ANALYSIS OF A THERMO-MECHANICAL MODEL FOR SHAPE MEMORY ALLOYS

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Abstract. This note addresses the mathematical study of a nonlinear model arising in the description of the macroscopic thermo-mechanical behavior of shape memory materials and previously introduced in [31]. In particular, we discuss the model derivation and investigate a system of PDE's coupled with a vectorial variational inequality. We develop the analysis in both the dissipative and the non-dissipative case, providing indeed a quantitative asymptotic connection between the two regimes. Moreover, we prove the global in time well-posedness for suitable initial and boundary value problems. As a by-product of the well-posedness analysis, we address a variable time-step discretization procedure, proving indeed its convergence and providing some a priori error bounds. Finally, we deal with the asymptotic behavior of the system for large times and establish the convergence of the trajectories to the solution of a suitable stationary problem.

Key words. shape memory alloys, well-posedness, discretization, long-time behavior

AMS subject classifications. 74C05, 35K55, 65M12, 35B40

1. Introduction. The present analysis is concerned with the evolution of four unknown fields θ, χ_1, χ_2, and u governed by the following system of equations and inclusion

$$\partial_t(c_s\theta - \ell\chi_1) - k\Delta\theta = r, \tag{1.1}$$

$$\operatorname{div}\sigma + b = 0, \tag{1.2}$$

$$\mathbb{A}(\varepsilon(u) + \beta\chi_2) = \sigma, \tag{1.3}$$

$$\mu\partial_t\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \gamma\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \begin{pmatrix}\frac{\ell}{\theta_*}(\theta-\theta_*)\\ \sigma:\beta\end{pmatrix} + \partial I_K\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \ni \begin{pmatrix}0\\ 0\end{pmatrix}. \tag{1.4}$$

These relations are asked to be fulfilled in the space-time domain $Q := \Omega\times(0,T)$ for some open and bounded subset $\Omega\subset\mathbb{R}^3$ with smooth boundary Γ and some reference time $T>0$. In addition c_s, ℓ, k, γ, and θ_* are positive parameters, $\mathbb{A}$ and β are respectively a 4-tensor and a 2-tensor, and μ is a non-negative constant (see below). Here, $u=(u_1,u_2,u_3)\in\mathbb{R}^3$ and $\varepsilon(u)$ denotes the 2-tensor

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{for } i,j=1,2,3,$$

while ∂I_K stands for the subdifferential of the indicator function of the non-empty, bounded, convex and closed subset K of $\mathbb{R}^2$, i.e. $I_K(x)=0$ if $x\in K$ and $I_K(x)=+\infty$ elsewhere. Namely, $\partial I_K:\mathbb{R}^2\to 2^{\mathbb{R}^2}$ is defined by

$$y\in\partial I_K(x) \ \text{ iff } \ x\in K \ \text{ and } \ y\cdot(w-x)\le 0 \ \ \forall w\in K.$$

Moreover $\sigma:\beta := \sigma_{ij}\beta_{ij}$ (summation convention) denotes the standard contraction product of 2-tensors, $(\operatorname{div}\sigma)_i := \partial(\sigma_{ij})/\partial x_j$ for $i=1,2,3$, and $r:Q\to\mathbb{R}$ and $b:Q\to\mathbb{R}^3$ are given functions.

The nonlinear system (1.1)-(1.4) arises in connection with the study of the thermo-mechanical behavior of shape memory materials. These are metallic alloys with an

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intrinsic ability of undergoing a thermo-elastic solid-solid transformation between crystallographic configurations with different physical and mechanical properties: the *austenite*, which is stable at higher temperatures, and variants of *martensite*, stable at lower temperatures [2, 21]. At the macroscopic level such a reversible phase transformation results in the so-called *shape memory effect*. Namely, shape memory alloys can be permanently deformed (up to 8% under traction) and then be forced to recover their original shape just by thermal means. This unusual macroscopic mechanical effect is nowadays exploited in several innovative devices. Indeed shape memory materials are actually used in order to realize a variety of actuators (also of microscopic size) and structures. The field of application of shape memory technologies ranges from bio-engineering to structures-engineering and aerospace sciences [13, 16, 30]. Here we are concerned with a macroscopic modelization previously introduced in [31] and able to describe the shape memory effect in a small deformation realm. We refer the reader to the forthcoming Section 2 for a derivation of the model as well as for some discussion on its thermodynamical consistency. As for the justification of this modeling perspective as well as some experimental validation one should refer to the original paper [31] as well as to [14, 20, 32]. For the purposes of this introductory discussion, let us remark that the scalar field θ in (1.1)-(1.4) represents the absolute temperature of the medium while the vector field u is its displacement. Hence, the 2-tensors σ and $\varepsilon(u)$ stand for the tension and the linearized strain, respectively. Finally, K is the admissible convex and closed range for the internal variables $[\chi_1, \chi_2]$ (see below).

The current literature on the mathematical modeling of shape memory alloys is quite rich and is beyond our purposes to provide here an exhaustive review. Indeed, let us just mention that the problem of describing the thermo-mechanical behavior of shape memory alloys has been tackled both from the microscopic [2, 4, 5] and the macroscopic viewpoint [1, 3, 6, 18, 19, 33]. Among the latter, we shall particularly mention the so-called Frémond model for shape memory alloys. This model was originally presented in [19] and analyzed in [9]. Indeed both the present model and Frémond's model [19] are formulated in the framework of Generalized Standard Materials by means of analogous free-energy and dissipation considerations. In particular, the phase relation of Frémond's model corresponds, in the present setting, to the choices $\mu > 0$ and $\gamma = 0$. On the contrary, the modeling considerations and the experimental evidences of [31] suggest to consider the degenerate case $\mu = 0$ (taking indeed $\gamma > 0$). We will refer to the situation $\mu > 0$ as the *dissipative case* and $\mu = 0$ as the *non-dissipative* one.

Form the mathematical point of view, the present situation appears to be more delicate with respect to the one of [9] because of the time-degeneracy of the phase relation (1.4). Moreover, let us stress that the momentum balance equation of [9] includes of a fourth order regularizing term that is actually not present in our situation.

The system (1.1)-(1.4) has to be supplied with suitable initial and boundary conditions. To this aim we ask for

$$\theta(\cdot, 0) = \theta^0 \quad \text{on} \quad \Omega, \tag{1.5}$$

$$\mu[\chi_1(\cdot, 0), \chi_2(\cdot, 0)] = \mu[\chi_1^0, \chi_2^0] \quad \text{on} \quad \Omega, \tag{1.6}$$

$$k\partial_\nu \theta + h(\theta - \theta_e) = f \quad \text{on} \quad \Gamma, \tag{1.7}$$

$$\sigma\nu = g \quad \text{on} \quad \Gamma_t, \tag{1.8}$$

$$u = 0 \quad \text{on} \quad \Gamma_0, \tag{1.9}$$

where θ^0, χ_1^0, χ_2^0 are initial values, $h, \theta_e > 0$, $f : \Gamma \to \mathbb{R}$, and $g : \Gamma_t \to \mathbb{R}^3$ are prescribed. Moreover, ν is the unit outward normal vector to the boundary, and $\{\Gamma_0, \Gamma_t\}$ is a partition of Γ into two disjoint subsets of positive surface measure.

This paper addresses the mathematical study of the system (1.1)-(1.9) in both the dissipative ($\mu > 0$) and non-dissipative ($\mu = 0$) regime. First of all, we shall comment on the model derivation and its thermodynamical consistency. Then, we investigate the dissipative situation and provide a well-posedness result for a global variational solution (Thm. 4.1). As a by-product of this analysis we provide a variable time-step discretization scheme which turns out to be stable and convergent. Moreover, we are in the position of providing an *a priori* bound of optimal order on the discretization error. Then, we prove an asymptotic result that connects the dissipative and the non-dissipative regimes. In particular, we prove that, as μ goes to zero, the solution of the dissipative model converges to the solution of the non-dissipative one (Thm. 4.2). Indeed, we also achieve some estimate in terms of μ on the distance between the latter two solutions. As a corollary of this asymptotic result, one obtains the global variational well-posedness for the non-dissipative case as well (Thm. 4.1). Then, we focus on the long-time behavior of solutions for the non-dissipative model. In particular, we prove that the model actually converges to a unique equilibrium which is characterized as the solution of a suitable elliptic problem (Thm. 4.3). Finally, we turn to the proof of a suitable maximum principle which entails an essential lower bound for the temperature in terms of data (Thm. 4.4). The latter in particular ensures that, starting from a positive datum, the temperature θ remains positive for all times.

This is the plan of the paper. We shall discuss the derivation of the model in Section 2. Then, we introduce the variational formulation of the problem in Section 3. Our main results are stated in Section 4, while Section 5 is devoted to the study of the dissipative case. In particular, it contains the details of the discretization method. The non-dissipative model is investigated in Section 6 and Section 7 focuses on the long-time behavior of solutions. The crucial proof of the positivity of the temperature is then given in Section 8.

2. Model. We devote this section to a derivation of the thermo-mechanical model in study [31]. Our aim is to possibly clarify the meaning of relations (1.1)-(1.4) and check for the thermodynamical consistency of the model. In particular, it is beyond our purposes to provide the reader with a full justification of this modeling perspective. Indeed, for a comprehensive discussion on the model as well as some experimental validation, the reader should refer to [31] where the model was introduced.

We will describe the thermo-mechanical evolution of a shape-memory material with respect to its smooth reference configuration $\Omega \subset \mathbb{R}^3$ by means of the absolute temperature θ, the (small) deformation $\varepsilon(u)$ (u is the displacement), and a pair of internal variables $[\chi_1, \chi_2]$ introduced below. In particular, for the purposes of this section, θ is *assumed* to be strictly positive (this will turn out to be Theorem 4.4 later on). Let us suppose from the very beginning that only two martensitic variants are present beside one austenite and indicate the respective local proportions as η_1, η_2, and η_A, respectively. This assumption is of course extremely reductive since, in some particular alloy, up to 24 martensitic variants have been detected. Nevertheless our somehow crude simplification is still suitable of describing the basic features of the physical phenomenon [9, 31]. We moreover assume that the phases possibly coexist at each point of the body, that no overlapping between different phases can occur, and that no void appears in the mixture. Hence, the phase proportions η_1, η_2, and

η_A are constrained to fulfill the obvious relations

$$0 \leq \ \eta_1, \eta_2, \eta_A \ \leq 1, \quad \eta_1 + \eta_2 + \eta_A = 1.$$

We exploit these relations in order to eliminate η_A by introducing the internal variables

$$\chi_1 := \eta_1 + \eta_2, \quad \chi_2 := \eta_1 - \eta_2.$$

Of course the set $\{\chi_1 = 1\}$ corresponds to the situation where no austenite is present, the set $\{\chi_1 = \chi_2\}$ corresponds to the set where just the first variant of martensite is present etc. Owing to the above discussion it is clear that $[\chi_1, \chi_2]$ are constrained in the triangle

$$K := \{[x_1, x_2] \in \mathbb{R}^2 : 0 \leq |x_2| \leq x_1 \leq 1\}. \tag{2.1}$$

In order to deduce the differential relations governing the evolutions of the state quantities above we will follow the approach via *microscopic motions* originally proposed by M. Frémond. The basic novelty of the latter theory is to take into account the thermo-mechanical effect of the microscopic rearrangements of the phases at the macro-scales. In particular, one admits that the microscopic movements of the substance might give rise to some thermal macroscopic effect which influences the overall energy balance of the body. We will not review here the full theory of Thermo-Mechanics of Continua with microscopic motions and just refer the reader to the recent monograph [20] for both a comprehensive discussion and a specific application to the description of shape memory materials.

To the aim of dealing with microscopic motions, let us postulate from the very beginning that the proper quantities describing such micro-movements are $[\dot{\chi}_1, \dot{\chi}_2]$ where of course the dot denotes time differentiation (as it is customary, at this stage we assume that all the quantities occurring in the analysis are as smooth as needed in order to go through the differentiations).

Hence, it seems convenient to regard the vector $(u, [\dot{\chi}, \dot{\chi}_2])$ as an *actual rigid velocity vector*. Moreover, let us assume that there exists a suitable linear space of virtual rigid body velocities R (see [20] for a full discussion). Finally, we suppose that, for all times $t \in [0, T]$, the virtual power of the internal forces of the body with respect to the generic smooth subdomain $D \subset \Omega$ and virtual rigid body velocities $(v, c) \in R$ is

$$P_{int}(D, v, c) := -\int_D \sigma : \varepsilon(v) - \int_D B \cdot c.$$

The first term above is classical while the second one describes the power of microscopic internal forces. In the latter the quantity $B(\cdot) \in \mathbb{R}^2$ comes into play and an obvious dimensional argument entails that it shall be regarded as an *energy density*. In particular, B represents and vector energy density per units of $[\dot{\chi}_1, \dot{\chi}_2]$ (see [20, Sec. 13.3, p. 360]). We now introduce the virtual power of the external and acceleration forces as

$$P_{ext}(D, v, c) := \int_D b \cdot v + \int_{\partial D} g \cdot v \, d\mathcal{H}^{n-1}, \qquad P_{acc}(D, v, c) := \int_D \rho \zeta \cdot v.$$

Here, b represents an action density at distance (body force) while g is an action density at contact (traction) and we use a standard notation for the Hausdorff measure.

Moreover, $\zeta = \ddot{u}$ is the macroscopic acceleration and ρ is the material density (no microscopic accelerations are considered). By recalling the *Virtual Power Principle* [23], choosing arbitrarily the regular and connected domain D and the virtual rigid body velocities $(v, c) \in R$ we deduce from the relation

$$P_{acc}(D, v, c) = P_{int}(D, v, c) + P_{ext}(D, v, c),$$

two systems of momentum balance equations, namely [20, Sec. 2.4, p. 5]

$$\rho\ddot{u} = \operatorname{div}\sigma + b \qquad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$\sigma\nu = g, \qquad \text{on } \Gamma \times (0, T), \tag{2.3}$$

which stands for the *macroscopic momentum balance*, and

$$B = 0 \qquad \text{in } \Omega \times (0, T), \tag{2.4}$$

which corresponds to the *microscopic momentum balance*. Of course, ν stands for the unit normal vector field pointing outward Γ.

Letting now e denote the internal energy density of the system, Q the entropy flux, we can follow [20, Sec. 13.4, p. 361] and deduce that, in our situation, the energy balance is expressed by

$$\dot{e} + \operatorname{div}(\theta Q) - r = \sigma : \varepsilon(\dot{u}) + B \cdot [\dot{\chi}_1, \dot{\chi}_2] \qquad \text{in } \Omega \times (0, T), \tag{2.5}$$

$$-\theta Q \cdot \nu = \pi \qquad \text{on } \Gamma \times (0, T), \tag{2.6}$$

where r and π denote some volume and surface heat source densities, respectively. In particular, we note that the right hand side of (2.5) takes into account the contribution to the energy balance provided by both macroscopic and microscopic movements.

The next step is to define the quantities e, Q, σ, and B in terms of the state variables in such a way that the Second Principle of Thermodynamics, in the form of the Clausius-Duhem inequality, is fulfilled. In particular, the latter reduces in our case to

$$\dot{s} + \operatorname{div} Q - \frac{r}{\theta} \geq 0 \qquad \text{in } \Omega \times (0, T), \tag{2.7}$$

where s is the entropy of the system and r/θ represents an external entropy source density. In order to accomplish the above requirement we will exploit the Ginzburg-Landau theory by introducing the *free energy density* $\psi = \psi(\theta, [\chi_1, \chi_2], \varepsilon(u))$ and the *pseudo-potential of dissipation* $\phi = \phi(\nabla\theta, [\dot{\chi}_1, \dot{\chi}_2])$ and defining

$$s := -\frac{\partial\psi}{\partial\theta}, \qquad e := \psi + \theta s, \qquad Q := -\frac{\partial\phi}{\partial(\nabla\theta)}, \tag{2.8}$$

$$\sigma := \frac{\partial\psi}{\partial(\varepsilon(u))}, \tag{2.9}$$

$$B := [B_1, B_2], \quad B_j := \frac{\partial\psi}{\partial\chi_j} + \frac{\partial\phi}{\partial(\dot{\chi}_j)} \quad j = 1, 2, \tag{2.10}$$

The above choice splits B into a *non-dissipative* and a *dissipative* part, respectively, and is inspired by thermodynamic considerations (see below and [20]). Moreover, the latter notions of derivative are intended to be properly generalized in case ψ, ϕ are non-smooth functions. At the present stage, our only requirement on the potentials ψ and ϕ is that [20, 27], ϕ is convex, non-negative, and vanishes in 0.

We shall now check for the thermodynamic consistency of this class of models by recalling (2.5) and the above definitions in order to compute that

$$\dot{s} + \operatorname{div} Q - \frac{r}{\theta} = \frac{1}{\theta}\left(\theta\dot{s} + \operatorname{div}(\theta Q) - r\right) - \frac{1}{\theta} Q\cdot\nabla\theta$$
$$= \frac{1}{\theta}\left(\left[\frac{\partial\phi}{\partial\dot{\chi}_1}, \frac{\partial\phi}{\partial\dot{\chi}_2}\right]\cdot[\dot{\chi}_1,\dot{\chi}_2] - Q\cdot\nabla\theta\right) \geq 0,$$

where we used the properties of the pseudo-potential ϕ. Hence, the Clausius-Duhem inequality (2.7) easily follows from the positivity of θ. As a consequence, the general positivity proof implied by Theorem 4.4 entails the thermodynamic consistency of the whole class of models.

We now come to our actual choice of ψ [31]. In particular, we let

$$\psi(\theta, [\chi_1,\chi_2], \varepsilon(u)) = -c_s\theta\ln\theta + \frac{1}{2}(\varepsilon(u)+\beta\chi_2) : \mathbb{A}(\varepsilon(u)+\beta\chi_2)$$
$$+\ell\frac{\theta_* - \theta_{**}}{2\theta_*}(\chi_1^2+\chi_2^2) + \frac{\ell}{\theta_*}(\theta-\theta_*)\chi_1 + I_K(\chi_1,\chi_2). \tag{2.11}$$

In the latter expression, the first term is purely caloric and c_s represents a specific heat density. The second term corresponds to the mechanical energy. In particular $\mathbb{A}$ is the elasticity tensor and the extra term $\beta\chi_2$ represent the mechanical effect of the presence of the two different martensitic variants. Indeed, one assumes that the mechanical potential of the material is

$$\frac{1}{2}(\varepsilon(u)+\beta_1\eta_1+\beta_2\eta_2) : \mathbb{A}(\varepsilon(u)+\beta_1\eta_1+\beta_2\eta_2),$$

where β_j are *transformation strain tensors* encoding the mechanical effect of the martensite-austenite phase change and are assumed to verify $\beta_1 = -\beta_2 =: \beta$, in order to take into account the so-called *self-accommodating* properties of the two martensitic variants [20, 31]. Note that the thermal expansion of the system is neglected.

The indicator function I_K forces $[\chi_1,\chi_2]$ to take solely admissible values in K and the term $\frac{\ell}{\theta_*}(\theta-\theta_*)\chi_1$ classically represents the phase-temperature interaction. In particular, ℓ is a latent heat density related to the martensite-austenite transformation and $\theta_* > 0$ is the critical martensite-to-austenite transition temperature.

The modeling novelty of this framework with respect to the original Frémond model for shape memory alloys [9, 19] consists in including into ψ the term

$$\ell\frac{\theta_* - \theta_{**}}{2\theta_*}(\chi_1^2+\chi_2^2), \tag{2.12}$$

where a second critical transition temperature $0 < \theta_{**} < \theta_*$ is introduced for the austenite-to-martensite transformation. By referring to the zero-stress situation, one observes that in Frémond's model [9] no austenite is present for temperatures θ below θ_* nor martensites for $\theta > \theta_*$. This simplification is however fairly crude and experiments suggest that one should consider a suitable temperature range where the three phases may coexist in the zero-stress situation [14, 32]. The present model [31] extends Frémond's approach in the direction of including some description of this effect. In particular, looking back to (2.4) in the zero-stress equilibrium ($[\dot{\chi}_1,\dot{\chi}_2] = 0$, $\sigma = 0$) the internal variables are asked to fulfill

$$\ell\frac{\theta_* - \theta_{**}}{\theta_*}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \begin{pmatrix}\frac{\ell}{\theta_*}(\theta-\theta_*)\\ 0\end{pmatrix} + \partial I_K\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \ni \begin{pmatrix}0\\ 0\end{pmatrix},$$

which entails that

$$\chi_1 = \min\left\{\max\left\{\frac{\theta_* - \theta}{\theta_* - \theta_{**}}, 0\right\}, 1\right\}, \qquad \chi_2 = 0.$$

Hence, no martensites are present for $\theta > \theta_*$, no austenite is allowed for $\theta < \theta_{**}$, and possibly all the phases are admissible for intermediate temperatures [31] (see also [20, Rem. 13.4, p. 364]).

As for the pseudo-potential of dissipation we will ask for

$$\phi(\nabla\theta, [\dot\chi_1, \dot\chi_2]) := \frac{k}{2\theta}|\nabla\theta|^2 + \frac{\mu}{2}\dot\chi_1^2 + \frac{\mu}{2}\dot\chi_2^2.$$

Here $k > 0$ stands for a constant thermal conductivity coefficient and $\mu \geq 0$ measures some dissipation effect on the phase variables. In particular, the heat flux $q := \theta Q = -k\nabla\theta$ is of Fourier type.

Finally, the balance relations (2.5), (2.2), and (2.4) read as follows

$$c_s\dot\theta - \ell\dot\chi_1 - k\Delta\theta - r = -\frac{\ell}{\theta_*}(\theta - \theta_*)\dot\chi_1 + \mu\dot\chi_1^2 + \mu\dot\chi_2^2, \tag{2.13}$$

$$\rho\ddot u = \operatorname{div}\sigma + b, \tag{2.14}$$

$$\mu\begin{pmatrix}\dot\chi_1\\ \dot\chi_2\end{pmatrix} + \ell\frac{\theta_* - \theta_{**}}{\theta_*}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \begin{pmatrix}\frac{\ell}{\theta_*}(\theta - \theta_*)\\ \sigma : \beta\end{pmatrix} + \partial I_K\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \ni \begin{pmatrix}0\\ 0\end{pmatrix}. \tag{2.15}$$

We shall focus our attention from the very beginning on the *quasi-static* situation where the inertial term $\rho\ddot u$ is negligible in (2.14). Indeed, let us stress that the latter approximation of the momentum balance equation is rather standard in connection with the Frémond model [9, 10, 12, 24] and translates the believe that the mechanical evolution takes place on some faster time scale when compared with the thermal evolution. On the other hand, the reader is referred to [8] where the full momentum problem is considered for Frémond's model (see also [15, 22, 25] and the recent monograph [26] for the analysis of mechanical evolution under different non-convex settings).

In order to deduce the system (1.1)-(1.4) from (2.9) and (2.13)-(2.15) we now apply some further modification to the balance relations by means of suitable small perturbation assumptions. At first, one supposes to be interested in a temperature range close to the critical temperature θ_* and neglects the first term in the right hand side of (2.13). By setting for the sake of notational simplicity $\gamma := \ell(\theta_* - \theta_{**})/\theta_*$, we readily check that the system (1.1)-(1.4) in the non-dissipative regime $\mu = 0$ follows directly from (2.9) and (2.13), (2.15), and (2.14) in its quasi-static form.

As for the non-dissipative regime $\mu > 0$ we shall additionally assume to be interested in a situation where the phase evolution is suitably slow, i.e. $\mu\dot\chi_j^2 = 0$ in the energy balance equation (2.13). On the other hand we retain the dissipation term $\mu[\dot\chi_1, \dot\chi_2]$ in (2.15). This is of course again an assumption of small perturbation type.

Finally, as for boundary conditions (1.7)-(1.9), one assumes to know g in (2.3) just on Γ_t and imposes the body to be clamped on Γ_0. Moreover, we choose $\pi := f + h(\theta_e - \theta)$ where f is a prescribed surface heat source density, $h > 0$ is a thermal exchange coefficient, and $\theta_e > 0$ is a given external temperature.

3. Variational Formulation. We start by fixing some notations. Let

$$H := L^2(\Omega), \quad \mathcal{H} := H \times H \times H, \quad V := H^1(\Omega),$$
$$\mathbb{H} := \{\sigma : \Omega \to \mathbb{R}^{3\times 3}_{symm} \text{ measurable, such that } \sigma : \sigma \in L^1(\Omega)\},$$

where $\mathbb{R}^{3\times3}_{symm}$ denotes of course the space of 3×3 symmetric tensors. All the above spaces are endowed with their respective natural scalar products. In particular, we will use the symbols $(\cdot,\cdot)$ and $\|\cdot\|$ for all products and norms in the above L^2−type spaces. Moreover, the notation $(\cdot,\cdot)_\Gamma$ will stand for the scalar product in both $L^2(\Gamma)$ and $(L^2(\Gamma))^3$, $|\cdot|$ denotes any Euclidean norm, $\|\cdot\|_E$ will stand for the norm in the generic normed space E, and $[\cdot,\cdot]$ denotes the generic pair. We introduce the following Hilbert space

$$\mathcal{V} := \{v \in V^3, \text{ such that } v = 0 \text{ on } \Gamma_0\},$$

endowed with the standard norm, and set, for any $u, v \in \mathcal{V}$,

$$a(u,v) := (\mathbb{A}\varepsilon(u), \varepsilon(v)),$$

where $\varepsilon : \mathcal{V} \to \mathbb{H}$ stands for the linearized strain tensor. Following the classic linear elasticity theory, we ask $\mathbb{A} = (a_{ijkh})$ to be symmetric and positive definite on $\mathbb{R}^{3\times3}_{symm}$, namely

$$a_{ijkh} = a_{ijhk} = a_{khij} \quad \forall i,j,h,k = 1,2,3, \quad \text{and} \quad \mathbb{A}\sigma : \sigma > 0 \quad \forall \sigma \in \mathbb{R}^{3\times3}_{symm}/\{0\}.$$

Namely, for all $\sigma, \tau \in \mathbb{R}^{3\times3}_{symm}$ one has that $\mathbb{A}\sigma : \tau = \sigma : \mathbb{A}\tau = \mathbb{A}^{\frac12}\sigma : \mathbb{A}^{\frac12}\tau$, where $\mathbb{A}^{\frac12}$ stands for the well-defined square root of $\mathbb{A}$. In particular, since $\beta \neq 0$, we readily compute that $\mathbb{A}^{\frac12}\beta \neq 0$ as well. Moreover, recalling the Γ_0 has a positive surface measure and thanks to Korn's inequality (see, e.g. [17, Thm. 3.3, p. 115]), there exists a positive constant $c_\mathcal{V}$ depending on $\mathbb{A}$ such that

$$a(v,v) \;=\; \|\mathbb{A}^{\frac12}\varepsilon(v)\|^2 \;\geq\; c_\mathcal{V}\|v\|^2_\mathcal{V} \quad \forall v \in \mathcal{V}.$$

Finally, let the notation $\langle\cdot,\cdot\rangle$ stand for the duality pairing between V' and V or $\mathcal{V}'$ and $\mathcal{V}$ where the prime denotes the topological duals. Since the special triangular form of K specified above is not needed for our analysis, let K be an arbitrary non-empty, bounded, convex, and closed subset of $\mathbb{R}^2$, and define the (convex and closed) set $\mathcal{K} := \big\{[x_1,x_2] \in (L^2(\Omega))^2, \text{ such that } [x_1,x_2] \in K \text{ a.e. in } \Omega\big\}$. For almost every $t \in (0,T)$ let us define the functionals $F(t) : V \to V'$ and $G(t) : \mathcal{V} \to \mathcal{V}'$ as

$$\langle F(t),\varphi\rangle := \big(r(t),\varphi\big) + \big(f(t),\varphi\big)_\Gamma \quad \forall \varphi \in V,$$
$$\langle G(t),v\rangle := \big(b(t),v\big) + \big(g(t),v\big)_\Gamma \quad \forall v \in \mathcal{V}.$$

We shall make precise our variational formulation of (1.1)-(1.9) by posing the following problem.

Problem P_μ: To find $\theta \in H^1(0,T;H) \cap L^\infty(0,T;V)$, $[\chi_1,\chi_2] \in (H^1(0,T;H) \cap L^\infty(Q))^2$, and $u \in H^1(0,T;\mathcal{V})$ such that $\mu(\chi_1,\chi_2) \in (W^{1,\infty}(0,T;H))^2$ and

$$\big((c_s\theta - \ell\chi_1)_t, \varphi\big) + k(\nabla\theta, \nabla\varphi) + h(\theta - \theta_e, \varphi)_\Gamma = \langle F, \varphi\rangle$$
$$\forall \varphi \in V, \text{ a.e. in } (0,T), \tag{3.1}$$

$$a(u,v) + (\mathbb{A}\beta\chi_2, \varepsilon(v)) = \langle G, v\rangle \quad \forall v \in \mathcal{V}, \text{ a.e. in } (0,T), \tag{3.2}$$

$$\mathbb{A}(\varepsilon(u) + \beta\chi_2) = \sigma \quad \text{a.e. in } Q, \tag{3.3}$$

$$\mu\partial_t\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \gamma\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \partial I_K\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \ni \begin{pmatrix}\frac{\ell}{\theta_*}(\theta_* - \theta)\\ -\sigma : \beta\end{pmatrix} \quad \text{a.e. in } Q, \tag{3.4}$$

$$c_s\theta(\cdot,0) - \ell\chi_1(\cdot,0) = c_s\theta^0 - \ell\chi_1^0 \quad \text{a.e. in } \Omega, \tag{3.5}$$

$$\mu\big[\chi_1(\cdot,0), \chi_2(\cdot,0)\big] = \mu\big[\chi_1^0, \chi_2^0\big] \quad \text{a.e. in } \Omega. \tag{3.6}$$

REMARK 3.1. Let us stress that the above regularity requirements and (3.3) entail in particular that $\sigma \in H^1(0,T;\mathbb{H})$. Namely, relation (3.4) makes sense.

4. Main results. We shall assume the following.
(A1) $F \in L^2(0,T;H) + W^{1,1}(0,T;V')$, $G \in H^1(0,T;\mathcal{V}')$,
(A2) $\theta^0 \in V$,
(A3) $[\chi_1^0, \chi_2^0] \in \mathcal{K}$.
In particular, the first in (A1) entails that there exist $F_1 \in L^2(0,T;H)$ and $F_2 \in W^{1,1}(0,T;V')$ such that $F = F_1 + F_2$. We observe that, whenever $r \in L^2(0,T;H)$, $f \in W^{1,1}(0,T;L^2(\Gamma))$, $b \in H^1(0,T;\mathcal{H})$, and $g \in H^1(0,T;(L^2(\Gamma))^3)$, the regularities in (A1) follow. As a consequence of (A1)-(A3), we introduce $u^0 \in \mathcal{V}$ as the unique solution to (3.2) at time $t=0$, $\sigma^0 \in \mathbb{H}$ via u^0 and (3.3), and finally

$$\mu[\chi_{1,\mu,t}(0), \chi_{2,\mu,t}(0)] := \left[\frac{\ell}{\theta_*}(\theta_* - \theta^0) - \gamma\chi_1^0, -\sigma^0 : \beta - \gamma\chi_2^0\right].$$

In particular, we observe that the left hand side above is bounded in $H \times H$, uniformly with respect to μ, and that relation (3.4) is fulfilled also for $t = 0$.

We are now in the position of stating our results.

THEOREM 4.1 (Well-posedness). *Let $\mu \geq 0$. Under the assumptions (A1)-(A3), there exists a unique solution to problem P_μ. Moreover, given two sets of data $(F_i, G_i, \theta_i^0, \chi_{1,i}^0, \chi_{2,i}^0)$ fulfilling (A1)-(A3) and two external temperatures $\theta_{e,i}$, for $i = 1,2$, the respective solutions $(\theta_i, \chi_{1,i}, \chi_{2,i}, u_i)$ to the corresponding problems P_μ fulfill*

$$\|\theta_1 - \theta_2\|^2_{L^2(0,T;H)} + \sup_{t\in[0,T]} \left\| \int_0^t \nabla(\theta_1 - \theta_2) \right\|^2 + \sup_{t\in[0,T]} \left\| \int_0^t (\theta_1 - \theta_2) \right\|^2_{L^2(\Gamma)}$$
$$+ \sum_{j=1}^{2} \left(\mu\|\chi_{j,1} - \chi_{j,2}\|^2_{C([0,T];H)} + \|\chi_{j,1} - \chi_{j,2}\|^2_{L^2(0,T;H)} \right) + \|u_1 - u_2\|^2_{L^2(0,T;\mathcal{V})}$$
$$\leq c_0 \left(\int_0^T \left\| \int_0^t (F_{1,1} - F_{1,2}) \right\|^2 dt + \|F_{2,1} - F_{2,2}\|^2_{L^1(0,T;V')} + \|G_1 - G_2\|^2_{L^2(0,T;\mathcal{V}')} \right.$$
$$\left. + \|c_s(\theta_1^0 - \theta_2^0) - \ell(\chi_{1,1}^0 - \chi_{1,2}^0)\|^2 + \mu\|\chi_{2,1}^0 - \chi_{2,2}^0\|^2 + |\theta_{e,1} - \theta_{e,2}|^2 \right), \tag{4.1}$$

where c_0 depends on c_s, k, h, ℓ, γ, Γ, $\mathbb{A}^{\frac{1}{2}}\beta$, θ_*, and $c_{\mathcal{V}}$ but is independent of μ.

THEOREM 4.2 (Dissipation asymptotics). *Under the assumptions (A1)-(A3), the solution $(\theta_\mu, \chi_{1,\mu}, \chi_{2,\mu}, u_\mu)$ to problem P_μ converges as $\mu \to 0$ to the solution $(\theta, \chi_1, \chi_2, u)$ of problem P_0 at least weakly in the respective natural spaces.*

Moreover, we will address the study of the long-time behavior of the solution to problem P_0. Indeed, the reader should notice that the above stated well-posedness result is actually independent of the choice of the reference time T. Hence, in particular, the solution $(\theta, \chi_1, \chi_2, u)$ to problem P_0 may be uniquely extended for all times. Let now the $\omega-$limit set be defined as

$$\omega(\theta,\chi_1,\chi_2,u) := \Big\{ (\theta_\infty, \chi_{1,\infty}, \chi_{2,\infty}, u_\infty) \in H \times H \times H \times \mathcal{V} \quad \text{such that there exists}$$
$$\text{a sequence of positive real numbers } \{t_n\} \text{ with } t_n \longrightarrow +\infty \text{ and}$$
$$(\theta(t_n), \chi_1(t_n), \chi_2(t_n), u(t_n)) \longrightarrow (\theta_\infty, \chi_{1,\infty}, \chi_{2,\infty}, u_\infty) \text{ in } H \times H \times H \times \mathcal{V} \Big\}.$$

To the end of establishing a long-time behavior result, we need some further assumptions on the data. In particular, we ask for

(A4) $F \in L^2(0,+\infty;H)$, $G \in H^1(0,+\infty,\mathcal{V}')$.

Hence, the following holds true.

THEOREM 4.3 (Long-time behavior). *Under assumptions (A2)-(A4), the $\omega-$limit set $\omega\big(\theta,\chi_1,\chi_2,u\big)$ reduces to the unique solution to the problem*

$$\theta_\infty = \theta_e \quad a.e.\ in\ \ \Omega, \tag{4.2}$$

$$a(u_\infty,v) + (\mathbb{A}\beta\chi_{2,\infty},\varepsilon(v)) = 0 \quad \forall v \in \mathcal{V}, \tag{4.3}$$

$$\mathbb{A}(\varepsilon(u_\infty) + \beta\chi_{2,\infty}) = \sigma_\infty \quad a.e.\ in\ \ \Omega, \tag{4.4}$$

$$\gamma\begin{pmatrix}\chi_{1,\infty}\\ \chi_{2,\infty}\end{pmatrix} + \partial I_K\begin{pmatrix}\chi_{1,\infty}\\ \chi_{2,\infty}\end{pmatrix} \ni \begin{pmatrix}\frac{\ell}{\theta_*}(\theta_* - \theta_e)\\ -\sigma_\infty : \beta\end{pmatrix} \quad a.e.\ in\ \ \Omega. \tag{4.5}$$

Namely, the whole trajectory converges to $(\theta_\infty,\chi_{1,\infty},\chi_{2,\infty},u_\infty)$ as $t \to +\infty$.

We close this section by stating precisely a maximum principle for the temperature θ which entails its positivity in the frame of our concrete modeling situation. To this aim, of course some sign assumption on the external heat sources is needed and we will ask for

(A5) $\langle F(t),v\rangle \geq 0$ for a.e. $t \in (0,T)$ and all $v \in V$ with $v \geq 0$ a.e. in Ω.

The latter follows for instance when $f,\, r \geq 0$ almost everywhere in their respective domains and could clearly be weakened. One has the following.

THEOREM 4.4 (Lower bound). *Let $\mu \geq 0$. Under assumptions (2.1), (A1)-(A3), and (A5), let $\theta_d \in \mathbb{R}$ be such that*

$$\gamma + \frac{\ell}{\theta_*}(\theta_d - \theta_*) \leq 0. \tag{4.6}$$

Then, the unique solution $(\theta,[\chi_1,\chi_2],u)$ to Problem P_μ fulfills

$$\inf\{\inf\theta_0,\, \theta_e,\, \theta_*,\theta_d\} \leq \theta(x,t) \quad for\ a.e.\ \ (x,t) \in Q, \tag{4.7}$$

where $\inf\theta_0$ stands for the essential infimum of θ_0 on Ω.

Clearly, in order to deduce from the above stated lower bound (4.7) a positivity result for θ (and consequently a proof of the thermodynamical consistency of the model, see Section 2) one has to start from a positive initial datum θ_0 and ask for the existence of a positive constant θ_d fulfilling (4.6). Let us stress that this second requirement is compatible with our modeling situation since, owing to the discussion of Section 2,

$$\gamma + \frac{\ell}{\theta_*}(\theta_d - \theta_*) = \frac{\ell}{\theta_*}(\theta_* - \theta_{**}) + \frac{\ell}{\theta_*}(\theta_d - \theta_*) = \frac{\ell}{\theta_*}(\theta_d - \theta_{**}),$$

and it suffices to choose $0 < \theta_d \leq \theta_{**}$ in order to achieve (4.6).

5. Dissipative problem. Throughout this section, the dissipation parameter μ is fixed and strictly positive.

5.1. Continuous dependence. Let us denote by $(\theta_i,\chi_{1,i},\chi_{2,i},u_i)$ for $i = 1,2$, two solutions to problem P_μ associated to the given two sets of data $(F_i,G_i,\theta_i^0,\chi_{1,i}^0,$ $\chi_{2,i}^0)$, and $\theta_{e,i}$, for $i = 1,2$. We set $\overline{\theta} := \theta_1 - \theta_2$, $\overline{u} := u_1 - u_2$ and so on. Let us take the integral on $(0,t)$, for $t \in (0,T]$, of relation (3.1) written for $(\theta_1,\chi_{1,1},\chi_{2,1},u_1)$

and subtract the same relation for $(\theta_2, \chi_{1,2}, \chi_{2,2}, u_2)$, choose $\varphi := \overline{\theta}$, and integrate on $(0,t)$, for $t \in (0,T]$. We readily obtain that

$$c_s \int_0^t \|\overline{\theta}\|^2 + \frac{k}{2} \left\| \int_0^t \nabla \overline{\theta} \right\|^2 + \frac{h}{2} \left\| \int_0^t \overline{\theta} \right\|^2_{L^2(\Gamma)} \leq \int_0^t \left(\|c_s \overline{\theta}^0 - \ell \overline{\chi}_1^0\| + \left\| \int_0^s \overline{F}_1 \right\| \right) \|\overline{\theta}\| \, ds$$
$$+ \int_0^t \left\langle \int_0^s \overline{F}_2, \overline{\theta} \right\rangle ds + h \int_0^t (s\overline{\theta}_e, \overline{\theta})_\Gamma ds + \ell \int_0^t (\overline{\theta}, \overline{\chi}_1). \tag{5.1}$$

Let us now take the difference between relation (3.4) written for $(\theta_1, \chi_{1,1}, \chi_{2,1}, u_1)$ and the same relation for $(\theta_2, \chi_{1,2}, \chi_{2,2}, u_2)$, multiply the corresponding relation by $[\overline{\chi}_1, \overline{\chi}_2]$, exploit the monotonicity of the subdifferential and integrate on $\Omega \times (0,t)$, for $t \in (0,T]$. We get

$$\sum_{j=1}^2 \left(\frac{\mu}{2} \|\overline{\chi}_j(t)\|^2 + \gamma \int_0^t \|\overline{\chi}_j\|^2 \right) \leq \sum_{j=1}^2 \frac{\mu}{2} \|\overline{\chi}_j^0\|^2 - \frac{\ell}{\theta_*} \int_0^t (\overline{\theta}, \overline{\chi}_1) - \int_0^t (\overline{\sigma} : \beta, \overline{\chi}_2). \tag{5.2}$$

As for the last term in the above right hand side we take advantage of (3.2)-(3.3) and readily compute that

$$-\overline{\sigma} : \beta = -\mathbb{A}^{\frac{1}{2}} \varepsilon(\overline{u}) : \mathbb{A}^{\frac{1}{2}} \beta - |\mathbb{A}^{\frac{1}{2}} \beta|^2 \overline{\chi}_2, \tag{5.3}$$

$$\int_0^t \|\mathbb{A}^{\frac{1}{2}} \varepsilon(\overline{u})\|^2 = - \int_0^t (\mathbb{A}^{\frac{1}{2}} \varepsilon(\overline{u}) : \mathbb{A}^{\frac{1}{2}} \beta, \overline{\chi}_2) + \int_0^t \langle \overline{G}, \overline{u} \rangle. \tag{5.4}$$

Hence, by choosing $1 < \rho < (\gamma + |\mathbb{A}^{\frac{1}{2}} \beta|^2)/|\mathbb{A}^{\frac{1}{2}} \beta|^2$, we take the sum between (5.2) and (5.4) in order to obtain

$$\sum_{j=1}^2 \left(\frac{\mu}{2} \left(\|\overline{\chi}_j(t)\|^2 - \|\overline{\chi}_j^0\|^2 \right) + \gamma \int_0^t \|\overline{\chi}_j\|^2 \right) + \int_0^t \|\mathbb{A}^{\frac{1}{2}} \varepsilon(\overline{u})\|^2$$
$$\leq -\frac{\ell}{\theta_*} \int_0^t (\overline{\theta}, \overline{\chi}_1) - 2 \int_0^t (\mathbb{A}^{\frac{1}{2}} \varepsilon(\overline{u}) : \mathbb{A}^{\frac{1}{2}} \beta, \overline{\chi}_2) - |\mathbb{A}^{\frac{1}{2}} \beta|^2 \int_0^t \|\overline{\chi}_2\|^2 + \int_0^t \langle \overline{G}, \overline{u} \rangle$$
$$\leq -\frac{\ell}{\theta_*} \int_0^t (\overline{\theta}, \overline{\chi}_1) + (\rho - 1) |\mathbb{A}^{\frac{1}{2}} \beta|^2 \int_0^t \|\overline{\chi}_2\|^2 + \frac{1}{\rho} \int_0^t \|\mathbb{A}^{\frac{1}{2}} \varepsilon(\overline{u})\|^2 + \int_0^t \langle \overline{G}, \overline{u} \rangle.$$

Then, multiplying (5.1) by $1/\theta_*$ and taking the sum with the above relation, one has

$$\frac{c_s}{\theta_*} \int_0^t \|\overline{\theta}\|^2 + \frac{k}{2\theta_*} \left\| \int_0^t \nabla \overline{\theta} \right\|^2 + \frac{h}{2\theta_*} \left\| \int_0^t \overline{\theta} \right\|^2_{L^2(\Gamma)} + \frac{\mu}{2} \sum_{j=1}^2 \|\overline{\chi}_j(t)\|^2$$
$$+ \gamma \int_0^t \|\overline{\chi}_1\|^2 + \left(\gamma + (1 - \rho) |\mathbb{A}^{\frac{1}{2}} \beta|^2 \right) \int_0^t \|\overline{\chi}_2\|^2 + \frac{\rho - 1}{\rho} \int_0^t \|\mathbb{A}^{\frac{1}{2}} \varepsilon(\overline{u})\|^2$$
$$\leq \frac{1}{\theta_*} \int_0^t \left(\|c_s \overline{\theta}^0 - \ell \overline{\chi}_1^0\| + \left\| \int_0^s \overline{F}_1 \, ds \right\| \right) \|\overline{\theta}\| + \frac{1}{\theta_*} \int_0^t \left\langle \int_0^s \overline{F}_2 \, ds, \overline{\theta} \right\rangle$$
$$+ \frac{h}{\theta_*} \int_0^t (s \overline{\theta}_e, \overline{\theta})_\Gamma ds + \sum_{j=1}^2 \frac{\mu}{2} \|\overline{\chi}_j^0\|^2 + \int_0^t \langle \overline{G}, \overline{u} \rangle. \tag{5.5}$$

Finally, the assertion follows from an integration by parts.

5.2. Discretization. Let us introduce our variable time-step discretization of P_μ. To this aim we define the partition $\mathcal{P} := \{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\}$ with variable time-step $\tau_i := t_i - t_{i-1}$ and let $\tau := \max_{1\leq i \leq N} \tau_i$ denote the diameter of the partition $\mathcal{P}$. In the forthcoming analysis the following notation will be extensively used: being $\{w_i\}_{i=0}^N$ a vector, we denote by $w_{\mathcal{P}}$ and $\overline{w}_{\mathcal{P}}$ two functions of the time interval $[0,T]$ which interpolate the values of the vector $\{w_i\}$ piecewise linearly and backward constantly on the partition $\mathcal{P}$, respectively. Namely

$$w_{\mathcal{P}}(0) := w_0, \quad w_{\mathcal{P}}(t) := g_i(t) w_i + \big(1 - g_i(t)\big) w_{i-1},$$
$$\overline{w}_{\mathcal{P}}(0) := w_0, \quad \overline{w}_{\mathcal{P}}(t) := w_i \quad \text{for} \quad t \in (t_{i-1}, t_i], \quad i = 1, \ldots, N$$

where $g_i(t) := (t - t_{i-1})/\tau_i$ for $t \in (t_{i-1}, t_i]$, $i = 1, \ldots, N$. Moreover, given a vector $\{w_i\}_{i=0}^N$, we define another vector $\{\delta w_i\}_{i=1}^N$ as $\delta w_i := (w_i - w_{i-1})/\tau_i$.

Finally, we introduce some approximation of the data. Hence let $F = F_1 + F_2$ where $F_1 \in L^2(0,T;H)$ and $F_2 \in W^{1,1}(0,T;V')$ and set

$$F_{1,i} := \frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} F_1(s)\, ds \ \in H \quad \text{for} \quad i = 1, \ldots, N, \tag{5.6}$$
$$F_{2,1} := F_2(t_i) \ \in V' \quad \text{for} \quad i = 0, 1, \ldots, N, \tag{5.7}$$
$$G_i := G(t_i) \ \in \mathcal{V}' \quad \text{for} \quad i = 0, 1, \ldots, N. \tag{5.8}$$

Of course, owing to (A1), the latter positions are justified. In particular, let us remark that one has

$$\overline{F}_{1,\mathcal{P}} \to F_1 \quad \text{strongly in} \quad L^2(0,T;H), \tag{5.9}$$
$$F_{2,\mathcal{P}} \to F_2 \quad \text{strongly in} \quad W^{1,1}(0,T;V'), \tag{5.10}$$
$$G_{\mathcal{P}} \to G \quad \text{strongly in} \quad H^1(0,T;\mathcal{V}'), \tag{5.11}$$

whenever the diameter τ of partition $\mathcal{P}$ goes to 0.

Hence, we are interested in the following discrete problem.

Problem D_μ: To find $\{\theta_i\}_{i=0}^N \in V^{N+1}$, $\{[\chi_{1,i}, \chi_{2,i}]\}_{i=0}^N \in \mathcal{K}^{N+1}$, and $\{u_i\}_{i=1}^N \in \mathcal{V}^N$ such that, for all $i = 1, \ldots, N$,

$$\big((c_s \delta\theta_i - \ell \delta\chi_{1,i}), \varphi\big) + k(\nabla\theta_i, \nabla\varphi) + h(\theta_i - \theta_e, \varphi)_\Gamma = \langle F_i, \varphi \rangle \quad \forall \varphi \in V, \tag{5.12}$$
$$a(u_i, v) + (\mathbb{A}\beta\chi_{2,i}, \varepsilon(v)) = \langle G_i, v \rangle \quad \forall v \in \mathcal{V}, \tag{5.13}$$
$$\mathbb{A}(\varepsilon(u_i) + \beta\chi_{2,i}) = \sigma_i \quad \text{a.e. in} \quad \Omega, \tag{5.14}$$
$$\mu \begin{pmatrix} \delta\chi_{1,i} \\ \delta\chi_{2,i} \end{pmatrix} + \gamma \begin{pmatrix} \chi_{1,i} \\ \chi_{2,i} \end{pmatrix} + \begin{pmatrix} \frac{\ell}{\theta_*}(\theta_i - \theta_*) \\ \sigma_i : \beta \end{pmatrix} + \partial I_K \begin{pmatrix} \chi_{1,i} \\ \chi_{2,i} \end{pmatrix} \ni \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{a.e. in} \quad \Omega, \tag{5.15}$$
$$\theta_0 = \theta^0 \quad \text{a.e. in} \quad \Omega, \tag{5.16}$$
$$\mu[\chi_{1,0}, \chi_{2,0}] = \mu[\chi_1^0, \chi_2^0] \quad \text{a.e. in} \quad \Omega. \tag{5.17}$$

5.3. Discrete well-posedness. We prove the following lemma.

LEMMA 5.1. *Under the assumptions (A1)-(A3), (5.6)-(5.8), and for all τ sufficiently small, problem D_μ has a unique solution.*

Proof. We proceed by induction. Namely, we assume to know the solution of the problem up to level $i-1$ and solve for level i. In particular, we are concerned with the problem of finding $\theta_i \in V$, $[\chi_{1,i}, \chi_{2,i}] \in \mathcal{K}$, and $u_i \in \mathcal{V}$ such that

$$\big((c_s\theta_i - \ell\chi_{1,i}, \varphi) + \tau_i k(\nabla\theta_i, \nabla\varphi) + \tau_i h(\theta_i - \theta_e, \varphi)_\Gamma = \langle F_i^*, \varphi\rangle \quad \forall\varphi \in V, \tag{5.18}$$

$$a(u_i, v) + (\mathbb{A}\beta\chi_{2,i}, \varepsilon(v)) = \langle G_i, v\rangle \quad \forall v \in \mathcal{V}, \tag{5.19}$$

$$\mathbb{A}(\varepsilon(u_i) + \beta\chi_{2,i}) = \sigma_i \quad \text{a.e. in } \Omega, \tag{5.20}$$

$$\mu\begin{pmatrix}\chi_{1,i}\\ \chi_{2,i}\end{pmatrix} + \tau_i\begin{pmatrix}\gamma\chi_{1,i} + \frac{\ell}{\theta_*}(\theta_i - \theta_*)\\ \gamma\chi_{2,i} + \sigma_i : \beta\end{pmatrix} + \partial I_K\begin{pmatrix}\chi_{1,i}\\ \chi_{2,i}\end{pmatrix} \ni \mu\begin{pmatrix}\chi_{1,i-1}\\ \chi_{2,i-1}\end{pmatrix} \quad \text{a.e. in } \Omega, \tag{5.21}$$

where we collected in the right hand sides of (5.18)-(5.19) and (5.21) the quantities known at level i and let $F_i^* := \tau_i F_i + c_s\theta_{i-1} - \ell\chi_{1,i-1}$. We shall stress that the latter scheme is of course fully implicit.

Let us now fix $[\tilde\chi_1, \tilde\chi_2] \in \mathcal{K}$. It is then straightforward to find the unique solutions $\theta \in V$ and $u \in \mathcal{V}$ to (5.18) with $\tilde\chi_1$ instead of $\chi_{1,i}$ and (5.19) with $\tilde\chi_2$ instead of $\chi_{2,i}$, respectively. Hence, we have implicitly defined a mapping $T_1 : \mathcal{K} \to V \times \mathcal{V}$ as $T_1[\tilde\chi_1, \tilde\chi_2] := [\theta, u]$. On the other hand, for all $(\tilde\theta, \tilde u, \tilde\chi_2) \in V \times \mathcal{V} \times H$ there exists a unique pair $[\chi_1, \chi_2] \in \mathcal{K}$ solving relation (5.21) with $(\tilde\theta, \tilde u)$ instead of (θ_i, u_i) and σ_i is defined by (5.20) with $\tilde\chi_2$ instead of $\chi_{2,i}$. Thus, one may define a mapping $T_2 : V \times \mathcal{V} \times H \to \mathcal{K}$ as $T_2(\tilde\theta, \tilde u, \tilde\chi_2) = [\chi_1, \chi_2]$.

Our next aim is to prove that, for sufficiently small τ, the mapping $T_3 : \mathcal{K} \to \mathcal{K}$ defined as $T_3[\tilde\chi_1, \tilde\chi_2] := T_2(T_1[\tilde\chi_1, \tilde\chi_2], \tilde\chi_2)$ is a contraction in $H \times H$. To this end let $[\tilde\chi_{1,j}, \tilde\chi_{2,j}] \in \mathcal{K}$, $[\theta_j, u_j] = T_1[\tilde\chi_{1,j}, \tilde\chi_{2,j}]$, $[\chi_{1,j}, \chi_{2,j}] := T_3[\tilde\chi_{1,j}, \tilde\chi_{2,j}]$ for $j = 1, 2$, and define $\overline\theta := \theta_1 - \theta_2$, $\overline u := u_1 - u_2$ etc. Hence, we readily check that

$$(c_s\overline\theta, \varphi) + \tau_i(\nabla\overline\theta, \nabla\varphi) + \tau_i h(\overline\theta, \varphi)_\Gamma = (\ell\overline{\tilde\chi}_1, \varphi) \quad \forall\varphi \in V,$$
$$a(\overline u, v) + (\mathbb{A}\beta\overline{\tilde\chi}_2, \varepsilon(v)) = 0 \quad \forall v \in \mathcal{V},$$
$$\sigma : \beta = \mathbb{A}\varepsilon(\overline u) : \beta + \mathbb{A}\beta\overline{\tilde\chi}_2 : \beta.$$

Namely, by choosing $[\varphi, v] = [\overline\theta, \overline u]$ above one gets that

$$\|\overline\theta\| \le \frac{\ell}{c_s}\|\overline{\tilde\chi}_1\|, \quad \|\mathbb{A}^{\frac12}\varepsilon(\overline u)\| \le c_1\|\overline{\tilde\chi}_2\|, \quad \|\overline\sigma : \beta\| \le c_1\|\overline{\tilde\chi}_2\|, \tag{5.22}$$

where c_1 just depends on $\mathbb{A}^{\frac12}\beta$. On the other hand, we exploit (5.21) and get that

$$\begin{pmatrix}\chi_{1,j}\\ \chi_{2,j}\end{pmatrix} = (1 + \partial I_K)^{-1}\left(\frac{\tau_i}{\mu + \tau_i\gamma}\right)\begin{pmatrix}\frac{\ell}{\theta_*}(\theta_* - \theta_j) + \mu\chi_{1,i-1}/\tau_i\\ -\sigma_j : \beta + \mu\chi_{2,i-1}/\tau_i\end{pmatrix} \quad \text{for } j = 1, 2.$$

In particular, also using (5.22), one computes that

$$\begin{aligned}\|\overline\chi_1\|^2 + \|\overline\chi_2\|^2 &\le \left(\frac{\tau_i}{\mu + \tau_i\gamma}\right)^2\left(\left\|\frac{\ell}{\theta_*}\overline\theta\right\|^2 + \|\overline\sigma : \beta\|^2\right)\\ &\le \left(\frac{\tau_i}{\mu + \tau_i\gamma}\right)^2\left(\frac{\ell^4}{c_s^2\theta_*^2}\|\overline{\tilde\chi}_1\|^2 + c_1^2\|\overline{\tilde\chi}_2\|^2\right).\end{aligned}$$

Finally it suffices to fix $\tau \le \mu/\max\{\ell^2/(c_s\theta_*), c_1\}$ in order to get that T_3 is actually a contraction in $H \times H$. The assertion follows from the fact that $T_3(H) \subset \mathcal{K}$ which is closed in $H \times H$. □

For the sake of later convenience, we rewrite the scheme (5.12)-(5.15) in more compact form as

$$\big(\partial_t(c_s\theta_{\mathcal{P}} - \ell\chi_{1,\mathcal{P}}),\varphi\big) + k(\nabla\overline{\theta}_{\mathcal{P}},\nabla\varphi) + h(\overline{\theta}_{\mathcal{P}} - \theta_e,\varphi)_\Gamma$$
$$= \langle \overline{F}_{\mathcal{P}},\varphi\rangle \quad \forall \varphi\in V, \ \text{ a.e. in } \ (0,T), \tag{5.23}$$
$$a(\overline{u}_{\mathcal{P}},v) + (\mathbb{A}\beta\overline{\chi}_{2,\mathcal{P}},\varepsilon(v)) = \langle \overline{G}_{\mathcal{P}},v\rangle \quad \forall v\in\mathcal{V}, \ \text{ a.e. in } \ (0,T), \tag{5.24}$$
$$\mathbb{A}(\varepsilon(\overline{u}_{\mathcal{P}}) + \beta\overline{\chi}_{2,\mathcal{P}}) = \overline{\sigma}_{\mathcal{P}} \quad \text{a.e. in } \ Q, \tag{5.25}$$
$$\mu\partial_t\begin{pmatrix}\chi_{1,\mathcal{P}}\\ \chi_{2,\mathcal{P}}\end{pmatrix} + \gamma\begin{pmatrix}\overline{\chi}_{1,\mathcal{P}}\\ \overline{\chi}_{2,\mathcal{P}}\end{pmatrix} + \partial I_K\begin{pmatrix}\overline{\chi}_{1,\mathcal{P}}\\ \overline{\chi}_{2,\mathcal{P}}\end{pmatrix} \ni \begin{pmatrix}\frac{\ell}{\theta_*}(\theta_* - \overline{\theta}_{\mathcal{P}})\\ -\overline{\sigma}_{\mathcal{P}}:\beta\end{pmatrix} \quad \text{a.e. in } \ Q. \tag{5.26}$$

REMARK 5.2. In order to completely justify the above notations one could consider, for instance, $u_0 := u^0$ and $\sigma_0 := \sigma^0$ where u^0 and σ^0 are defined above.

5.4. Stability. Our approximation scheme fulfills some suitable *conditional stability* property. In particular, this subsection brings to the proof of the following lemma.

LEMMA 5.3. *Under the assumptions (A1)-(A3), (5.6)-(5.8), and for all τ sufficiently small, let $\{\theta_i\}_{i=0}^N \in V^{N+1}$, $\{[\chi_{1,i},\chi_{2,i}]\}_{i=0}^N \in \mathcal{K}^{N+1}$, and $\{u_i\}_{i=1}^N \in \mathcal{V}^N$ be the unique solution to problem D_μ. Then there exists a positive constant c_2 just depending on c_s, k, h, θ_e, Γ, θ_*, γ, $\mathbb{A}^{\frac12}\beta$, $c_{\mathcal{V}}$, θ^0, $[\chi_1^0,\chi_2^0]$, $\|F_1\|_{L^2(0,T;H)}$, $\|F_2\|_{W^{1,1}(0,T;V')}$, and $\|G\|_{H^1(0,T;\mathcal{V}')}$ such that*

$$\|\theta_{\mathcal{P}}\|_{H^1(0,T;H)\cap C^0([0,T];V)} + \|[\chi_{1,\mathcal{P}},\chi_{2,\mathcal{P}}]\|_{(H^1(0,T;H))^2}$$
$$+\sqrt{\mu}\|[\chi_{1,\mathcal{P}},\chi_{2,\mathcal{P}}]\|_{(W^{1,\infty}(0,T;H))^2} + \|u\|_{H^1(0,T;\mathcal{V})} \le c_2. \tag{5.27}$$

In particular, c_2 is independent of μ and τ.

Proof. Henceforth we will denote by c any positive constant, possibly depending on data but neither on μ nor $\mathcal{P}$. In particular, c may vary from line to line.

Let us take the difference between relation (5.15) written at level i and the same relation at level $i-1$. By defining $[\delta\chi_{1,0},\delta\chi_{2,0}] := [\chi_{1,\mu,t}(0),\chi_{2,\mu,t}(0)]$ we are entitled to do so for $i=1,\dots,N$. Next, we multiply the resulting relation by $[\delta\chi_{1,i},\delta\chi_{2,i}]$, integrate in space and sum for $i=1,\dots,m$ for some $m=1,\dots,N$. By exploiting the monotonicity of the subdifferential, we obtain that

$$\sum_{j=1}^2\left(\frac{\mu}{2}\|\delta\chi_{j,m}\|^2 - \frac{\mu}{2}\|\delta\chi_j(0)\|^2 + \gamma\sum_{i=1}^m \tau_i\|\delta\chi_{j,i}\|^2\right)$$
$$\le -\frac{\ell}{\theta_*}\sum_{i=1}^m \tau_i(\delta\theta_i,\delta\chi_{1,i}) - \sum_{i=1}^m \tau_i(\delta\sigma_i:\beta,\delta\chi_{2,i}). \tag{5.28}$$

We now take the difference between relation (5.13) written at level i and the same relation at level $i-1$, choose $v := \delta u_i$, and sum for $i=1,\dots,m$. One readily gets that

$$\sum_{i=1}^m \tau_i\|\mathbb{A}^{\frac12}\varepsilon(\delta u_i)\|^2 = \sum_{i=1}^m \tau_i\langle\delta G_i,\delta u_i\rangle - \sum_{i=1}^m \tau_i(\mathbb{A}^{\frac12}\varepsilon(\delta u_i):\mathbb{A}^{\frac12}\beta,\delta\chi_{2,i}). \tag{5.29}$$

On the other hand, owing to (5.14), it may be easily computed that

$$\delta\sigma_i:\beta = \mathbb{A}^{\frac12}\varepsilon(\delta u_i):\mathbb{A}^{\frac12}\beta + |\mathbb{A}^{\frac12}\beta|^2\delta\chi_{2,i} \quad \forall i=1,\dots,N. \tag{5.30}$$

Hence, taking into account (5.30) and adding (5.28) to (5.29), we may again choose a suitable ρ such that $1<\rho<(\gamma+|\mathbb{A}^{\frac{1}{2}}\beta|^2)/|\mathbb{A}^{\frac{1}{2}}\beta|^2$ and deduce that

$$\begin{aligned}&\sum_{j=1}^{2}\left(\frac{\mu}{2}\|\delta\chi_{j,m}\|^2-\frac{\mu}{2}\|\delta\chi_j(0)\|^2\right)\\&+\sum_{i=1}^{m}\tau_i\left(\gamma\|\delta\chi_{1,i}\|^2+(\gamma+|\mathbb{A}^{\frac{1}{2}}\beta|^2-\rho|\mathbb{A}^{\frac{1}{2}}\beta|^2)\|\delta\chi_{2,i}\|^2\right)+\frac{\rho-1}{\rho}\sum_{i=1}^{m}\tau_i\|\mathbb{A}^{\frac{1}{2}}\varepsilon(\delta u_i)\|^2\\&\leq-\frac{\ell}{\theta_*}\sum_{i=1}^{m}\tau_i(\delta\theta_i,\delta\chi_{1,i})+\sum_{i=1}^{m}\tau_i\langle\delta G_i,\delta u_i\rangle.\end{aligned}\tag{5.31}$$

Next, we test relation (5.12) by $\varphi=\tau_i\delta\theta_i$ and take the sum for $i=1,\ldots,m$. Due to (A2) we obtain that

$$\begin{aligned}&c_s\sum_{i=1}^{m}\tau_i\|\delta\theta_i\|^2+\frac{k}{2}\|\nabla\theta_m\|^2+\frac{h}{2}\|\theta_m\|^2_{L^2(\Gamma)}\leq c+h(\theta_e,\theta_m)_\Gamma+\ell\sum_{i=1}^{m}\tau_i(\delta\theta_i,\delta\chi_{1,i})\\&+\sum_{i=1}^{m}\tau_i(F_{1,i},\delta\theta_i)+\langle F_{2,m},\theta_m\rangle-\langle F_{2,1},\theta^0\rangle-\sum_{i=2}^{m}\langle F_{2,i}-F_{2,i-1},\theta_{i-1}\rangle.\end{aligned}\tag{5.32}$$

Finally, it suffices to take the sum between (5.31) and (5.32) multiplied by $1/\theta_*$, consider (A3), and perform some standard computations in order to obtain that

$$\begin{aligned}&\mu\sum_{j=1}^{2}\|\delta\chi_{j,m}\|^2+\|\theta_m\|_V^2+\sum_{i=1}^{m}\tau_i\left(\sum_{j=1}^{2}\|\delta\chi_{j,i}\|^2+\|\delta\theta_i\|^2+\|\delta u_i\|^2_{\mathcal{V}}\right)\\&\leq c\sum_{i=2}^{m-1}\tau_i\langle\delta F_{2,i},\theta_{i-1}\rangle\\&+c\left(1+\|F_{2,m}\|^2_{V'}+\|F_{2,1}\|^2_{V'}+\sum_{i=1}^{m}\tau_i\left(\|\delta_i G\|^2_{\mathcal{V}'}+\|F_{1,i}\|^2\right)\right).\end{aligned}\tag{5.33}$$

Finally, from (5.6) and (5.8) and an application of the discrete Gronwall lemma we readily obtain the bounds of (5.27). □

5.5. Convergence. Let us now consider the limit as the diameter τ of partition $\mathcal{P}$ goes to zero. We are actually in the position of proving the following.

LEMMA 5.4. *Under the assumptions (A1)-(A3), (5.6)-(5.8) and for all τ sufficiently small, let $\{\theta_i\}_{i=0}^N\in V^{N+1}$, $\{[\chi_{1,i},\chi_{2,i}]\}_{i=0}^N\in\mathcal{K}^{N+1}$, and $\{u_i\}_{i=1}^N\in\mathcal{V}^N$ be* *the unique solution to problem D_μ. Then the following convergences hold*

$$\theta_{\mathcal{P}} \longrightarrow \theta_\mu \quad \textit{weakly star in} \quad H^1(0,T;H) \cap L^\infty(0,T;V) \quad \textit{and strongly in} \quad C([0,T];H), \tag{5.34}$$

$$\overline{\theta}_{\mathcal{P}} \longrightarrow \theta_\mu \quad \textit{weakly star in} \quad L^\infty(0,T;V) \quad \textit{and strongly in} \quad L^\infty(0,T;H), \tag{5.35}$$

$$\chi_{j,\mathcal{P}} \longrightarrow \chi_{j,\mu} \quad \textit{weakly star in} \quad W^{1,\infty}(0,T;H), \quad \textit{and strongly in} \quad C([0,T];H), \quad j=1,2, \tag{5.36}$$

$$\overline{\chi}_{j,\mathcal{P}} \longrightarrow \chi_{j,\mu} \quad \textit{strongly in} \quad L^\infty(0,T;H), \quad j=1,2, \tag{5.37}$$

$$u_{\mathcal{P}} \longrightarrow u_\mu \quad \textit{weakly in} \quad H^1(0,T;\mathcal{V}) \quad \textit{and strongly in} \quad C([0,T];\mathcal{V}), \tag{5.38}$$

$$\overline{u}_{\mathcal{P}} \longrightarrow u_\mu \quad \textit{strongly in} \quad L^\infty(0,T;\mathcal{V}), \tag{5.39}$$

where $(\theta_\mu, \chi_{1,\mu}, \chi_{2,\mu}, u_\mu)$ is the unique solution to problem P_μ.

In particular, let us stress that the latter lemma entails the proof of the existence statement of Theorem 4.1.

Proof. Taking into account the estimate (5.27) and well known compactness results, we readily find a quadruple $(\theta_\mu, \chi_{1,\mu}, \chi_{2,\mu}, u_\mu)$ such that, possibly taking not relabeled subsequences, the weak and weak-star convergences of Lemma 5.4 hold true together with the following

$$\theta_{\mathcal{P}} \longrightarrow \theta_\mu \quad \text{strongly in} \quad C([0,T];H), \tag{5.40}$$

$$\overline{\theta}_{\mathcal{P}} \longrightarrow \theta_\mu \quad \text{strongly in} \quad L^\infty(0,T;H). \tag{5.41}$$

We now turn to the proof of some strong convergence for $\chi_{1,\mathcal{P}}, \chi_{2,\mathcal{P}}$, and $u_{\mathcal{P}}$ by a direct Cauchy argument. To this aim, let $\mathcal{P}_m$ denote the extracted sequence of partitions. We denote by $\theta_m := \theta_{\mathcal{P}_m}$, $u_m := u_{\mathcal{P}_m}$ etc. By taking the difference between (5.26) written for $\mathcal{P}_n$ and the same relation for $\mathcal{P}_m$, multiply it by $[\overline{\chi}_{1,n} - \overline{\chi}_{1,m}, \overline{\chi}_{2,n} - \overline{\chi}_{2,m}]$, integrate on Ω, and exploit the monotonicity of the subdifferential we readily obtain that

$$\frac{\mu}{2}\frac{d}{dt}\sum_{j=1}^{2} \|(\chi_{j,n} - \chi_{j,m})(t)\|^2 + \gamma \sum_{j=1}^{2} \|(\overline{\chi}_{j,n} - \overline{\chi}_{j,m})(t)\|^2$$
$$\leq \frac{\ell}{\theta_*}\|(\overline{\theta}_n - \overline{\theta}_m)(t)\| \, \|(\overline{\chi}_{1,n} - \overline{\chi}_{1,m})(t)\| - \big((\overline{\sigma}_n - \overline{\sigma}_m)(t) : \beta, (\overline{\chi}_{2,n} - \overline{\chi}_{2,m})(t)\big), \tag{5.42}$$

for almost every $t \in (0,T)$. In particular, we made a crucial use of the fact that, given any vector $\{w_i\}_{i=0}^N \in H^{N+1}$, one has that $(w'_{\mathcal{P}}, \overline{w}_{\mathcal{P}}) \geq (w'_{\mathcal{P}}, w_{\mathcal{P}})$ since of course the residual term $(w'_{\mathcal{P}}, \overline{w}_{\mathcal{P}} - w_{\mathcal{P}})$ is non-negative. On the other hand, by taking the difference of the corresponding relations (5.24) with $v = \overline{u}_n - \overline{u}_m$ and of (5.25) we readily check that

$$\|\mathbb{A}^{\frac12}\varepsilon(\overline{u}_n - \overline{u}_m)\|^2 = -\big(\mathbb{A}^{\frac12}\varepsilon(\overline{u}_n - \overline{u}_m) : \mathbb{A}^{\frac12}\beta, (\overline{\chi}_{2,n} - \overline{\chi}_{2,m})\big) + \langle \overline{G}_n - \overline{G}_m, \overline{u}_n - \overline{u}_m \rangle, \tag{5.43}$$

$$-\big((\overline{\sigma}_n - \overline{\sigma}_m) : \beta, \overline{\chi}_{2,n} - \overline{\chi}_{2,m}\big) = -(\mathbb{A}^{\frac12}\varepsilon(\overline{u}_n - \overline{u}_m) : \mathbb{A}^{\frac12}\beta, \overline{\chi}_{2,n} - \overline{\chi}_{2,m}) - |\mathbb{A}^{\frac12}\beta|^2 \|\overline{\chi}_{2,n} - \overline{\chi}_{2,m}\|^2. \tag{5.44}$$

Hence, owing to the latter relation, taking the sum of (5.42) and (5.43), and integrating on $(0,t)$ for some $t \in (0,T]$ we easily infer that

$$\sum_{j=1}^{2}\left(\frac{\mu}{2}\|(\chi_{j,n}-\chi_{j,m})(t)\|^2+\gamma\int_0^t\|\overline{\chi}_{j,n}-\overline{\chi}_{j\,m}\|^2\right)+\int_0^t\|\mathbb{A}^{\frac12}\varepsilon(\overline{u}_n-\overline{u}_m)\|^2$$
$$\le\frac{\ell}{\theta_*}\int_0^t\|\overline{\theta}_n-\overline{\theta}_m\|\,\|\overline{\chi}_{1,n}-\overline{\chi}_{1,m}\|-2\int_0^t(\mathbb{A}^{\frac12}\varepsilon(\overline{u}_n-\overline{u}_m):\mathbb{A}^{\frac12}\beta,\overline{\chi}_{2,n}-\overline{\chi}_{2,m})$$
$$-|\mathbb{A}^{\frac12}\beta|^2\int_0^t\|\overline{\chi}_{2,n}-\overline{\chi}_{2,m}\|^2+\int_0^t\langle\overline{G}_n-\overline{G}_m,\overline{u}_n-\overline{u}_m\rangle$$
$$\le\frac{\gamma}{2}\int_0^t\|\overline{\chi}_{1,n}-\overline{\chi}_{1,m}\|^2+(\rho-1)|\mathbb{A}^{\frac12}\beta|^2\int_0^t\|\overline{\chi}_{2,n}-\overline{\chi}_{2,m}\|^2+\frac{1}{\rho}\int_0^t\|\mathbb{A}^{\frac12}\varepsilon(\overline{u}_n-\overline{u}_m)\|^2$$
$$+\frac{\ell^2}{2\theta_*^2\gamma}\int_0^t\|\overline{\theta}_n-\overline{\theta}_m\|^2+\int_0^t\langle\overline{G}_n-\overline{G}_m,\overline{u}_n-\overline{u}_m\rangle,$$

for some $1<\rho<(\gamma+|\mathbb{A}^{\frac12}\beta|^2)/|\mathbb{A}^{\frac12}\beta|^2$. Hence, in particular,

$$\sum_{j=1}^{2}\left(\mu\|\chi_{j,n}-\chi_{j,m}\|^2_{C([0,t];H)}+\int_0^t\|\overline{\chi}_{j,n}-\overline{\chi}_{j\,m}\|^2\right)+\int_0^t\|\overline{u}_n-\overline{u}_m\|^2_{\mathcal{V}}$$
$$\le c\left(\int_0^t\|\overline{\theta}_n-\overline{\theta}_m\|^2+\int_0^t\|\overline{G}_n-\overline{G}_m\|^2_{\mathcal{V}'}\right),\tag{5.45}$$

where c depends on ℓ, θ_*, γ and $c_{\mathcal{V}}$. Finally it suffices to exploit (5.11) and (5.41) in order to obtain that $[\chi_{1,\mathcal{P}},\chi_{2,\mathcal{P}}]$ is a Cauchy sequence in $(C([0,T];H))^2$. Namely, we checked the strong convergences in (5.36)-(5.37). The strong convergences of (5.38)-(5.39) are now an easy consequence of (5.11), (5.36)-(5.37), and (5.43).

We prove that indeed the quadruple $(\theta_\mu,\chi_{1,\mu},\chi_{2,\mu},u_\mu)$ solves problem P_μ. Let us introduce the new auxiliary variables $[\overline{\xi}_{1,\mathcal{P}},\overline{\xi}_{2,\mathcal{P}}]\in\partial I_K(\overline{\chi}_{1,\mathcal{P}},\overline{\chi}_{2,\mathcal{P}})$ almost everywhere in Q such that (5.26) reduces to the following equality

$$\mu\partial_t\begin{pmatrix}\chi_{1,\mathcal{P}}\\ \chi_{2,\mathcal{P}}\end{pmatrix}+\gamma\begin{pmatrix}\overline{\chi}_{1,\mathcal{P}}\\ \overline{\chi}_{2,\mathcal{P}}\end{pmatrix}+\begin{pmatrix}\frac{\ell}{\theta_*}(\overline{\theta}_{\mathcal{P}}-\theta_*)\\ \overline{\sigma}_{\mathcal{P}}:\beta\end{pmatrix}+\begin{pmatrix}\overline{\xi}_{1,\mathcal{P}}\\ \overline{\xi}_{2,\mathcal{P}}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\text{a.e. in } Q.$$

Hence, moving from (5.27), it is straightforward to possibly extract a further subsequence such that the convergences of Lemma 5.4 hold and there exists $[\xi_{1,\mu},\xi_{2,\mu}]$ such that

$$\xi_{j,\mathcal{P}}\to\xi_{j,\mu}\quad\text{weakly in } L^2(0,T;H),\quad\text{for } j=1,2.$$

Owing to the above proved convergences and (5.9)-(5.11) it is now possible to pass to the limit as the diameter τ of partition $\mathcal{P}$ goes to zero and check that $(\theta_\mu,\chi_{1,\mu},\chi_{2,\mu},u_\mu,\xi_{1,\mu},\xi_{2,\mu})$ fulfills relations (3.1)-(3.3), (3.5)-(3.6), and

$$\mu\partial_t\begin{pmatrix}\chi_{1,\mu}\\ \chi_{2,\mu}\end{pmatrix}+\gamma\begin{pmatrix}\chi_{1,\mu}\\ \chi_{2,\mu}\end{pmatrix}+\begin{pmatrix}\xi_{1,\mu}\\ \xi_{2,\mu}\end{pmatrix}\ni\begin{pmatrix}\frac{\ell}{\theta_*}(\theta_*-\theta_\mu)\\ -\sigma_\mu:\beta\end{pmatrix}\qquad\text{a.e. in } Q.\tag{5.46}$$

Moreover, it is straightforward to check that

$$\int_0^T(\overline{\xi}_{j,\mathcal{P}},\overline{\chi}_{j,\mathcal{P}})\to\int_0^T(\xi_{j,\mu},\chi_{j,\mu})\qquad\text{for } j=1,2.$$

In particular, classical results on maximal monotone operators [7, Prop. 2.5, p. 27] entail that $[\xi_{1,\mu}, \xi_{2,\mu}] \in \partial I_K(\chi_{1,\mu}, \chi_{2,\mu})$ almost everywhere in Q and the assertion of the Lemma follows. Before closing this proof, we observe that the convergences stated in the Lemma hold for all the sequence of partitions and not just for a subsequence since the solution to problem P_μ is unique. □

5.6. Error estimates. For the sake of completeness, we state here an a priori bound on the discretization error. To this aim, recalling from (A1) that $F := F_1 + F_2$, we sharpen our regularity requirements by asking for

$$F_1 \in BV([0,T]; H), \tag{5.47}$$

where the above notation refers to the space of real functions valued in H with bounded variation. We have the following estimate.

LEMMA 5.5. *Under the assumptions (A1)-(A3), (5.6)-(5.8), and (5.47) let* $(\theta, [\chi_1, \chi_2], u)$ *and* $\{\theta_i\}_{i=0}^N \in V^{N+1}$, $\{[\chi_{1,i}, \chi_{2,i}]\}_{i=0}^N \in \mathcal{K}^{N+1}$, *and* $\{u_i\}_{i=1}^N \in \mathcal{V}^N$ *be the unique solutions to problem* P_μ *and* D_μ, *respectively. Hence there exists a positive constant* c_3 *depending on* c_2 *such that, possibly taking* τ *small enough, one has that*

$$\begin{aligned}\|\theta - \theta_{\mathcal{P}}\|_{L^2(0,t;H)} &+ \sup_{s\in[0,t]} \left\| \int_0^s \nabla(\theta - \overline{\theta}_{\mathcal{P}}) \right\| + \sup_{s\in[0,t]} \left\| \int_0^s (\theta - \overline{\theta}_{\mathcal{P}}) \right\|_{L^2(\Gamma)} \\ &+ \mu \sum_{j=1}^2 \|\chi_j - \chi_{j,\mathcal{P}}\|_{C([0,t];H)} + \sum_{j=1}^2 \|\chi_j - \chi_{j,\mathcal{P}}\|_{L^2(0,t,H)} \\ &+ \|u - u_{\mathcal{P}}\|_{C([0,t];\mathcal{V})} \leq c_3 \tau \quad \forall t \in [0,T].\end{aligned} \tag{5.48}$$

We will not give here the detailed proof of the latter result. Indeed, the argument follows closely the lines of the proof of the continuous dependence estimate (4.1). The additional intricacy related to the fact that the continuous and the discrete solutions do not solve the same equations may be overcome by the same techniques of the two papers [34, 35] where indeed the abstract analysis of [28, 29] is applied in a similar context. On the other hand, some comment is in order. We point out that the latter estimate is *optimal* with respect to the order of convergence, since the backward Euler scheme is used in order to approximate time derivatives in problem P_μ. Moreover, no a priori constraints between consecutive time-steps are imposed. Hence (5.48) ensures the possibility of implementing an adaptive procedure as in [29]. Finally, let us remark that c_3 depends exponentially on T since the Gronwall lemma is used in the proof of (5.48).

6. Non-dissipative problem. We now proceed to the proof of Theorem 4.1 for $\mu = 0$ and Theorem 4.2. To this aim $(\theta_\mu, \chi_{1,\mu}, \chi_{2,\mu}, u_\mu)$ will denote a sequence of solutions to problem P_μ as μ converges toward 0.

6.1. Continuous dependence. First of all, we address the continuous dependence claim in the non-dissipative situation. To this end, it suffices to follow exactly the lines of the corresponding proof of Subsection 5.1 with the choice of the parameter $\mu = 0$.

6.2. A priori estimates. We shall prove the following.

LEMMA 6.1. *Under the assumptions (A1)-(A3), let* $(\theta_\mu, \chi_{1,\mu}, \chi_{2,\mu}, u_\mu)$ *be solutions to problem* P_μ. *Then, there exists a positive constant* c_4 *just depending on*

c_s, k, h, θ_e, Γ, γ, $\mathbb{A}^{\frac{1}{2}}\beta$, $c_{\mathcal{V}}$, θ_*, θ^0, $[\chi_1^0, \chi_2^0]$, $\|F_1\|_{L^2(0,T;H)}$, $\|F_2\|_{W^{1,1}(0,T;V')}$, *and* $\|G\|_{H^1(0,T;\mathcal{V}')}$ *and such that*

$$\|\theta_\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|[\chi_{1,\mu}, \chi_{2,\mu}]\|_{(H^1(0,T;H))^2} + \|u_\mu\|_{H^1(0,T;\mathcal{V})} \leq c_4. \tag{6.1}$$

In particular, c_4 *is independent of* μ.

Proof. This argument is just sketched here since it is very close to that of Lemma 5.3. Indeed, the above stated result represent the continuous version of the stability estimates for the discrete scheme. Namely, the key a priori estimate consists in taking the sum between (3.1) with $\varphi := \theta_t/\theta_*$, the time derivative of (3.4) multiplied by $[\chi_{1,\mu,t}, \chi_{2,\mu,t}]$, and the time derivative of (3.2) with $v = u_{\mu,t}$. We shall remark that the latter choices of test functions are not admissible at the present stage. However, the above calculation is to be intended at an appropriate approximation level (for instance that of the discrete scheme). We prefer to skip the details of this discussion for the sake of clarity. Next, taking the integral on $(0,t)$ for some $t \in (0,T]$ of the upcoming relation, the assertion of Lemma 6.1 follows along the same lines of the proof of Lemma 5.3. $\square$

6.3. Passage to the limit. Let us now finally turn to the proof of the Theorem 4.2. Precisely, we prove the following.

LEMMA 6.2. *Under the assumptions (A1)-(A3), let* $(\theta_\mu, \chi_{1,\mu}, \chi_{2,\mu}, u_\mu)$ *be the solution to problem* P_μ. *Then the following convergences hold*

$$\begin{aligned} \theta_\mu \longrightarrow \theta \quad & \textit{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V) \\ & \textit{and strongly in } C([0,T];H), \qquad (6.2) \\ \chi_{j,\mu} \longrightarrow \chi_j \quad & \textit{weakly in } H^1(0,T;H) \\ & \textit{and strongly in } C([0,T];H), \quad j=1,2, \qquad (6.3) \\ u_\mu \longrightarrow u \quad & \textit{weakly in } H^1(0,T;\mathcal{V}) \\ & \textit{and strongly in } C([0,T];\mathcal{V}), \qquad (6.4) \end{aligned}$$

where $(\theta, \chi_1, \chi_2, u)$ *is the unique solution to* P_0.

Proof. This argument is very close to that of Lemma 5.4. Thanks to Lemma 6.1 we are in the position of finding a quadruple $(\theta, \chi_1, \chi_2, u)$ such that, possibly taking not relabeled subsequences and owing to well known compactness results, the following convergences hold

$$\begin{aligned} \theta_\mu \longrightarrow \theta \quad & \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V) \\ & \text{and strongly in } C([0,T];H), \qquad (6.5) \\ \chi_{j,\mu} \longrightarrow \chi_j \quad & \text{weakly in } H^1(0,T;H), \quad j=1,2, \qquad (6.6) \\ u_\mu \longrightarrow u \quad & \text{weakly in } H^1(0,T;\mathcal{V}). \qquad (6.7) \end{aligned}$$

We now turn to the proof of a direct Cauchy argument. To this aim, we fix two parameters μ_n, μ_m of the extracted subsequence and denote by $(\theta_r, \chi_{1,r}, \chi_{2,r}, u_r)$ the solution to problem P_{μ_r}, for $r = n, m$ (σ_n, σ_m are defined accordingly). Next, we take the difference between (3.4) written for μ_1 and the same relation for μ_2, multiply it by $(\chi_{1,n} - \chi_{1,m}, \chi_{2,n} - \chi_{2,m})$, and integrate in space. We readily obtain that, for almost all $t \in (0,T)$,

$$\gamma \sum_{j=1}^{2} \|(\chi_{j,n} - \chi_{j,m})(t)\|^2 \le \frac{\ell}{\theta_*} \|(\theta_n - \theta_m)(t)\| \, \|(\chi_{1,n} - \chi_{1,m})(t)\| \\ -\big(\beta : (\sigma_n - \sigma_m)(t), (\chi_{2,n} - \chi_{2,m})(t)\big) \\ + \sum_{j=1}^{2} \|(\mu_n \chi_{j,n,t} - \mu_m \chi_{j,m,t})(t)\| \, \|(\chi_{j,n} - \chi_{j,m})(t)\|. \tag{6.8}$$

Once again we readily compute that

$$\|\mathbb{A}^{\frac{1}{2}} \varepsilon(u_n - u_m)\|^2 = -(\mathbb{A}^{\frac{1}{2}} \varepsilon(u_n - u_m) : \mathbb{A}^{\frac{1}{2}} \beta, \chi_{2,n} - \chi_{2,m}) \quad \text{a.e. in } (0,T), \tag{6.9}$$
$$-(\sigma_n - \sigma_m) : \beta(\chi_{2,n} - \chi_{2,m})$$
$$= -\mathbb{A}^{\frac{1}{2}} \varepsilon(u_n - u_m) : \mathbb{A}^{\frac{1}{2}} \beta(\chi_{2,n} - \chi_{2,m}) - |\mathbb{A}^{\frac{1}{2}} \beta|^2 (\chi_{2,n} - \chi_{2,m})^2 \quad \text{a.e. in } Q. \tag{6.10}$$

Finally, taking the sum between (6.8) and (6.9) and using (6.10), we readily obtain that

$$\gamma \|(\chi_{1,n} - \chi_{1,m})(t)\|^2 + \big(\gamma + |\mathbb{A}^{\frac{1}{2}} \beta|^2 - \rho |\mathbb{A}^{\frac{1}{2}} \beta|^2\big) \|(\chi_{2,n} - \chi_{2,m})(t)\|^2 \\ + \frac{\rho - 1}{\rho} \|\mathbb{A}^{\frac{1}{2}} \varepsilon(u_n - u_m)(t)\|^2 \le \frac{\ell}{\theta_*} \|(\theta_n - \theta_m)(t)\| \, \|(\chi_{1,n} - \chi_{1,m})(t)\| \\ + \sum_{j=1}^{2} \|(\mu_n \chi_{j,n,t} - \mu_m \chi_{j,m,t})(t)\| \, \|(\chi_{j,n} - \chi_{j,m})(t)\|,$$

where once again ρ is such that $1 < \rho < (\gamma + |\mathbb{A}^{\frac{1}{2}} \beta|^2)/|\mathbb{A}^{\frac{1}{2}} \beta|^2$. Now, it suffices to recall (5.27) and (6.5) in order to infer the strong convergences

$$\chi_{j,\mu} \longrightarrow \chi_j \quad \text{strongly in } C([0,T]; H) \quad \text{for } j = 1, 2,$$
$$u_\mu \longrightarrow u \quad \text{strongly in } C([0,T]; \mathcal{V}).$$

Moving from the above positions, the proof of this lemma may be concluded exactly as that of Lemma 5.4. □

6.4. Error control. The Cauchy argument devised in the latter subsection may be used in order to achieve some quantitative control on the distance between the dissipative and the non-dissipative regimes. In particular, we have the following.

LEMMA 6.3. *Under assumptions (A1)-(A3), let* $(\theta_\mu, \chi_{1,\mu}, \chi_{2,\mu}, u_\mu)$ *and* $(\theta, \chi_1, \chi_2, u)$ *denote the unique solutions to problems* P_μ *and* P_0*, respectively. Then, there exists a constant* c_5 *with the same dependences of* c_2 *(in particular independent of* μ*) such that*

$$\|\theta - \theta_\mu\|_{L^2(0,T;H)} + \sup_{t \in [0,T]} \left\| \int_0^t \nabla(\theta - \theta_\mu) \right\| + \sup_{t \in [0,T]} \left\| \int_0^t (\theta - \theta_\mu) \right\|_{L^2(\Gamma)} \\ + \sum_{j=1}^{2} \|\chi_j - \chi_{j,\mu}\|_{L^2(0,T;H)} + \|u - u_\mu\|_{L^2(0,T;\mathcal{V})} \le c_5 \sqrt{\mu}.$$

The proof of the latter lemma follows the same lines of Subsections 5.1 and 6.3 and is therefore omitted.

6.5. Discretization. Let us comment here the possible discretization of problem P_0. First of all we observe that the limit procedure of Section 5 is completely independent of μ. On the other hand, the positivity of μ is exploited in order to implement the contraction argument. Namely, one could choose $\mu = \mu(\tau) > 0$ such that $\lim_{\tau\to 0^+} \mu(\tau) = 0$ and prove that the resulting discrete solution exists and converges indeed to a solution to problem P_0.

It is however remarkable that we would also be in the position of providing a variable time-step discretization scheme for problem P_0 as well. Namely, we could directly work at the non-dissipative level $\mu = 0$ and prove Lemmas 5.1, 5.3, and 5.4 (and hence Theorem 4.1) directly for problem D_0. On the other hand, we prefer to analyze here the discretization of the dissipative problem because the well-posedness proof for problem D_0 relies on some non-constructive technique (Schauder fixed point) and hence shows a merely theoretical interest (while the scheme for problem D_μ is effectively computable). Finally, we are interested in establishing the asymptotic connection within the dissipative and the non-dissipative models at all levels, namely the continuous and the discrete ones.

As for the discretization error estimate (5.48) one could actually prove that an analogous bound holds true in the case $\mu = 0$. In particular, in the latter non-dissipative case we will be forced to replace τ with $\sqrt{\tau}$, i.e. we reduce ourselves to a sub-optimal convergence rate.

7. Long-time behavior. Let us now turn to the proof of Theorem 4.3. In particular, let us recall that the dissipation parameter μ is set to be zero throughout this section. Namely, we will carry out the long-time behavior analysis in the non-dissipative regime. Owing to Theorem 4.1 it is a standard matter to check for the existence and uniqueness of a quadruple $(\theta, \chi_1, \chi_2, u)$ such that, for each $T \in (0, +\infty)$, one has that $\theta \in H^1(0,T;H) \cap L^\infty(0,T;V)$, $[\chi_1, \chi_2] \in (H^1(0,T;H))^2$, $u \in H^1(0,T;\mathcal{V})$, fulfilling conditions (3.1)-(3.5) with $\mu = 0$. We proceed by establishing some lemmas.

LEMMA 7.1. *Under the assumptions (A2)-(A4), there exists a positive constant* c_6 *depending on* c_s, k, h, θ_e, Γ, γ, $\mathbb{A}^{\frac{1}{2}}\beta$, $c_{\mathcal{V}}$, θ_*, θ^0, $[\chi_1^0, \chi_2^0]$, $\|F\|_{L^2(0,+\infty;H)}$, *and* $\|G\|_{H^1(0,+\infty;\mathcal{V}')}$ *such that*

$$\int_0^t \left(\|\theta_t\|^2 + \sum_{j=1}^2 \|\chi_{j,t}\|^2 + \|u_t\|_{\mathcal{V}}^2 \right) + \|\theta(t) - \theta_e\|_V^2 \le c_6 \qquad \forall t > 0. \tag{7.1}$$

We do not provide here a detailed proof of the latter estimate. Indeed, the argument of Lemma 6.1 (together with the long-time assumption (A4)) may be easily adapted to ensure the validity of (7.1).

A first consequence of Lemma 7.1 is that the set

$$\big\{(\theta(t), \chi_1(t), \chi_2(t), u(t)),\ t > 0\big\} \text{ is bounded in } V \times H \times H \times \mathcal{V}.$$

Therefore, there exists a sequence $t_n \to +\infty$ and a quadruple $(\theta_\infty, \chi_{1,\infty}, \chi_{2,\infty}, u_\infty)$ such that

$$\begin{aligned} \theta(t_n) &\longrightarrow \theta_\infty \quad \text{strongly in } H,\\ \chi_j(t_n) &\longrightarrow \chi_{j,\infty} \quad \text{weakly in } H,\ j = 1,2,\\ u(t_n) &\longrightarrow u_\infty \quad \text{weakly in } \mathcal{V}. \end{aligned}$$

Indeed, by observing that relations (3.2)-(3.4) are actually fulfilled everywhere in time and almost everywhere in Ω and arguing as in the proof of Lemma 6.2, we easily check that the above devised direct Cauchy argument entails that the latter convergences are strong. In particular, the set $\omega(\theta,\chi_1,\chi_2,u)$ is non-empty.

Consider now any $(\theta_\infty,\chi_{1,\infty},\chi_{2,\infty},u_\infty)\in\omega(\theta,\chi_1,\chi_2,u)$. Hence, there is a sequence $\{t_n\}$ of positive real numbers such that $t_n\to+\infty$ and

$$(\theta(t_n),\chi_1(t_n),\chi_2(t_n),u(t_n)) \longrightarrow (\theta_\infty,\chi_{1,\infty},\chi_{2,\infty},u_\infty) \quad \text{in } H\times H\times H\times\mathcal{V}. \tag{7.2}$$

For n and $t\geq 0$, we define

$$\theta_n(t):=\theta(t_n+t),\quad \chi_{j,n}(t):=\chi_j(t_n+t)\quad j=1,2,\quad u_n(t):=u(t_n+t).$$

We can introduce a pair of auxiliary functions $[\xi_{1,n},\xi_{2,n}]$ such that the functions $\theta_n,\chi_{1,n},\ \chi_{2,n},u_n,\sigma_n,\xi_{1,n},\xi_{2,n}$ solve relations

$$\gamma\begin{pmatrix}\chi_{1,n}\\ \chi_{2,n}\end{pmatrix}+\begin{pmatrix}\frac{\ell}{\theta_*}(\theta_n-\theta_*)\\ \sigma_n:\beta\end{pmatrix}+\begin{pmatrix}\xi_{1,n}\\ \xi_{2,n}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\qquad \text{a.e. in } \Omega\times(0,T), \tag{7.3}$$

$$\begin{pmatrix}\xi_{1,n}\\ \xi_{2,n}\end{pmatrix}\in\partial I_K(\chi_1,\chi_2)\qquad \text{a.e. in } \Omega\times(0,T), \tag{7.4}$$

as well as the relations (3.1)-(3.3), for all $T\in(0,+\infty)$. However, note that, in (3.1)-(3.2) F and G have to be replaced by $F_n:=F(\cdot+t_n)$ and $G_n:=G(\cdot+t_n)$, respectively. We also point out the initial condition $\theta_n(\cdot,0)=\theta(\cdot,t_n)$ almost everywhere in Ω.

Owing to Lemma 7.1 it is not difficult to prove some estimates for the functions $\theta_n,\chi_{1,n},\ \chi_{2,n},\ u_n,\xi_{1,n}$, and $\xi_{2,n}$ which are uniform with respect to n. The proof of the next result is omitted since it is analogous to the proof of Lemma 5.3.

LEMMA 7.2. *Let $T>0$. Under the above assumptions, letting $\xi_{1,n}$ and $\xi_{2,n}$ be as in (7.3)-(7.4), there exists a positive constant c_7 with the same dependencies of c_6 such that*

$$\|\theta_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}+\|u_n\|_{H^1(0,T;\mathcal{V})} + \sum_{j=1}^{2}\|\chi_{j,n}\|_{H^1(0,T;H)}+\sum_{j=1}^{2}\|\xi_{j,n}\|_{H^1(0,T;H)}\leq c_7. \tag{7.5}$$

Another consequence of Lemma 7.1 is to allow the identification of the limit of $\theta_n,\chi_{1,n},\chi_{2,n}$, and u_n as $n\to+\infty$. More precisely, we have the following

LEMMA 7.3. *Under the above assumptions, for every $T>0$ there holds*

$$\theta_n\longrightarrow\theta_\infty\qquad \textit{strongly in } H^1(0,T;H), \tag{7.6}$$

$$\chi_{j,n}\longrightarrow\chi_{j,\infty}\qquad \textit{strongly in } H^1(0,T;H),\quad j=1,2, \tag{7.7}$$

$$u_n\longrightarrow u_\infty\qquad \textit{strongly in } H^1(0,T;\mathcal{V}). \tag{7.8}$$

Proof. Taking into account (7.1) it is straightforward to check that

$$\int_0^T\|\theta_{n,t}\|^2=\int_{t_n}^{t_n+T}\|\theta_t\|^2\longrightarrow 0\quad \text{as } n\longrightarrow+\infty. \tag{7.9}$$

An analogous computation applies to $\chi_{1,n,t}, \chi_{2,n,t}$, and $u_{n,t}$, as well. Hence, we easily deduce that

$$\begin{aligned}\|\theta_n(t)-\theta_\infty\| &\le \|\theta_n(t)-\theta_n(0)\| + \|\theta(t_n)-\theta_\infty\| \\ &\le T^{1/2}\|\theta_{n,t}\|_{L^2(0,T;H)} + \|\theta(t_n)-\theta_\infty\|\end{aligned}$$

Owing to (7.2) and (7.9), the right hand side of the latter inequality goes to zero as $n \to +\infty$. Hence, (7.6) is proved. A similar argument ensures that (7.7)-(7.8) hold true. □

After these preliminaries, we may prove Theorem 4.3 by passing to the limit as $n \to +\infty$ in the equations (3.1)-(3.3) for the quadruple $(\theta_n, u_n, \chi_{1,n}, \chi_{2,n})$ and data $[F_n, G_n]$ and in relations (7.3)-(7.4). Thanks to the above lemmas and well known compactness results we find a subsequence (not relabeled) of $\theta_n, \chi_{1,n}, \chi_{2,n}$, and u_n and a pair $(\xi_{1,\infty}, \xi_{2,\infty})$ such that, in addition to (7.6)-(7.8), the following convergences hold

$$\theta_n \longrightarrow \theta_\infty \qquad \text{weakly star in } L^\infty(0,T;V), \tag{7.10}$$

$$\xi_{j,n} \longrightarrow \xi_{j,\infty} \qquad \text{weakly in } H^1(0,T;H), \quad j=1,2. \tag{7.11}$$

The above proved convergences and (A4) are sufficient in order to pass to the limit in (3.1)-(3.3). In particular, it turns out that $\theta_\infty = \theta_e$ almost everywhere in Ω. As far as relations (7.3)-(7.4) are concerned, we observe that we also have

$$\begin{pmatrix} \frac{\ell}{\theta_*}(\theta_n-\theta_*) \\ \sigma_n : \beta \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{\ell}{\theta_*}(\theta_\infty-\theta_*) \\ \sigma_\infty : \beta \end{pmatrix} \quad \text{strongly in } \ (C([0,T];H))^2.$$

Hence, we just need to identify the limit of $[\xi_{1,n}, \xi_{2,n}]$. Indeed, from (7.7) and (7.11), one easily infers that

$$(\xi_{i,n}, \chi_{i,n}) \longrightarrow (\xi_{i,\infty}, \chi_{i,\infty}) \quad \text{a.e. in } (0,T), \quad \text{for } i=1,2.$$

The classical theory of maximal monotone operators (see, e.g., [7, Prop. 2.5, p. 27]) then entails that $[\xi_{1,\infty}, \xi_{2,\infty}] \in \partial I_K(\chi_{1,\infty}, \chi_{2,\infty})$ almost everywhere in Ω and we have finally proved (4.2)-(4.5). In order to conclude the proof of Theorem 4.3 we provide the following stronger result.

LEMMA 7.4. *Let the external temperatures* $\theta_{e,1}, \theta_{e,2}$ *be given and let* $(\chi_{1,i}, \chi_{2,i}, u_i) \in H \times H \times \mathcal{V}$ *fulfill*

$$\begin{aligned} a(u_i, v) + \big(\mathbb{A}\beta\chi_{2,i}, \varepsilon(v)\big) = 0 \quad \forall v \in \mathcal{V}, \\ \mathbb{A}\big(\varepsilon(u_i) + \beta\chi_{2,i}\big) = \sigma_i \quad \textit{a.e. in } \Omega, \\ \gamma\begin{pmatrix}\chi_{1,i}\\ \chi_{2,i}\end{pmatrix} + \partial I_K\begin{pmatrix}\chi_{1,i}\\ \chi_{2,i}\end{pmatrix} \ni \begin{pmatrix} \frac{\ell}{\theta_*}(\theta_* - \theta_{e,i}) \\ -\sigma_i : \beta\end{pmatrix} \quad \textit{a.e. in } \Omega, \end{aligned}$$

for $i=1,2$. *Then, there exists a positive constant* c_8 *depending just on* ℓ, θ_*, γ, $\mathbb{A}^{\frac12}\beta$, *and* $c_{\mathcal{V}}$ *such that*

$$\sum_{j=1}^{2} \|\chi_{j,1} - \chi_{j,2}\| + \|u_1 - u_2\|_{\mathcal{V}} \le c_8\, |\theta_{e,1} - \theta_{e,2}|. \tag{7.12}$$

Once again the proof of the latter lemma may be easily achieved by adapting the argument of Subsection 5.1. A consequence of the above continuous dependence result is that, since $(\theta_\infty, \chi_{1,\infty}, \chi_{2,\infty}, u_\infty)$ are uniquely determined, we readily check that the $\omega-$limit set reduces to a point and the whole trajectory $(\theta(t), \chi_1(t), \chi_2(t), u(t))$ converges to $(\theta_\infty, \chi_{1,\infty}, \chi_{2,\infty}, u_\infty)$ as $t \to +\infty$. In particular, this concludes the proof of Theorem 4.3.

8. Lower bound for the temperature. Let us now turn to the proof of Theorem 4.4. This argument is very close to that of [11] and will be just sketched referring to the latter paper for details. We will start by checking (4.7) in the dissipative case $\mu > 0$. In this situation we claim that

$$\left(\gamma\chi_1 + \frac{\ell}{\theta_*}(\theta - \theta_*)\right)^+ \geq -\mu\chi_{1,t} \quad \text{a.e. in } Q, \tag{8.1}$$

where we used the standard notation for the positive part. Indeed, $\chi_{1,t} = 0$ almost everywhere on the measurable set $\{\chi_1 = 1\}$. On the other hand, for almost every $(x,t) \in \{\chi_1 < 1\}$, one readily checks from (3.4) that (see [11])

$$\left(\mu\chi_{1,t} + \gamma\chi_1 + \frac{\ell}{\theta_*}(\theta - \theta_*)\right)(x,t) \geq 0.$$

Let us now consider

$$\underline{\theta} := \inf\{\inf\theta_0,\, \theta_e,\, \theta_*,\, \theta_d\} \in \mathbb{R},$$

(the case $\underline{\theta} = -\infty$ being obvious), choose $\varphi = -(\theta - \underline{\theta})^- \in V$ in (3.1), and take the integral on $(0,t)$ for $t \in (0,T]$ obtaining

$$\begin{aligned} \frac{c_s}{2}\|(\theta - \underline{\theta})^-(t)\|^2 + k\int_0^t \|\nabla((\theta - \underline{\theta})^-)\|^2 - h\int_0^t (\theta - \theta_e, (\theta - \underline{\theta})^-)_\Gamma \\ = -\int_0^t \langle F, (\theta - \underline{\theta})^-\rangle - \ell\int_0^t (\chi_{1,t}, (\theta - \underline{\theta})^-). \end{aligned} \tag{8.2}$$

Owing to (A5) and (8.1) one gets that the above right hand side may be controlled as follows

$$\begin{aligned} &-\int_0^t \langle F, (\theta - \underline{\theta})^-\rangle - \ell\int_0^t (\chi_{1,t}, (\theta - \underline{\theta})^-) \\ &\leq \frac{\ell}{\mu}\int_0^t \left(\left(\gamma\chi_1 + \frac{\ell}{\theta_*}(\theta - \theta_*)\right)^+, (\theta - \underline{\theta})^-\right) \\ &\leq \frac{\ell}{\mu}\int_0^t \left(\left(\gamma + \frac{\ell}{\theta_*}(\underline{\theta} - \theta_*)\right)^+, (\theta - \underline{\theta})^-\right) = 0, \end{aligned}$$

since $\underline{\theta} \leq \theta_d$ and (4.6) holds . Hence, looking back to (8.2) and considering that $\underline{\theta} \leq \theta_e$ as well, we readily check that $\theta \geq \underline{\theta}$ almost everywhere in Q. The proof of the lower bound for the temperature in the non-dissipative case $\mu = 0$ simply follows by approximation owing to (6.5).

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