_{\mathrm{SA}}$ is not admissible here for the sake of proving the global existence of solutions as it will be detailed in Section 4. Finally, the value $\varepsilon_L > 0$ represents a maximal amount of transformation strain obtainable by martensitic reorientation in a uniaxial test and I_{ε_L} is the indicator function of the closed ball centered at 0 with radius ε_L. Namely, $I_{\varepsilon_L}(\boldsymbol{\varepsilon}^{tr}) = 0$ iff $|\boldsymbol{\varepsilon}^{tr}| \le \varepsilon_L$ and $I_{\varepsilon_L}(\boldsymbol{\varepsilon}^{tr}) = \infty$ elsewhere.

The evolution of the material will be prescribed by means of the specification of the *rate-independent* dissipation related to the phase transformation via the corresponding *pseudo-potential of dissipation*

$$\varphi(\dot{\boldsymbol{\varepsilon}}^{tr}) = R|\dot{\boldsymbol{\varepsilon}}^{tr}|$$

where $R > 0$ is the so called transformation radius.

Note that the material is assumed to have constant mass density, to present the same specific heat and elastic behavior in all phases, and to show no thermal dilation. These are of course crude simplifications which are however motivated by simplicity and partly justified by the good quantitative agreement of the model to experimental data. In particular, Figure 2.1 illustrates (thin lines) the outcome of two thermo-cycle experiment with different (fixed) tensions for a commercial NiTi wire (straight annealed *NiTi FWM #1* superelastic wire with diameter=0.1 mm and length=50 mm [45]).

These experimental data were obtained by the SMA Group at the Institute of Physics of the Academy of Sciences of the Czech Republic in Prague within an international action on SMA modeling validation [46]. Bold lines in Figure 2.1 are obtained by partly fitting the experimental data for the Souza-Auricchio model (hence for $f = f_{\mathrm{SA}}$) by the SMA group in Pavia. For the sake of later consideration let us report here the values of the material parameters:

$$\begin{gathered} c = 5.2 \text{ MPa/K}, \quad E = 53.6 \text{ GPa}, \quad \nu = 0.33, \quad E_h = 750 \text{ MPa}, \quad \varepsilon_L = 0.058, \\ b = 5.6 \text{ MPa/K}, \quad \theta_M = 235 \text{ K}, \quad R = 90 \text{ MPa}. \end{gathered} \tag{2.4}$$

All these values are fitted from the experiments in Figure 2.1 but the specific heat c which is taken instead from [40].

Some validation of the Souza-Auricchio model is illustrated in Figure 2.2 where the fitted parameters are used in a pure tension test. In particular, experimental (thin) and numerical results (bold line) for complete and incomplete tension cycles are compared.

Given the expression of the free energy (2.2), the corresponding entropy s and internal energy e are given by the formulas

$$s = -\frac{\partial \psi}{\partial \theta} = c \log \theta - f'(\theta)|\boldsymbol{\varepsilon}^{tr}|, \tag{2.5}$$

$$e = \psi + \theta s = c\theta + \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{tr}) : \mathbb{C}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{tr}) + \frac{E_h}{2}|\boldsymbol{\varepsilon}^{tr}|^2 + (f(\theta) - \theta f'(\theta))|\boldsymbol{\varepsilon}^{tr}| + I_{\varepsilon_L}(\boldsymbol{\varepsilon}^{tr}). \tag{2.6}$$

From the expression of the entropy (2.5) it is already clear that some restriction on f has to be required for the model to comply with Thermodynamics. Indeed, as the temperature-entropy relation needs to be strictly increasing, by computing

$$\frac{\partial s}{\partial \theta} = \frac{c}{\theta} - f''(\theta)|\boldsymbol{\varepsilon}^{tr}| > 0, \tag{2.7}$$

FIG. 2.2. *Comparison between experimental results and numerical predictions in a cyclic tension test. Material parameters are fitted from the thermo-thermo-cycling experiment illustrated in Figure 2.1.*

we find that the value of $f''(\theta)$ cannot be too large (and, in particular, cannot be a Dirac delta as for the choice $f_{\rm SA}$).

For the stress $\boldsymbol{\sigma}$ and transformation strain $\boldsymbol{\varepsilon}^{tr}$ we prescribe the constitutive equations

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}, \tag{2.8}$$

$$\mathbf{0} \in \frac{\partial \varphi}{\partial \dot{\boldsymbol{\varepsilon}}^{tr}} + \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^{tr}}, \tag{2.9}$$

which, owing to the specific choice for the free energy in (2.2), read as

$$\boldsymbol{\sigma} = \mathbb{C}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{tr}) = \mathbb{C}\boldsymbol{\varepsilon}^{el}, \tag{2.10}$$

$$\boldsymbol{\sigma} \in R\partial|\dot{\boldsymbol{\varepsilon}}^{tr}| + f(\theta)\partial|\boldsymbol{\varepsilon}^{tr}| + E_h\boldsymbol{\varepsilon}^{tr} + \partial I_{\varepsilon_L}(\boldsymbol{\varepsilon}^{tr}). \tag{2.11}$$

The process is governed by the mechanical equilibrium equation (quasi-static) and by the energy balance, that is

$$-\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}, \tag{2.12}$$

$$e_t + \operatorname{div}\boldsymbol{q} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t + r_0(x,t) \tag{2.13}$$

in $\Omega \times (0,T)$, where $\boldsymbol{q}$ is the heat flux vector which we assume to obey Fourier's law $\boldsymbol{q} = -\kappa\nabla\theta$ with a constant heat conductivity κ, and $r_0(x,t) \geq 0$ are given heat sources (the Joule heating, e.g.). From the construction it is clear that every regular solution of the system with positive temperature θ satisfies the Clausius-Duhem inequality

$$s_t + \operatorname{div}\left(\frac{\boldsymbol{q}}{\theta}\right) - \frac{r_0(x,t)}{\theta} \geq 0. \tag{2.14}$$

Indeed, for $\theta > 0$ the latter is equivalent to the inequality

$$-\theta_t s + \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t - \psi_t - \frac{\boldsymbol{q}\cdot\nabla\theta}{\theta} \geq 0.$$

By the chain rule, we have $\psi(\theta, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{tr})_t = -\theta_t s + \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t + \dot{\boldsymbol{\varepsilon}}^{tr} : \partial_{\boldsymbol{\varepsilon}^{tr}}\psi$, hence, by virtue of (2.9), the left hand side of (2) equals $R|\dot{\boldsymbol{\varepsilon}}^{tr}| + \kappa|\nabla\theta|^2/\theta \geq 0$.

2.1. One-dimensional model. Let us now comment on the one-dimensional model we shall be considering in the following. We shall assume that the process takes place in a domain $\Omega \subset \mathbb{R}^3$ of the form $\Omega = (0,\ell)\times\omega$, where $\ell > 0$ is fixed and $\omega \subset \mathbb{R}^2$ is a very small bounded connected domain (i.e., the cross section of a wire). We assume that all state variables depend only on $x \in [0,\ell]$ and $t \in [0,T]$, where $T > 0$ is some final time of the process and that we are concerned with an *uniaxial traction test*, namely the only non-zero component of the stress is

$$\sigma := \sigma_{11}. \tag{2.15}$$

Moreover, we assume that the body is so thin (or so long) that the displacement out of the x-direction is negligible. This amounts to say that $u_2 = u_3 = 0$. Hence, owing to (2.15), the only non-zero components of the displacements are

$$\varepsilon := \varepsilon_{11}, \quad \varepsilon^{tr} := \varepsilon^{tr}_{11}.$$

Correspondingly, the free energy of the medium can be rewritten as

$$\psi(\theta,\varepsilon,\varepsilon^{tr}) = c\theta(1-\log\theta) + \frac{E}{2}(\varepsilon-\varepsilon^{tr})^2 + \frac{E_h}{2}(\varepsilon^{tr})^2 + f(\theta)|\varepsilon^{tr}| + I_{\varepsilon_L}(\varepsilon^{tr}).$$

Hence, the balance equations and the constitutive relation (2.11) read

$$\sigma_x = 0, \tag{2.16}$$
$$e_t - \kappa\theta_{xx} = \sigma\varepsilon_t + r(\theta,x,t), \tag{2.17}$$
$$\sigma \in R\partial|\dot\varepsilon^{tr}| + f(\theta)\partial|\varepsilon^{tr}| + E_h\varepsilon^{tr} + \partial I_{\varepsilon_L}(\varepsilon^{tr}). \tag{2.18}$$

in $(0,\ell)\times(0,T)$. We offer two motivations for the θ-dependence of the heat source r in the energy balance (2.17). At first, given an external temperature distribution $\theta_\Gamma(x,t)$, the Robin boundary condition along the wire leads after 1D reduction to the formula

$$r(\theta,x,t) = r_0(x,t) - h_0(\theta - \theta_\Gamma(x,t)) - h_1(\theta^4 - \theta_\Gamma^4(x,t))$$

with $r_0 \ge 0$, $\theta_\Gamma(x,t) \ge \theta_* > 0$, and with constants $h_0, h_1 \ge 0$. Secondly, the θ-dependence in r allows the possibility of prescribing the inequality $r(\theta,x,t)H(\theta_* - \theta) \ge 0$, where H is the Heaviside function. This in turn ensures the validity of a maximum principle giving the positivity of the temperature.

We complement the system (2.16)-(2.18) with the boundary conditions

$$u(0,t) = 0, \quad \sigma(\ell,t) = \tau(t), \tag{2.19}$$
$$\theta_x(0,t) = \theta_x(\ell,t) = 0 \tag{2.20}$$

which in particular entail that the wire is thermally insulated from the actuators, it is fixed at $x = 0$, and a known traction $\tau : [0,T] \to \mathbb{R}$ is applied at $x = \ell$.

3. The constitutive relation. Before formulating our existence result we shall devote a preliminary discussion on the specific form of the material constitutive relation (2.18). Our aim here is to advance an equivalent formulation of the functional relation $(\theta,\sigma) \mapsto \varepsilon^{tr}$ prescribed by (2.18) in terms of (a function of) elementary hysteresis operators [12, 25, 44]. In particular, the functions $f = f(\theta)$ and σ are supposed to be known functions of time throughout this section. Indeed, for given functions $f, \sigma \in W^{1,1}(0,T)$, $f(t) \ge 0$ for all $t \in [0,T]$, we consider the differential inclusion for the unknown function η

$$\sigma(t) \in R\partial|\dot\eta(t)| + f(t)\partial|\eta(t)| + \eta(t) + \partial I_{[-1,1]}(\eta(t)) \tag{3.1}$$

with a given initial condition $\eta(0) = \eta_0 \in [-1,1]$. Not all initial conditions are admissible. We easily check the implications

$$\begin{cases} \eta_0 = 1 & \Rightarrow \sigma(0) - f(0) - 1 \ge -R, \\ \eta_0 = -1 & \Rightarrow \sigma(0) + f(0) + 1 \le R, \\ \eta_0 = 0 & \Rightarrow \sigma(0) \in [-f(0) - R, f(0) + R], \\ \eta_0 \in (0,1) & \Rightarrow \sigma(0) - f(0) - \eta_0 \in [-R,R], \\ \eta_0 \in (-1,0) & \Rightarrow \sigma(0) + f(0) - \eta_0 \in [-R,R]. \end{cases} \tag{3.2}$$

We first prove that there exists a most one solution to (3.1) for any given admissible initial condition. Let η_1, η_2 be two solutions and let $m_i(t) \in \partial|\dot\eta_i(t)| = \text{sign } \dot\eta_i(t)$, $i = 1,2$ be arbitrary selections. We use $f(t) \ge 0$ and deduce that

$$R(m_1(t) - m_2(t))(\eta_1(t) - \eta_2(t)) + (\eta_1(t) - \eta_2(t))^2 \le 0 \quad \text{a.e.},$$

whence the implication $\eta_1(t) > \eta_2(t) \Rightarrow \dot\eta_1(t) < \dot\eta_2(t)$ holds for a.e. $t \in [0,T]$. Interchanging the roles of η_1 and η_2, we obtain

$$(\dot\eta_1(t) - \dot\eta_2(t))(\eta_1(t) - \eta_2(t)) \le 0 \quad \text{a.e.},$$

hence $\eta_1 \equiv \eta_2$.

We now construct an explicit solution to (3.1) in terms of the so-called *play operator* $\mathfrak{p}_R$ with threshold $R > 0$. Recall that for every given function $v \in W^{1,1}(0,T)$ and every $z_0 \in [-R,R]$ there exists a unique solution $\xi \in W^{1,1}(0,T)$ to the problem

$$\begin{cases} |v(t) - \xi(t)| \le R & \forall t \in [0,T], \\ v(0) - \xi(0) = z_0, \\ \dot\xi(t)(v(t) - \xi(t) - y) \ge 0 & \text{a.e. in } (0,T),\ \forall y \in [-R,R]. \end{cases} \tag{3.3}$$

The operator $\mathfrak{p}_R : [-R, R] \times W^{1,1}(0,T) \to W^{1,1}(0,T)$ is defined as the solution mapping $\xi = \mathfrak{p}_R[z_0, v]$ of (3.3). It is Lipschitz continuous and admits a Lipschitz continuous extension to $\mathfrak{p}_R : [-R, R] \times C[0,T] \to C[0,T]$. The following scaling property

$$\mathfrak{p}_{cR}[cz_0, cv] = c\mathfrak{p}_R[z_0, v] \quad \forall c \in \mathbb{R} \tag{3.4}$$

is an immediate consequence of the definition.

We now state a classical comparison result by Krasnosel'skii & Pokrovskii [24].

Lemma 3.1. *Let* $v_1(t) \le v_2(t)$ *for all* $t \in [0,T]$*, let* $z_{01} \ge z_{02}$*, and let* $\xi_i = \mathfrak{p}_R[z_{0i}, v_i]$ *for* $i = 1, 2$*. Then* $\xi_1(t) \le \xi_2(t)$ *for all* $t \in [0,T]$*.*

Proof. From the inequality

$$(\dot{\xi}_1(t) - \dot{\xi}_2(t))(v_1(t) - v_2(t) - \xi_1(t) + \xi_2(t)) \ge 0 \quad \text{a.e.}$$

we infer the implication

$$\xi_1(t) > \xi_2(t) \;\Rightarrow\; \dot{\xi}_1(t) \le \dot{\xi}_2(t),$$

which yields in turn that

$$(\dot{\xi}_1(t) - \dot{\xi}_2(t))\, H(\xi_1(t) - \xi_2(t)) \le 0 \quad \text{a.e.},$$

where H is the Heaviside function. Hence,

$$(\xi_1(t) - \xi_2(t))^+ \le (\xi_1(0) - \xi_2(0))^+ = (v_1(0) - v_2(0) - z_{01} + z_{02})^+ = 0$$

for all t and the assertion follows. ■

We are now ready to state and prove the main result of this section.

Proposition 3.2. *Let* $f, \sigma \in W^{1,1}(0,T)$*,* $f(t) \ge 0$ *for* $t \in [0,T]$*, and* $\eta_0 \in [-1, 1]$ *be given satisfying the compatibility conditions* (3.2)*. Then,* (3.1) *admits a unique solution* η *given by the formula*

$$\eta = Q(\mathfrak{p}_R[z_{0-}, \sigma - f]) - Q(-\mathfrak{p}_R[z_{0+}, \sigma + f]), \tag{3.5}$$

where $Q : \mathbb{R} \to [0,1]$ *is the projection defined as* $Q(y) = \max\{0, \min\{y, 1\}\}$ *for* $y \in \mathbb{R}$*, and the values* $z_{0\pm}$ *are chosen as* $z_{0-} = \sigma(0) - f(0) - \eta_0$*,* $z_{0+} = \sigma(0) + f(0) - \eta_0$*.*

The relation between the initial condition η_0 on the one hand and $z_{0\pm}$ on the other hand is not one-to-one for other choices of $z_{0\pm}$ are possible for the same η_0.

Proof. We already know that (3.1) admits at most one solution for a given initial condition. A straightforward computation shows that $\mathfrak{p}_R[z_{0+}, \sigma + f](0) = \mathfrak{p}_R[z_{0-}, \sigma - f](0) = \eta_0$, hence $\eta(0) = \eta_0$. Moreover, by Lemma 3.1 we have

$$\mathfrak{p}_R[z_{0+}, \sigma + f](t) \ge \mathfrak{p}_R[z_{0-}, \sigma - f](t)$$

for all $t \in [0,T]$, since $f(t) \ge 0$. Hence $\eta^+ = Q(\mathfrak{p}_R[z_{0-}, \sigma - f])$, and $\eta^- = Q(-\mathfrak{p}_R[z_{0+}, \sigma + f])$. It remains to check that the function η defined by (3.5) satisfies the inclusion (3.1) almost everywhere. We define the set $M \subset (0,T)$ as the set of all Lebesgue points of all functions $\sigma, f, \xi_- = \mathfrak{p}_R[z_{0-}, \sigma - f], \xi_+ = \mathfrak{p}_R[z_{0+}, \sigma + f], \eta$, and consider any $t \in M$. We distinguish again the five cases as above.

- $\eta(t) = 1$: Then $\dot{\eta}(t) = 0$, $\xi_-(t) \ge 1$, hence $\sigma(t) - f(t) \ge 1 - R$, and (3.1) follows;
- $\eta(t) = -1$: Then $\dot{\eta}(t) = 0$, $\xi_+(t) \le -1$, hence $\sigma(t) + f(t) \le -1 + R$, and (3.1) follows again;
- $\eta(t) = 0$: Then $\xi_-(t) \le 0 \le \xi_+(t)$, hence $-f(t) - R \le \sigma(t) \le f(t) + R$, and (3.1) follows provided $\dot{\eta}(t) = 0$. The case $\dot{\eta}(t) \ne 0$ can only occur if $f(t) = 0$ and $\sigma(t) = \operatorname{sign}(\dot{\eta}(t))R$, hence (3.1) holds again;
- $\eta(t) \in (0,1)$: Then $\eta(t) = \xi_-(t)$, hence $\dot{\eta}(t)(\sigma(t) - f(t) - \eta(t) - y) \ge 0$ for all $y \in [-R, R]$, which is equivalent to (3.1);
- $\eta(t) \in (-1,0)$: Then $\eta(t) = \xi_+(t)$, hence $\dot{\eta}(t)(\sigma(t) + f(t) - \eta(t) - y) \ge 0$ for all $y \in [-R, R]$, which is equivalent to (3.1).

The proof of Proposition 3.2 is complete. ■

4. Main existence result. We shall now exploit the results of Section 3 in order to provide a useful reformulation of the system (2.16)-(2.18). Equation (2.16) with boundary conditions (2.19) has a unique solution $\sigma(x,t) = \tau(t)$ for all $(x,t) \in (0,\ell)\times(0,T)$. We prescribe initial conditions $\theta_0(x) > 0$ for θ and $\varepsilon_0^{tr}(x) \in [-\varepsilon_L, \varepsilon_L]$ for ε^{tr}, satisfying the compatibility conditions analogous to (3.2), that is,

$$\left\{\begin{array}{lcll} \varepsilon_0^{tr}(x) & = & \varepsilon_L & \Rightarrow \tau(0) - f(\theta_0(x)) - E_h\varepsilon_L \ge -R, \\ \varepsilon_0^{tr}(x) & = & -\varepsilon_L & \Rightarrow \tau(0) + f(\theta_0(x)) + E_h\varepsilon_L \le R, \\ \varepsilon_0^{tr}(x) & = & 0 & \Rightarrow \tau(0) \in [-f(\theta_0(x)) - R, f(\theta_0(x)) + R], \\ \varepsilon_0^{tr}(x) & \in & (0,\varepsilon_L) & \Rightarrow \tau(0) - f(\theta_0(x)) - E_h\varepsilon_0^{tr}(x) \in [-R,R], \\ \varepsilon_0^{tr}(x) & \in & (-\varepsilon_L, 0) & \Rightarrow \tau(0) + f(\theta_0(x)) - E_h\varepsilon_0^{tr}(x) \in [-R,R]. \end{array}\right. \tag{4.1}$$

We now divide the constitutive relation (2.18) by $E_h\varepsilon_L$ and obtain

$$\frac{\tau(t)}{E_h\varepsilon_L} \in \frac{R}{E_h\varepsilon_L}\partial\Big|\frac{\dot\varepsilon^{tr}}{\varepsilon_L}\Big| + \frac{f(\theta(x,t))}{E_h\varepsilon_L}\partial\Big|\frac{\varepsilon^{tr}}{\varepsilon_L}\Big| + \frac{\varepsilon^{tr}}{\varepsilon_L} + \partial I_{[-1,1]}\Big(\frac{\varepsilon^{tr}}{\varepsilon_L}\Big).$$

By Proposition 3.2 and identity (3.4) we thus have

$$\begin{aligned} \varepsilon^{tr}(x,t) &= \varepsilon_L Q\Big(\frac{1}{E_h\varepsilon_L}\mathfrak{p}_R[z_{0-}(x), \tau - f(\theta(x,\cdot))](t)\Big) \\ &\quad - \varepsilon_L Q\Big(-\frac{1}{E_h\varepsilon_L}\mathfrak{p}_R[z_{0+}(x), \tau + f(\theta(x,\cdot))](t)\Big), \end{aligned} \tag{4.2}$$

with

$$z_{0-}(x) = \tau(0) - f(\theta_0(x)) - E_h\varepsilon_0^{tr}(x), \quad z_{0+}(x) = \tau(0) + f(\theta_0(x)) - E_h\varepsilon_0^{tr}(x). \tag{4.3}$$

The energy balance (2.17) can be written in the form

$$\Big(c\theta + (f(\theta) - \theta f'(\theta))|\varepsilon^{tr}| + \frac{E_h}{2}(\varepsilon^{tr})^2\Big)_t - \kappa\theta_{xx} = \tau(t)\varepsilon_t^{tr} + r(\theta,x,t) \tag{4.4}$$

or, alternatively,

$$\big(c - \theta f''(\theta)|\varepsilon^{tr}|\big)\theta_t - \kappa\theta_{xx} = \theta f'(\theta)|\varepsilon^{tr}|_t + R|\varepsilon_t^{tr}| + r(\theta,x,t). \tag{4.5}$$

We shall start by collecting our assumptions on data in the following hypothesis.

HYPOTHESIS 4.1. *There exist constants $\theta_* > 0$, $r^* > 0$, $C_f > 0$ and an increasing function $C : (0,\infty) \to (0,\infty)$ with the following properties.*

(i) *The data have the regularity $\varepsilon_0^{tr} \in C[0,\ell]$, $\theta_0 \in W^{1,2}(0,\ell)$, $\tau \in W^{1,\infty}(0,T)$, $\theta_0 \ge \theta_*$, $\varepsilon_0^{tr}(x) \in [-\varepsilon_L, \varepsilon_L]$ for all $x \in [0,\ell]$, and the compatibility condition (4.1) holds.*

(ii) *$f : (0,\infty) \to [0,\infty)$ is a nondecreasing convex function of class $C^{1,1}$ such that $f(\theta) = 0$ for $0 < \theta < \theta_*$, $f'(\theta) \le C_f$ for all $\theta > 0$, and*

$$c_0 := \inf_{\theta>0}\Big(c - \theta f''(\theta)\varepsilon_L + \frac{f'(\theta)}{E_h}(\theta f'(\theta) - R)\Big) > 0, \tag{4.6}$$

$$c_1 := \inf_{\theta>0}\Big(c - \theta f''(\theta)\varepsilon_L\Big) > 0. \tag{4.7}$$

(iii) *$r : (0,\infty)\times(0,\ell)\times(0,T) \to \mathbb{R}$ is a measurable function such that*
- *$r(\theta,x,t) \le r^*$, $|r_t(\theta,x,t)| \le r^*$ a.e.,*
- *$r(\theta,x,t)H(\theta_* - \theta) \ge 0$ a.e.,*
- *$|r(\theta_1,x,t) - r(\theta_2,x,t)| \le C(\max\{\theta_1,\theta_2\})|\theta_1 - \theta_2|$ a.e. for all $\theta_1, \theta_2 > 0$*

where we recall that H is the Heaviside function.

The ultimate motivation for introducing the restrictions (4.6)-(4.7) is that of preserving the parabolicity of the problem. Restriction (4.7) is nothing but the former (2.7) and ensures that the coefficient of θ_t in the energy balance (4.5) is positive. On the other hand, (4.6) entails that the overall thermomechanical evolution of the body is such that energy gets dissipated with time.

An example for a function f fulfilling the assumptions is

$$f'(\theta)=\begin{cases}0 & \text{for } \theta\in(0,(1-\delta)\theta_M],\\ \dfrac{b}{2\delta\theta_M}(\theta-(1-\delta)\theta_M) & \text{for } \theta\in((1-\delta)\theta_M,(1+\delta)\theta_M),\\ b & \text{for } \theta\geq(1+\delta)\theta_M\end{cases} \tag{4.8}$$

for $\delta\in(0,1)$. Note that f coincides with the original function $f_{\rm SA}$ from (2.3) of the Souza-Auricchio model out of the temperature interval centered in θ_M with radius $\rho=\delta\theta_M$. This radius cannot be taken arbitrarily small without violating (4.6)-(4.7). Indeed, by exploiting the actual material parameters from (2.4), the restrictions (4.6)-(4.7) entail $\rho\geq 7.5$ K.

Our main existence result reads as follows.

Theorem 4.2. *Let Hypothesis* 4.1 *hold. Then system* (4.4), (2.20), (4.2) *with* $z_{0\pm}$ *given by* (4.3) *admits a solution* $\theta\in C([0,\ell]\times[0,T])$ *such that* $\theta_t,\theta_{xx}\in L^2((0,\ell)\times(0,T))$, *and* $\theta(x,t)>0$ *for all* $(x,t)\in[0,\ell]\times[0,T]$. *If moreover no compression takes place, that is* $\varepsilon_0^{tr}(x)\geq 0$ *for all* $x\in[0,\ell]$ *and* $\tau(t)\geq 0$ *for all* $t\in[0,T]$, *then condition* (4.6) *can be replaced by*

$$c_0:=\inf_{\theta>0}\Big(c-\theta f''(\theta)\varepsilon_L-f'(\theta)\varepsilon_L\Big)>0. \tag{4.9}$$

The proof of this result is detailed in Section 5.

4.1. Counterexamples to global existence. Conditions (4.6), (4.7), (4.9) are probably not optimal. The following examples show, however, that if the material parameters are completely arbitrary, and, in particular, these conditions are not fulfilled, a global solution to the full thermodynamic system may fail to exist.

In the examples below, we consider the function f given by (4.8), and space homogeneous heat source $r=r(t)\geq 0$ and initial conditions θ_0 and ε_0^{tr}. The physically relevant solutions θ and ε^{tr} are homogeneous in space as well. Namely the problem reduces to a (nonlinear) system of ODEs. We construct solutions to system (4.2), (4.4) which exists only for (small) finite time. No general uniqueness proof is available, hence the existence of a global pathological solution cannot be rigorously excluded. Nevertheless, in the class of piecewise monotone functions of time, the play operator can be represented by a Lipschitz continuous superposition (Nemytskii) operator, and uniqueness follows from the general theory of ODEs.

Example 4.3. We shall start by directly showing the the Souza-Auricchio model may not admit global weak solutions. This follows from the fact that the choice $f=f_{\rm SA}$ violates (4.7). As $f_{\rm SA}$ is not smooth, strong solvability is clearly out of range and we resort in checking that physically reasonable weak solutions cannot exist globally in $(0,T)$. To this aim, let us fix $\theta_0<\theta_M$ and $r(t)=1$ and assume that $\theta\in L^1(0,T)$ solves weakly (4.5), (4.2) along with the choices $\varepsilon^{tr}(x,t)=\varepsilon^{tr}(x,0)=\varepsilon_L$ and $\tau(t)=E_h\varepsilon_L+f(\theta(t))$, namely

$$(c\theta(t)-b\theta_M H(\theta(t)-\theta_M)\varepsilon_L)_t=1 \quad \text{in the distributional sense.}$$

We have that

$$c\theta(t)-b\theta_M H(\theta(t)-\theta_M)\varepsilon_L=C\theta_0+t \quad \text{a.e.}$$

As the latter right-hand side is continuous whereas the mapping $\theta\mapsto c\theta+b\theta_M H(\theta-\theta_M)$ is not, the function $t\mapsto\theta(t)$ has necessarily to jump over the *non-monotonicity* region $(\theta_M,(1+(b\varepsilon_L)/c)\theta_M)$. This behavior is however unphysical as it results in the entropy decrease

$$s|_{\theta=(1+(b\varepsilon_L)/c)\theta_M)}-s|_{\theta=\theta_M}=c\log\left(\left(1+\frac{b\varepsilon_L}{c}\right)\theta_M\right)-b\varepsilon_L-c\log(\theta_M)<0.$$

Example 4.4. We shall now assume to be in the smooth situation $f\in C^{1,1}$ of Theorem 4.2 and check what happens if (4.7) is violated. Assume that $r(t)=r_c>0$ is constant, and consider the maximal solution θ_c to the ODE

$$\Big(c-\theta_c f''(\theta_c)\varepsilon_L\Big)\dot\theta_c=r_c,\quad \theta_c(0)=(1-\delta)\theta_M.$$

For $\theta\in((1-\delta)\theta_M,(1+\delta)\theta_M)$, we have $\theta f''(\theta)=b\theta/(2\delta\theta_M)$. Assume that

$$\frac{(1-\delta)\varepsilon_L}{2\delta}<\frac{c}{b}<\frac{(1+\delta)\varepsilon_L}{2\delta}. \tag{4.10}$$

The second inequality above follows by requiring (4.7) not to be satisfied whereas the first inequality is true for all δ close to 1. Then θ_c is uniquely defined and increasing in an interval $[0,T_c)$, $\theta_c(T_c-) = 2\delta c\theta_M/(b\varepsilon_L) < (1+\delta)\theta_M$. Set $\varepsilon^{tr}(x,0) = \varepsilon_L$ and $\tau(t) = E_h\varepsilon_L + f(\theta_c(t))$. Then $\theta(x,t) = \theta_c(t)$ and $\varepsilon^{tr}(x,t) \equiv \varepsilon_L$ satisfy the system (4.4), (4.2) in the whole interval $[0,T_c)$ of existence of θ_c. Let us estimate T_c. The equation for $p = (\theta_c/\theta_M) - (1-\delta)$ reads

$$(C-p)\dot{p} = r^*, \quad p(0) = 0,$$

with

$$C = \frac{2\delta c}{\varepsilon_L b} - 1 + \delta, \quad r^* = \frac{2\delta r_c}{\varepsilon_L b\theta_M}.$$

We have $p(t) = C - \sqrt{C - 2r^* t}$, $T_c = C/2r^*$. The interval of existence is very small if r_c is large and the system is not globally solvable. This amounts to a counterexample to the strong solvability claimed in Theorem 4.2. On the other hand, possibly discontinuous weak solutions to the problem would not represent a suitable alternative as they encounter the same entropy-temperature non-monotonicity phenomenon described in Example 4.3.

Example 4.5. The counterexample to (4.6) is a bit more involved. Unlike in Example 4.4, a failure may occur even if ε_L/δ is arbitrarily small with respect to c/b, provided E_h is sufficiently small or R is sufficiently large and also compression is allowed. Assume that

$$R > E_h\varepsilon_L\,. \tag{4.11}$$

We choose $\varepsilon_0^{tr} = -\varepsilon_L$, $\theta_0 = (1-\delta)\theta_M$, $r(t) = r_c > 0$, $\tau(t) = R - E_h\varepsilon_L$ for $t \geq 0$. For $t > 0$, we have $\varepsilon^{tr}(t) = (1/E_h)(\tau(t) + f(\theta(t)) - R) = (1/E_h)f(\theta(t)) - \varepsilon_L$ as long as $f(\theta) < E_h\varepsilon_L$. We now show that the process stops before this threshold can be reached if E_h is sufficiently small or R is sufficiently large.

For $t > 0$, θ satisfies the equation

$$\big(cE_h - \theta f''(\theta)(E_h\varepsilon_L - f(\theta)) + \theta(f'(\theta))^2 - Rf'(\theta)\big)\dot{\theta} = E_h r_c. \tag{4.12}$$

For $\theta \in ((1-\delta)\theta_M, (1+\delta)\theta_M)$, the left hand side of (4.12) can be written as

$$E_h\Big(c - \frac{(1-\delta)b\varepsilon_L}{2\delta}\Big) - \frac{b(E_h\varepsilon_L + R)}{2\delta\theta_M}(\theta - (1-\delta)\theta_M) + \frac{b^2}{12\delta^2\theta_M^2}\theta(\theta - (1-\delta)\theta_M)^2\,.$$

This is a cubic polynomial in $(\theta - (1-\delta)\theta_M)$. Assuming still that the lower bound in (4.10) holds, the absolute term is positive and the polynomial admits a smallest positive root $\theta_{crit} \in ((1-\delta)\theta_M, (1+\delta)\theta_M)$ if E_h is sufficiently small or R is sufficiently large. We can then neglect quadratic and cubic terms in $(\theta - (1-\delta)\theta_M)$, which do not have substantial influence on the process. The dynamics is approximately determined by the equation

$$\Big(cE_h - \frac{(1-\delta)bE_h\varepsilon_L}{2\delta} - \frac{b(E_h\varepsilon_L + R)}{2\delta\theta_M}(\theta - (1-\delta)\theta_M)\Big)\dot{\theta} = E_h r_c,$$

that is,

$$\Big(C - (\theta - (1-\delta)\theta_M)\Big)\dot{\theta} = r^*,$$

where

$$C = E_h\theta_M\frac{2\delta c - (1-\delta)b\varepsilon_L}{b(E_h\varepsilon_L + R)}, \quad r^* = \frac{2\delta E_h\theta_M r_c}{b(E_h\varepsilon_L + R)}.$$

The solution θ is defined only on a small time interval $[0,T_c)$ and $\theta_{crit} \approx \theta(T_c-) = (1-\delta)\theta_M + C$. We have to check that $f(\theta) < E_h\varepsilon_L$ on $[0,T_c)$. We have

$$f(\theta) = \frac{b}{4\delta\theta_M}(\theta - (1-\delta)\theta_M)^2 \leq \frac{b}{4\delta\theta_M}C^2 \leq \frac{b}{4\delta\theta_M}\left(E_h\theta_M\frac{2\delta c - (1-\delta)b\varepsilon_L}{b(E_h\varepsilon_L + R)}\right)^2,$$

hence it suffices to choose E_h small enough or R large enough in such a way that

$$\frac{b}{4\delta\theta_M}\left(E_h\theta_M\frac{2\delta c - (1-\delta)b\varepsilon_L}{b(E_h\varepsilon_L + R)}\right)^2 \leq E_h\varepsilon_L.$$

Again, discontinuous solutions realizing a non-monotone temperature-entropy evolution shall not be considered to be admissible alternatives.

5. Proof of Theorem 4.2. We proceed by space semidiscretization. We extend the function f by zero for $\theta < 0$. Let $n \in \mathbb{N}$ be fixed for the moment, let $x_j = jd_n$, $d_n = \ell/n$, $j = 0, 1, \ldots, n$ be a uniform partition of the interval $[0, \ell]$, and let

$$\theta_j^0 = \theta_0(x_j),\ (\varepsilon_0^{tr})_j = \varepsilon_0^{tr}(x_j)$$

be the discretized initial data. We consider the system

$$d_n\dot{\theta}_j = \varrho_j - \Big(c\theta_j + (f(\theta_j) - \theta_j f'(\theta_j))|\varepsilon_j^{tr}| + \frac{E_h}{2}(\varepsilon_j^{tr})^2 - \tau(t)\varepsilon_j^{tr}\Big), \tag{5.1}$$

$$\dot{\varrho}_j = \frac{\kappa}{d_n^2}(\theta_{j+1} - 2\theta_j + \theta_{j-1}) - \dot{\tau}(t)\varepsilon_j^{tr} + r_j(\theta_j, t) \tag{5.2}$$

for $j = 1, \ldots, n$, where $\theta_0 = \theta_1$, $\theta_{n+1} = \theta_n$, $d_n r_j(\theta, t) = \int_{x_{j-1}}^{x_j} r(\theta, x, t)\,\mathrm{d}x$, and

$$\varepsilon_j^{tr}(t) = \varepsilon_L Q\Big(\frac{1}{E_h\varepsilon_L}\mathfrak{p}_R[z_{0j-}, \tau - f(\theta_j)](t)\Big) - \varepsilon_L Q\Big(-\frac{1}{E_h\varepsilon_L}\mathfrak{p}_R[z_{0j+}, \tau + f(\theta_j)](t)\Big), \tag{5.3}$$

with

$$z_{0j-} = \tau(0) - f(\theta_j^0) - E_h(\varepsilon_0^{tr})_j, \quad z_{0j+} = \tau(0) + f(\theta_j^0) - E_h(\varepsilon_0^{tr})_j. \tag{5.4}$$

For (5.1), (5.2) we prescribe initial conditions

$$\theta_j(0) = \theta_j^0, \tag{5.5}$$

$$\varrho_j(0) = c\theta_j^0 + (f(\theta_j^0) - \theta_j^0 f'(\theta_j^0))|(\varepsilon_0^{tr})_j| + \frac{E_h}{2}(\varepsilon_0^{tr})_j^2 - \tau(0)(\varepsilon_0^{tr})_j. \tag{5.6}$$

This is a system of the form $\dot{X}(t) = \Phi(X(t), t)$ for $X = (\theta_1, \ldots, \theta_n, \varrho_1, \ldots, \varrho_n)$ where the mapping Φ is Lipschitz continuous in $C([0,T]; \mathbb{R}^{2n})$ in the sense that

$$|\Phi(X_1(t), t) - \Phi(X_2(t), t)| \le C \max_{s\in[0,t]} |X_1(s) - X_2(s)| \tag{5.7}$$

for every $t \in [0, T]$ and a positive constant C. We may thus rewrite system (5.1)-(5.3) along with conditions (5.5)-(5.6) as a fixed point $G(X) = X$ for the mapping

$$G : X(t) \mapsto X_0 + \int_0^t \Phi(X(t'), t')\,\mathrm{d}t'.$$

From the inequality

$$|G(X_1)(t) - G(X_2)(t)| \le C \int_0^t \max_{s\in[0,t']} |X_1(s) - X_2(s)|\,\mathrm{d}t'$$

which directly follows from (5.7), we easily deduce the existence of a unique fixed point by the Banach contraction principle.

We now let n tend to ∞. The right hand side of (5.1) is absolutely continuous, hence we can differentiate (5.1) in t and multiply by $\dot{\theta}_j$. We multiply also (5.2) by $\dot{\theta}_j$, sum the two equations and sum up over j to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^n \Big(\frac{d_n}{2}\dot{\theta}_j^2 + \frac{\kappa}{2d_n^2}(\theta_j - \theta_{j-1})^2\Big) + c\sum_{j=1}^n \dot{\theta}_j^2 = B_n, \tag{5.8}$$

where

$$\begin{aligned}
B_n &= \sum_{j=1}^n \Big(-\frac{\mathrm{d}}{\mathrm{d}t}\Big((f(\theta_j) - \theta_j f'(\theta_j))|\varepsilon_j^{tr}| + \frac{E_h}{2}(\varepsilon_j^{tr})^2 - \tau(t)\varepsilon_j^{tr}\Big) - \dot{\tau}(t)\varepsilon_j^{tr} + r_j(\theta_j, t)\Big)\,\dot{\theta}_j \\
&= \sum_{j=1}^n \Big(\theta_j f''(\theta_j)|\varepsilon_j^{tr}|\dot{\theta}_j^2 + \big(\theta_j f'(\theta_j)|\varepsilon_j^{tr}|^{\cdot} + R|\dot{\varepsilon}_j^{tr}| + r_j(\theta_j, t)\big)\dot{\theta}_j\Big) \\
&= \sum_{j=1}^n \Big(\theta_j f''(\theta_j)|\varepsilon_j^{tr}|\dot{\theta}_j^2 + \big(\theta_j f'(\theta_j)|\varepsilon_j^{tr}|^{\cdot} + R|\dot{\varepsilon}_j^{tr}|\big)\dot{\theta}_j - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\theta_j} (r^* - r_j(\theta, t))\,\mathrm{d}\theta + r^*\dot{\theta}_j \\
&\quad - \int_0^{\theta_j} (r_j)_t(\theta, t))\,\mathrm{d}\theta\Big)
\end{aligned}$$

where we have also used (5.3) and Proposition 3.2 in order to deduce that

$$f(\theta_j)|\varepsilon_j^{tr}|^{\cdot} + E_h \varepsilon_j^{tr}\dot\varepsilon_j^{tr} - \tau(t)\dot\varepsilon_j^{tr} = -R|\dot\varepsilon_j^{tr}| \text{ a.e.}$$

We have the implication

$$\dot\varepsilon_j^{tr} \neq 0 \;\Rightarrow\; |\varepsilon_j^{tr}|^{\cdot} = \frac{1}{E_h}(\pm\dot\tau - f'(\theta_j))\dot\theta_j,$$

hence

$$|\varepsilon_j^{tr}|^{\cdot}\dot\theta_j \leq \frac{1}{E_h}(|\dot\tau||\dot\theta_j| - f'(\theta_j))\dot\theta_j^2.$$

Using conditions (4.6) and (4.7), we thus obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^n \Big(\frac{d_n}{2}\dot\theta_j^2 + \frac{\kappa}{2d_n^2}(\theta_j-\theta_{j-1})^2 + \int_0^{\theta_j}(r^* - r_j(\theta,t))\,\mathrm{d}\theta\Big) + \min\{c_0,c_1\}\sum_{j=1}^n \dot\theta_j^2\\
&\leq \sum_{j=1}^n \Big(r^*(|\dot\theta_j| + |\theta_j|) + \frac{(\theta_j f'(\theta_j) + R)}{E_h}|\dot\tau||\dot\theta_j|\Big). \qquad (5.9)
\end{aligned}$$

We argue similarly if $\varepsilon_0^{tr}(x) \geq 0$ for all x, $\tau(t)\geq 0$ for all t, and (4.9) holds instead of (4.6). Then, by Lemma 3.1, we have

$$\varepsilon_j^{tr}(t) = \varepsilon_L Q\Big(\frac{1}{E_h\varepsilon_L}\mathfrak{p}_R[z_{0j-}, \tau - f(\theta_j)](t)\Big) \geq 0,$$

and the implication

$$\dot\varepsilon_j^{tr} \neq 0 \;\Rightarrow\; \dot\varepsilon_j^{tr} = \frac{1}{E_h}(\dot\tau - f'(\theta_j))\dot\theta_j$$

holds so that

$$\dot\varepsilon_j^{tr}\dot\theta_j \leq \frac{1}{E_h}|\dot\tau||\dot\theta_j|.$$

Then, we have that

$$\begin{aligned}
B_n &= \sum_{j=1}^n \Big(\theta_j f''(\theta_j)|\varepsilon_j^{tr}|\dot\theta_j^2 + \Big((\theta_j f'(\theta_j) - f(\theta_j))|\varepsilon_j^{tr}|^{\cdot} - E_h\varepsilon_j^{tr}\dot\varepsilon_j^{tr} + \tau(t)\dot\varepsilon_j^{tr} + r_j(\theta_j,t)\Big)\dot\theta_j\Big)\\
&\leq \sum_{j=1}^n \Big(\theta_j f''(\theta_j)\varepsilon_L\dot\theta_j^2 + \Big(\frac{1}{E_h}(\theta_j f'(\theta_j) - f(\theta_j))|\dot\tau| + \varepsilon_L(|\dot\tau| + f'(\theta_j)|\dot\theta_j|) + \frac{1}{E_h}\tau|\dot\tau|\Big)|\dot\theta_j|\\
&\qquad + r_j(\theta_j,t)\dot\theta_j\Big)
\end{aligned}$$

and we obtain a counterpart of (5.9) with different constants on the right hand side.

We have $\dot\theta_j(0) = 0$, and

$$\sum_{j=1}^n \frac{1}{d_n}(\theta_j^0 - \theta_{j-1}^0)^2 \leq \int_0^\ell (\theta_0')^2\,\mathrm{d}x,$$

hence there exist constants C_1, C_2 independent of n such that

$$\sup_{t\in(0,T)}\sum_{j=1}^n \Big(d_n^2\dot\theta_j^2(t) + \frac{1}{d_n}(\theta_j - \theta_{j-1})^2(t)\Big) + d_n\sum_{j=1}^n\int_0^T \dot\theta_j^2(t)\,\mathrm{d}t \leq C_1, \qquad (5.10)$$

$$\frac{1}{d_n}\sum_{j=1}^n\int_0^T (\theta_{j+1}(t) - 2\theta_j(t) + \theta_{j-1}(t))^2\,\mathrm{d}t \leq C_2. \qquad (5.11)$$

We now define piecewise constant and piecewise quadratic interpolations

$$
\begin{aligned}
\bar\theta^{(n)}(x,t) &= \theta_j,\\
\bar\varrho^{(n)}(x,t) &= \varrho_j,\\
\theta_0^{(n)}(x) &= \theta_j^0,\\
\varepsilon_0^{(n)}(x) &= (\varepsilon_0^{tr})_j,\\
\bar r(\tau,x,t) &= r_j(\tau,t)\\
\hat\theta^{(n)}(x,t) &= \frac{1}{2}(\theta_j+\theta_{j-1}) + \frac{1}{d_n}(x-x_{j-1})(\theta_j-\theta_{j-1})\\
&\quad + \frac{1}{2d_n^2}(x-x_{j-1})^2(\theta_{j+1}-2\theta_j+\theta_{j-1})
\end{aligned}
$$

for $x\in[x_{j-1},x_j)$, continuously extended to $x=\ell$. System (5.1), (5.2) then has the form

$$
d_n\bar\theta_t^{(n)} = \bar\varrho^{(n)} - \Big(c\bar\theta^{(n)} + (f(\bar\theta^{(n)}) - \bar\theta^{(n)}f'(\bar\theta^{(n)}))|\varepsilon^{(n)}| + \frac{E_h}{2}(\varepsilon^{(n)})^2 - \tau(t)\varepsilon^{(n)}\Big), \tag{5.12}
$$

$$
\bar\varrho_t^{(n)} = \kappa\hat\theta_{xx}^{(n)} - \dot\tau(t)\varepsilon^{(n)} + \bar r(\bar\theta^{(n)},x,t) \tag{5.13}
$$

where

$$
\varepsilon^{(n)} = \varepsilon_L Q\Big(\frac{1}{E_h\varepsilon_L}\mathfrak{p}_R[z_{0-}^{(n)},\tau - f(\bar\theta^{(n)})](t)\Big) - \varepsilon_L Q\Big(-\frac{1}{E_h\varepsilon_L}\mathfrak{p}_R[z_{0+}^{(n)},\tau + f(\bar\theta^{(n)})](t)\Big), \tag{5.14}
$$

with

$$
z_{0-}^{(n)} = \tau(0) - f(\theta_0^{(n)}) - E_h\varepsilon_0^{(n)}, \quad z_{0+}^{(n)} = \tau(0) + f(\theta_0^{(n)}) - E_h\varepsilon_0^{(n)}. \tag{5.15}
$$

From the estimates (5.10)-(5.11) it follows that the sequences $\hat\theta_t^{(n)}$ and $\hat\theta_{xx}^{(n)}$ are bounded in $L^2((0,\ell)\times(0,T))$. By compact embedding, a subsequence (still denoted by $\hat\theta^{(n)}$) converges uniformly to some $\theta\in C([0,\ell]\times[0,T])$. Using the inequality

$$
|\hat\theta^{(n)}(x,t) - \bar\theta^{(n)}(x,t)| \le 2|\theta_j(t)-\theta_{j-1}(t)| + \frac{1}{2}|\theta_{j+1}(t)-\theta_j(t)|
$$

for $x\in[x_{j-1},x_j)$, we obtain for all $(x,t)\in(0,\ell)\times(0,T)$ the estimate

$$
|\hat\theta^{(n)}(x,t) - \bar\theta^{(n)}(x,t)|^2 \le 5\sum_{j=1}^n |\theta_j(t)-\theta_{j-1}(t)|^2 \le 5C_1 d_n,
$$

hence also $\bar\theta^{(n)}$ converge uniformly to θ. Using the continuity of the play with respect to uniform convergence and the identity $\hat\theta_x^{(n)}(0,t) = \hat\theta_x^{(n)}(\ell,t) = 0$ for all n and t, we easily verify that θ is a solution of the original problem (4.4), (2.20), (4.2) with the prescribed initial conditions.

It remains to check that $\theta(x,t)\ge\theta_*$ for all $(x,t)\in(0,\ell)\times(0,T)$. To this end, we test equation (4.5) by $-(\theta_*-\theta)^+$. For $\theta_j<\theta_*$, we have $f(\theta)=f'(\theta)=0$ and $r(\theta,x,t)\ge 0$, hence

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^\ell ((\theta_*-\theta)^+)^2\,\mathrm{d}x + \kappa\int_0^\ell ((\theta_*-\theta)^+)_x^2\,\mathrm{d}x \ \le\ 0,
$$

and using the fact that $\theta_0(x)\ge\theta_*$ for all x, we conclude that $\theta(x,t)\ge\theta_*$ globally. This completes the proof of Theorem 4.2. ■

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