Analysis of a 1D thermoviscoelastic model with temperature-dependent viscosity

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Abstract

This note addresses the analysis of a thermoviscoelastic model with a temperature-dependent viscous modulus. We provide the well-posedness of a related initial and boundary value problem and detail a suitable fully implicit variable time-step discretization. The latter is proved to be conditionally stable and convergent. Moreover, some a priori error estimates of optimal order are established.

Key words: thermoviscoelasticity, well-posedness, discretization, error control.

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1 Introduction

The isothermal evolution of a polymer is generally considered to be viscoelastic. On the other hand, the mechanical behavior of polymers is known to be strongly temperature-dependent and a certain number of phenomena such as strain localization (shear bands or necking) cannot be modeled in a satisfactory way by means of classic models [18, 24]. Moreover, industrial polymers are known to experience large temperature variations in most (if not all) manufacturing processes (see [6] and the reference therein). The mathematical description of polymeric materials is therefore forced to take into account thermo-mechanical couplings. In particular, mechanically induced heat sources, thermal stresses, and strong temperature-dependence of the viscous and elastic moduli have to be considered.

The present analysis is devoted to the discussion of a one-dimensional thermoviscoelastic model of Kelvin-Voigt type. In particular, we shall be concerned with the system

$$c_s\dot{\theta} - k\theta_{xx} = -\alpha c\theta\dot{\varepsilon} + \nu(\theta)\dot{\varepsilon}^2, \tag{1.1}$$

$$\nu(\theta)\dot{\varepsilon} + c\varepsilon = \sigma + \alpha c(\theta - \theta_c), \tag{1.2}$$

posed in the space time domain $Q := \Omega \times (0,T)$ where $\Omega := (0,1)$ and $T > 0$ is a final reference time. In the latter relations $\theta > 0$ represents the absolute temperature of some thermoviscoelastic wire, ε is its strain, and σ is some given stress. System (1.1)-(1.2) arises naturally in the framework of Continuum Thermo-mechanics [8, 9] whenever one prescribes the *free energy density* and the *potential of dissipation* of the medium to be respectively

$$\begin{aligned} \psi(\theta,\varepsilon) &:= -c_s\theta\ln\theta - \alpha c(\theta-\theta_c)\varepsilon + \frac{c}{2}\varepsilon^2, \\ \phi(\theta,\theta_x,\dot\varepsilon) &:= \frac{k}{2\theta}\theta_x^2 + \frac{\nu(\theta)}{2}\dot\varepsilon^2. \end{aligned}$$

Here $c_s > 0$ is the specific heat, $k > 0$ is the thermal conductivity, $\alpha \geq 0$ is the thermal dilatation coefficient, $c \geq 0$ is the elastic modulus, $\theta_c > 0$ is some critical temperature, and $\nu : (0,+\infty) \to (0,+\infty)$ is the *temperature-dependent* viscous modulus which we assume to be bounded, smooth, and uniformly positive. We explicitly observe that the case of a purely viscous material $c = 0$ is included in the present framework. Relations (1.1)-(1.2) arise from the energy balance relation

$$\dot e + q_x = \sigma\dot\varepsilon,$$

where e is the internal energy density, q is the heat flux, and $\sigma\dot\varepsilon$ represents the mechanically induced heat sources (note that no external thermal source is considered for simplicity) and the the classical positions

$$q := \theta R, \quad R := -\frac{\partial\phi}{\partial\theta_x}, \quad \sigma := \sigma_{el} + \sigma_{vis} = \frac{\partial\psi}{\partial\varepsilon} + \frac{\partial\phi}{\partial\dot\varepsilon}.$$

Here R represents the entropy flux and σ_{el} and σ_{vis} are the elastic and the viscous components of the stress, respectively. It may be readily checked that the Second Law of Thermodynamics is fulfilled in the form of the Clausius-Duhem inequality.

We assume to be given the stress $\sigma : Q \to \mathbb{R}$. This is not very restrictive in one dimension since the equilibrium equation reads

$$\sigma_x + b = 0, \tag{1.3}$$

where b is some prescribed body force. Indeed, whenever we consider some traction test $\sigma(1) = g$ one readily computes from the latter relation that $\sigma(x,t) = \int_x^1 b(y,t)\,dy + g(t)$.

Finally, the system is complemented with the initial and boundary conditions

$$\theta(\cdot,0) = \theta^0, \quad \varepsilon(\cdot,0) = \varepsilon^0, \tag{1.4}$$

$$k\theta_x(0,\cdot) - h(\theta(0,\cdot) - \theta_e) = k\theta(1,\cdot) + h(\theta(1,\cdot) - \theta_e) = 0, \tag{1.5}$$

where θ^0 and ε^0 are initial data, $h \geq 0$ is a thermal exchange coefficient, and $\theta_e > 0$ is a constant environmental temperature. Note that the case $h = 0$ is included in the analysis. Namely, we are in the position of considering homogeneous Neumann boundary conditions on θ as well.

It is beyond our purposes to provide the reader with a comprehensive report on the vast mathematical literature on one-dimensional thermoviscoelasticity with constant viscosity. On the other hand, let us at least mention the pioneering papers [4, 5] where the global solvability of the problem was firstly addressed. Moreover, one-dimensional existence and uniqueness results have been obtained in a variety of different settings [12, 20, 23]. More recently, the modeling of the thermo-mechanical evolution of so-called shape memory materials has drawn new interest to the coupling of thermal and mechanical effects. As a consequence, some well-posedness results for thermoviscoelasticity can be recovered as special cases in the context of the study of solid-solid phase transformations, see [10, 11, 15, 16] and the monograph [3]. Existence and uniqueness models in three space dimensions are comparably less studied. One can refer, for instance, to [1, 2, 7, 21, 25] and the references therein. As for the numerical treatment of one-dimensional thermoviscoelasticity, one has to mention some previous results in somehow different settings [13, 14, 19].

The main novelty of this paper is that of addressing the temperature-dependent viscosity situation. Of course even quite ordinary materials show suitable temperature dependence in both the elastic and the viscous moduli. Our main focus here is however on materials which show a strongly distinguished mechanical behavior at different temperature regimes. This is especially the case of polymeric materials, among others. As a first step in the direction of describing the full temperature-dependent situation, we assume here that the only viscous modulus ν depends on the temperature θ while the elastic modulus is constant. Moreover, we impose ν to be uniformly positive for all temperatures. This of course prevents the system from degenerating into a purely elastic situation and crucially simplifies the analysis. On the other hand, we shall stress that no monotonicity is assumed on the viscous modulus.

In addition to some (by now quite classical) well-posedness analysis, our main aim consists in providing the detail of some effective fully implicit variable time-step discretization procedure. The latter turns out to be conditionally stable and convergent. Moreover, the discrete approximating temperature remains positive for all times. Finally, we are able to present some a priori error estimates of optimal order. Since no a priori constraints between consecutive time-steps are imposed, our error estimates ensure the possibility of implementing an adaptive time-stepping procedure.

This is the plan of the paper. We recall some notations and state our main assumptions in Section 2. The well-posedness results for the continuous problem are formulated in Section 3 where it is also directly checked that the temperature remains positive for all times. We give a proof of some continuous dependence result in Section 4. Then, our discrete scheme is introduced in Section 5 and proved to be (conditionally) uniquely solvable in Section 6. We establish the conditional stability of the approximation procedure in Section 7 and its convergence in Section 8. Finally, we present our a priori error bounds in Section 9.

2 Notations and assumptions

Let us start by setting some notations. We let

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := H^2(\Omega),$$

endowed with the usual scalar products. The reader is referred to [17] for definitions and properties of Sobolev spaces. Let also $(\cdot,\cdot)$ denote the scalar product in H, $|\cdot|$ stand for both the norm in H and the modulus in $\mathbb{R}$, and $\|\cdot\|_E$ denote the norm in the generic normed space E.

We are now in the position of stating our assumptions:

(A1) $\nu : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, bounded, and uniformly positive, namely

$$0 < \nu_* \le \nu(r) \le \nu^* \quad \text{for all} \quad r \in \mathbb{R},$$

for some given ν_*, ν^*.

(A2) $\sigma \in H^1(0,T;H) \cap L^\infty(0,T;V)$.

(A3) $\theta^0 \in V$, $\theta > 0$ on $[0,1]$, and $\varepsilon^0 \in V$.

Taking into account the above discussion, some sufficient requirements on b and g in view of (A2) are

$$b \in H^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;H), \quad g \in H^1(0,T).$$

3 Continuous problem

The continuous problem (CP) reads as follows:

CP: to find $\theta \in H^1(0,T;H) \cap C([0,T],V) \cap L^2(0,T;W)$ and $\varepsilon \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V)$ such that (1.1)-(1.2), (1.4)-(1.5) are almost everywhere fulfilled.

Indeed, given the above stated regularities, it is straightforward to check that (1.2) and (1.4) will actually be fulfilled everywhere. One has the following.

Theorem 3.1 (Existence and Uniqueness). *Under assumptions* (A1)-(A3) *there exists a unique solution to* CP.

Lemma 3.2 (Continuous dependence). *Let* $(\sigma_i, \theta_i^0, \varepsilon_i^0)$, $i = 1, 2$, *fulfill* (A2)-(A3) *and* $(\theta_i, \varepsilon_i)$ *be the respective solutions to* CP. *Then there exists a positive constant* C *depending on* $(\sigma_i, \theta_i^0, \varepsilon_i^0)$ *and on data such that*

$$\begin{aligned} &\|\theta_1 - \theta_2\|_{C([0,T];H) \cap L^2(0,T;V)} + \|\varepsilon_1 - \varepsilon_2\|_{H^1(0,T;H)} \\ &\leq C\Big(|\theta_1^0 - \theta_2^0| + |\varepsilon_1^0 - \varepsilon_2^0| + \|\sigma_1 - \sigma_2\|_{L^2(0,T;H)}\Big). \end{aligned} \tag{3.1}$$

Of course the latter (local) Lipschitz continuous dependence result allows us to consider some weaker notion of solution corresponding to weaker assumptions of data. On the other hand, let us mention that the uniqueness part of the statement of Theorem 3.1 will be an easy consequence of Lemma 3.2.

Moreover, we shall remark that the techniques here developed could be easily tailored in order to include in the energy balance equation (1.1) some distributed heat source term of the form $\ell(x,t,\theta(x,t))$ taking into account, for instance, the Joule effect, radiations, microwaves etc. Here $\ell : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, uniformly Lipschitz continuous with respect to θ, and such that there exists some parameter $\theta_\ell > 0$ such that, for all $\theta \leq \theta_\ell$ one has that $\ell(\cdot,\theta) \geq 0$ almost everywhere in Q. Indeed, with analogous assumptions, we could consider also some non-homogeneous boundary conditions of the form

$$k\theta_x(\cdot,0) - h(\theta(\cdot,0) - \theta_e(\cdot)) = g_0(\cdot), \quad k\theta(\cdot,1) + h(\theta(\cdot,1) - \theta_e(\cdot)) = -g_1(\cdot),$$

where g_0, $g_1 \geq 0$ are given heat fluxes and the variable external temperature $\theta_e(\cdot)$ is uniformly bounded away from zero and suitably behaved. We shall present the original case of (1.1)-(1.5) for the sake of clarity.

Before closing this section let us explicitly observe that, owing to (A3), the component θ of the solution of Problem CP remains positive for all times. Indeed, the function θ fulfills (1.4) and

$$c_s\dot{\theta} - k\theta_{xx} \geq -\alpha c\theta\dot{\varepsilon} \quad \text{a.e. in } Q. \tag{3.2}$$

Letting $\underline{\theta} := \min\{\min_{x\in[0,1]}\theta^0(x), \theta_e\}$, we define the function ζ by

$$\zeta(t) := -\left(\theta(t) - \underline{\theta}\exp\left(-\frac{\alpha c}{c_s}\int_0^t \|\dot{\varepsilon}(s)\|_{L^\infty(\Omega)}ds\right)\right)^- \quad \forall t \in [0,T].$$

Next, we multiply (3.2) by ζ and take the integral over $\Omega\times(0,t)$ for some $t\in(0,T]$ obtaining that

$$\frac{c_s}{2}|\zeta(t)|^2 + k\int_0^t |\zeta_x|^2 \leq \frac{c_s}{2}|\zeta(0)|^2 + \alpha c\int_0^t \|\dot{\varepsilon}(s)\|_{L^\infty(\Omega)}|\zeta(s)|^2 ds$$
$$+\alpha c\underline{\theta}\int_0^t \left(\|\dot{\varepsilon}(s)\|_{L^\infty(\Omega)} - \dot{\varepsilon}(s)\right)\exp\left(-\frac{\alpha c}{c_s}\int_0^s \|\dot{\varepsilon}(r)\|_{L^\infty(\Omega)}dr\right)\zeta(s)\,ds$$
$$\leq \frac{c_s}{2}|\zeta(0)|^2 + \alpha c\int_0^t \|\dot{\varepsilon}(s)\|_{L^\infty(\Omega)}|\zeta(s)|^2 ds.$$

Hence , since $\zeta(0) = 0$, the Gronwall lemma ensures that $\zeta \equiv 0$ and $\theta(t)$ turns out to be bounded from below by

$$\underline{\theta}\exp\left(-\frac{\alpha c}{c_s}\int_0^t \|\dot{\varepsilon}(s)\|_{L^\infty(\Omega)}ds\right) > 0$$

almost everywhere in Ω and for all times $t \in [0,T]$.

4 Continuous dependence

We shall be concerned with the proof of Lemma 3.2. Let us start by fixing a positive constant C_0 such that

$$\|\theta_i\|_{L^\infty(Q)}, \|\dot{\varepsilon}_i\|_{L^\infty(Q)} \leq C_0 \quad \text{for } i = 1,2. \tag{4.1}$$

Indeed, as it will be clear in the sequel (see Lemma 5.2), we are entitled to choose C_0 depending on data and on $(\sigma_i, \theta_i^0, \varepsilon_i^0)$, for $i = 1,2$ but independently of $(\theta_i, \varepsilon_i)$.

Next, let us set $\overline{\theta} := \theta_1 - \theta_2$, $\overline{\varepsilon} = \varepsilon_1 - \varepsilon_2$ etc. and denote by C any positive constant, possibly varying from line to line, and depending on data, on C_0, and on $(\sigma_i, \theta_i^0, \varepsilon_i^0)$, for $i = 1,2$. Let us take the difference between (1.1) written for $(\theta_1, \varepsilon_1)$ and the same relation for $(\theta_2, \varepsilon_2)$. We get

$$c_s\dot{\overline{\theta}} - k\overline{\theta}_{xx} = -\alpha c\dot{\varepsilon}_1\overline{\theta} - \alpha c\theta_2\dot{\overline{\varepsilon}} + \left(\nu(\theta_1) - \nu(\theta_2)\right)\dot{\varepsilon}_1^2 + \nu(\theta_2)(\dot{\varepsilon}_1 + \dot{\varepsilon}_2)\dot{\overline{\varepsilon}}.$$

Hence, by multiplying by $\overline{\theta}$, integrating in space and time, and taking advantage of (A1) and (4.1), we readily obtain that

$$\frac{c_s}{2}|\overline{\theta}(t)|^2 + k\int_0^t |\overline{\theta}_x|^2 \leq C\left(|\overline{\theta}^0|^2 + \int_0^t |\overline{\theta}|^2 + \int_0^t |\dot{\overline{\varepsilon}}|^2\right). \tag{4.2}$$

On the other hand, by taking the difference of the respective relations (1.2), we may check that

$$\nu(\theta_1)\dot{\overline{\varepsilon}} = \overline{\sigma} - \big(\nu(\theta_1) - \nu(\theta_2)\big)\dot{\varepsilon}_2 - c\overline{\varepsilon} + \alpha c\overline{\theta}.$$

We now multiply by $\dot{\overline{\varepsilon}}$, integrate in space and time, and apply Gronwall's lemma obtaining

$$\int_0^t |\dot{\overline{\varepsilon}}|^2 \leq C \left(|\overline{\varepsilon}^0|^2 + \int_0^t |\overline{\theta}|^2 + \int_0^t |\overline{\sigma}|^2 \right).$$

Whence, looking back to (4.2), the assertion follows from another application of Gronwall's lemma.

5 Approximation

We shall be concerned with a variable time-step discretization of CP. To this aim let us start by introducing the partition

$$\mathcal{P} := \{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\},$$

with variable time-step $\tau_i := t_i - t_{i-1}$ and let $\tau := \max_{1\leq i\leq N} \tau_i$ denote the diameter of the partition $\mathcal{P}$.

In the forthcoming analysis the following notation will be extensively used: letting $\{u_i\}_{i=0}^N$ be a vector, we denote by $u_{\mathcal{P}}$ and $\overline{u}_{\mathcal{P}}$ two functions of the time interval $[0,T]$ which interpolate the values of the vector $\{u_i\}$ piecewise linearly and backward constantly on the partition $\mathcal{P}$, respectively. Namely

$$\begin{aligned} u_{\mathcal{P}}(0) &:= u_0, \quad u_{\mathcal{P}}(t) := \gamma_i(t)u_i + \big(1-\gamma_i(t)\big)u_{i-1}, \\ \overline{u}_{\mathcal{P}}(0) &:= u_0, \quad \overline{u}_{\mathcal{P}}(t) := u_i, \qquad \text{for } t \in (t_{i-1}, t_i], \ i = 1, \ldots, N \end{aligned}$$

where

$$\gamma_i(t) := (t - t_{i-1})/\tau_i \quad \text{for } t \in (t_{i-1}, t_i], \ i = 1, \ldots, N.$$

Moreover, we shall define a second vector $\{\delta u_i\}_{i=1}^N$ as $\delta u_i := (u_i - u_{i-1})/\tau_i$ (this is nothing but a discrete derivative).

Finally we introduce some approximation of the data. In order to fix ideas (other choices are actually possible) we approximate σ by $\overline{\sigma}_{\mathcal{P}}$ where $\sigma_i := \sigma(t_i)$ for $i = 0, 1, \ldots, N$. Let us stress that indeed (A2) ensure that

$$\|\dot{\sigma}_{\mathcal{P}}\|_{L^2(0,T;H)}, \ \|\overline{\sigma}_{\mathcal{P}}\|_{L^\infty(0,T;V)} \ \text{are bounded independently of } \mathcal{P}, \tag{5.1}$$

$$\overline{\sigma}_{\mathcal{P}} \to \sigma \ \text{ strongly in } H^1(0,T;H) \ \text{ as } \tau \to 0. \tag{5.2}$$

Moreover and for the sake of completeness, we approximate $(\theta^0, \varepsilon^0)$ by some suitable $(\theta^0_{\mathcal{P}}, \varepsilon^0_{\mathcal{P}}) \in W \times V$ strongly converging to $(\theta^0, \varepsilon^0)$ in $V \times V$ as the diameter of the partition goes to zero and such that $\theta^0_{\mathcal{P}} \geq \min \theta^0$ (this may be achieved by standard singular perturbation procedures).

Finally, let us fix an arbitrary positive parameter θ_* such that $\theta_* \leq \underline{\theta}$ and introduce the function $\varphi : \mathbb{R} \to [0, +\infty)$ as

$$\varphi(r) := \max\{0, \min\{r, 2r - \theta_*\}\}.$$

Of course φ is Lipschitz continuous of constant 2 and it is bounded by $r^+ = \max\{r, 0\}$ independently of the choice of θ_*.

We are interested in the following fully implicit scheme

$$c_s\delta\theta_i - k\theta_{i,xx} = -\alpha c\varphi(\theta_i)\delta\varepsilon_i + \nu(\theta_i)(\delta\varepsilon_i)^2 \quad \text{for } i = 1, \ldots, N, \tag{5.3}$$

$$\nu(\theta_i)\delta\varepsilon_i + c\varepsilon_i = \sigma_i + \alpha c\big(\varphi(\theta_i) - \theta_c\big) \quad \text{for } i = 1, \ldots, N, \tag{5.4}$$

$$\theta_0 = \theta^0_{\mathcal{P}}, \quad \varepsilon_0 := \varepsilon^0_{\mathcal{P}}, \tag{5.5}$$

$$k\theta_{i,x}(0) - h(\theta_i(0) - \theta_e) = k\theta_{i,x}(1) + h(\theta_i(1) - \theta_e) = 0 \quad \text{for } i = 1, \ldots, N. \tag{5.6}$$

In particular, we shall consider the following discrete problem.

DP: to find $\{\theta_i, \varepsilon_i\}_{i=0}^N \in (W \times V)^{N+1}$ such that relations (5.3)-(5.6) are almost everywhere fulfilled.

As before, (5.4)-(5.6) will actually be fulfilled everywhere. We can prove the following:

Lemma 5.1 (Existence and Uniqueness). *Under the above assumptions,* DP *has a unique solution for all τ small enough.*

Lemma 5.2 (Conditional stability). *Under the above assumptions, whenever τ is chosen to be small enough, the solution to* DP *fulfills*

$$\|\theta_{\mathcal{P}}\|_{H^1(0,T;H)\cap C([0,T];V)\cap L^2(0,T;W)} + \|\varepsilon_{\mathcal{P}}\|_{W^{1,\infty}(0,T;V)} + \|e_{\mathcal{P}}\|_{H^1(0,T;H)\cap L^\infty(Q)} \leq C_{stab}, \tag{5.7}$$

where the positive constant C_{stab} depends just on data and $\overline{e}_{\mathcal{P}} := \dot{\varepsilon}_{\mathcal{P}}$. Moreover, one also has that $\theta_{\mathcal{P}} \geq \theta_*/2$ everywhere.

Lemma 5.3 (Convergence). *Under the above assumptions and letting the diameter τ of partition $\mathcal{P}$ go to zero we find (θ, ε) such that the following convergences hold*

$$\theta_{\mathcal{P}} \to \theta \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \text{ and strongly in } C([0,T];H) \cap L^2(0,T;V), \tag{5.8}$$

$$\varepsilon_{\mathcal{P}} \to \varepsilon \quad \text{weakly star in } W^{1,\infty}(0,T;V), \tag{5.9}$$

$$e_{\mathcal{P}} \to \dot{\varepsilon} \quad \text{weakly in } H^1(0,T;H). \tag{5.10}$$

Moreover, whenever θ_* is chosen to be small enough, the functions (θ, ε) are the unique solution to CP.

We shall stress that the existence part of the statement of Theorem 3.1 is indeed a consequence of Lemma 5.3.

Lemma 5.4 (Error control). *Under the above assumptions, whenever τ is chosen to be small enough, and letting (θ, ε) and $\{\theta_i, \varepsilon_i\}_{i=0}^N$ be the solutions to* CP *and* DP, *respectively, we have that*

$$\|\theta - \theta_{\mathcal{P}}\|_{L^2(0,T;H)} + \sup_{t\in[0,T]} \left\| \int_0^t (\theta - \overline{\theta}_{\mathcal{P}}) \right\|_V + \|\varepsilon - \varepsilon_{\mathcal{P}}\|_{H^1(0,T;H)} \leq C_{err}\left(|\theta^0 - \theta^0_{\mathcal{P}}| + |\varepsilon^0 - \varepsilon^0_{\mathcal{P}}| + \|\sigma - \overline{\sigma}_{\mathcal{P}}\|_{L^2(0,T;H)} + \tau\right), \tag{5.11}$$

$$\|\theta - \theta_{\mathcal{P}}\|_{C([0,T];H)\cap L^2(0,T;V)} \leq C_{err}\left(|\theta^0 - \theta^0_{\mathcal{P}}| + |\varepsilon^0 - \varepsilon^0_{\mathcal{P}}| + \|\sigma - \overline{\sigma}_{\mathcal{P}}\|_{L^2(0,T;H)} + \sqrt{\tau}\right) \tag{5.12}$$

for some positive constant C_{err} *depending on data. Moreover, if* $\nu \in W^{2,\infty}(\mathbb{R})$ *one has that*

$$\|\varepsilon - \varepsilon_{\mathcal{P}}\|_{H^1(0,T;V)} \le C^*_{err}\left(\|\varepsilon^0 - \varepsilon^0_{\mathcal{P}}\|_V + \|\sigma - \overline{\sigma}_{\mathcal{P}}\|_{L^2(0,T;V)} + \sqrt{\tau}\right) \tag{5.13}$$

for some positive constant C^*_{err} *depending on data and on* $\|\nu''\|_{L^\infty(\mathbb{R})}$.

We point out that the *a priori* estimate (5.11) is *optimal* with respect to the order of convergence, as we used a first order approximation of time derivatives. Moreover, the proof of Lemma 5.4 will show that C_{err} and C^*_{err} depends solely on data and, in particular, exponentially on T.

6 Discrete well-posedness

This section is concerned with the proof of Lemma 5.1. We will proceed by induction. Indeed, we assume to be given the solution $(\theta_j, \varepsilon_j) \in W \times V$ for all $0 \le j \le i-1$ and solve for $(\theta_i, \varepsilon_i) \in W \times V$. To this aim, let us define

$$K := \{\tilde{\theta} \in H \ : \ |\tilde{\theta}| \le \kappa \ \text{ a.e in } \ \Omega\},$$

where the constant $\kappa \ge 1$ satisfies

$$|\sigma_i|,\ |\varepsilon_{i-1}|,\ 2(|\theta_{i-1}| + 2\theta_e),\ \theta_c \ \le \kappa \ \text{ a.e. in } \ \Omega.$$

For any fixed $\tilde{\theta} \in K$ we may find a unique $\varepsilon \in H$ such that almost everywhere

$$\big(\nu(\tilde{\theta}) + \tau_i c\big)\varepsilon = \tau_i\big(\sigma_i + \alpha c\big(\varphi(\tilde{\theta}) - \theta_c\big)\big) + \nu(\tilde{\theta})\varepsilon_{i-1}.$$

Moreover, we readily check that

$$\nu_*\left|\frac{\varepsilon - \varepsilon_{i-1}}{\tau_i}\right| \le |\sigma_i| + \alpha c(|\tilde{\theta}| + \theta_c) + c\tau_i\left|\frac{\varepsilon - \varepsilon_{i-1}}{\tau_i}\right| + c|\varepsilon_{i-1}|.$$

In particular, for τ small enough, say $\tau \le \tau_*$, we can easily find a positive constant C_1 such that

$$\left|\frac{\varepsilon - \varepsilon_{i-1}}{\tau_i}\right| \le C_1\kappa \ \text{ a.e. in } \ \Omega.$$

We have implicitly defined the mapping $T_1 : K \to L^\infty(\Omega)$ as $T_1(\tilde{\theta}) := \varepsilon$.

On the other hand, given $(\tilde{\theta}, \tilde{\varepsilon}) \in H \times L^\infty(\Omega)$, we may find the unique almost everywhere solution $\theta \in W$ to

$$c_s\theta - \tau_i k\theta_{xx} = \tau_i\left(-\alpha c\varphi(\tilde{\theta})\left(\frac{\tilde{\varepsilon} - \varepsilon_{i-1}}{\tau_i}\right) + \nu(\tilde{\theta})\left(\frac{\tilde{\varepsilon} - \varepsilon_{i-1}}{\tau_i}\right)^2\right) + c_s\theta_{i-1},$$

$$k\theta_x(0) - h(\theta(0) - \theta_e) = k\theta_x(1) + h(\theta(1) - \theta_e) = 0.$$

In particular, via $T_2(\tilde{\theta}, \tilde{\varepsilon}) := \theta$, we have defined the mapping $T_2 : H \times L^\infty(\Omega) \to W$. Moreover, let us denote by $f := -\alpha c c_s^{-1}\varphi(\tilde{\theta})(\tilde{\varepsilon} - \varepsilon_{i-1})/\tau_i + c_s^{-1}\nu(\tilde{\theta})((\tilde{\varepsilon} - \varepsilon_{i-1})/\tau_i)^2$, $\mu := c_s^{-1}h$, fix $p > 1$, multiply the latter equation by $|\theta|^{p-2}\theta$ and integrate over Ω. Denoting by p' the conjugate exponent of p (i.e., $1/p + 1/p' = 1$) we readily get that

$$\int_\Omega |\theta|^p + \mu\tau_i \sum_{j=0}^{1} |\theta(j)|^p \le \int_\Omega \left(\tau_i f + \theta_{i-1}\right)\left(|\theta|^{p-2}\theta\right) + \mu\tau_i \sum_{j=0}^{1} \theta_e |\theta(j)|^{p-2}\theta(j)$$

$$\le \frac{1}{p'} \| \, |\theta|^{p-2}\theta \, \|^{p'}_{L^{p'}(\Omega)} + \frac{\mu\tau_i}{p'} \sum_{j=0}^{1} |\, |\theta(j)|^{p-2}\theta(j)|^{p'} + \frac{1}{p} \|\tau_i f + \theta_{i-1}\|^p_{L^p(\Omega)} + \frac{2\mu\tau_i}{p}\theta_e^p.$$

Hence

$$\|\theta\|_{L^p(\Omega)} \le \|\tau_i f + \theta_{i-1}\|_{L^p(\Omega)} + (2\mu\tau_i)^{1/p}\theta_e \le \tau_i \|f\|_{L^\infty(\Omega)} + \|\theta_{i-1}\|_{L^\infty(\Omega)} + (2\mu\tau_i)^{1/p}\theta_e,$$

and, since $r^{1/p} \le 1 + r$ for all $r \ge 0$, $p \ge 1$, there exists a positive constant C_2 just depending on data and in particular on C_1 such that

$$\|\theta\|_{L^\infty(\Omega)} \le \tau_i C_2 \kappa^2 + \|\theta_{i-1}\|_{L^\infty(\Omega)} + \theta_e + 2\mu\tau_i\theta_e.$$

Finally, it suffices to choose

$$\tau \le \tau_{**} := \min\{\tau_*, \, 1/(2C_2\kappa), 1/(2\mu)\},$$

In order to have that the mapping $T : K \to W$ defined as $T(\tilde\theta) := T_2(\tilde\theta, T_1(\tilde\theta)) = \theta$ takes values in the non-empty, closed, and convex set K.

We shall now prove that T is a contraction in H for sufficiently small τ. To this aim, let us fix $\tilde\theta_1, \tilde\theta_2 \in K$ and define $\varepsilon_j := T_1(\tilde\theta_j)$, $\theta_j := T(\tilde\theta_j)$ for $j = 1, 2$, and $\overline{\varepsilon} := \varepsilon_1 - \varepsilon_2$, $\overline{\theta} := \theta_1 - \theta_2$, $\overline{\overline{\theta}} := \tilde\theta_1 - \tilde\theta_2$. Next we may write

$$\nu(\tilde\theta_1)\varepsilon_2 + \tau_i c \varepsilon_2 = \tau_i\big(\sigma_i + \alpha c\big(\varphi(\tilde\theta_2) - \theta_c\big)\big) + \big(\nu(\tilde\theta_1) - \nu(\tilde\theta_2)\big)\varepsilon_2 + \nu(\theta_2)\varepsilon_{i-1},$$

and, by taking the difference between the latter relation and the analogous for $j = 1$, and recalling the boundedness properties of T_1, we readily obtain that

$$\nu_* \left| \frac{\varepsilon_1 - \varepsilon_2}{\tau_1} \right| = \frac{\nu_*}{\tau_i}|\overline{\varepsilon}| \le \big(2\alpha c + \|\nu'\|_{L^\infty(\mathbb{R})} C_1 \kappa\big)|\overline{\overline{\theta}}| + c|\overline{\varepsilon}| \quad \text{a.e. in } \Omega.$$

On the other hand, owing to the definition of T_2 we readily check that

$$|\overline{\theta}| \le \frac{\tau_i}{c_s}\left(2\alpha c C_1 \kappa |\overline{\overline{\theta}}| + \alpha c \kappa |\overline{\varepsilon}|/\tau_i + \|\nu'\|_{L^\infty(\mathbb{R})} C_1^2 \kappa^2 |\overline{\overline{\theta}}| + 2\nu^* C_1 \kappa |\overline{\varepsilon}|/\tau_i\right) \quad \text{a.e. in } \Omega,$$

so that, taking into account the above discussion, we may easily find a suitable small τ_{***} such that, for all $\tau \le \min\{\tau_{**}, \tau_{***}\}$, the mapping T is a contraction in H and therefore has a unique fixed point $\theta \in W$. In order to conclude the proof it suffices to observe that, since φ is Lipschitz continuous, one has that $\varepsilon := T_1(\theta) \in V$.

Let us moreover stress that the above constructed solution of DP depends continuously on data in suitable spaces. We omit the proof of this fact for the sake of simplicity.

7 Stability

Let us now turn to the proof of Lemma 5.2 by providing some a priori bounds on the solution of DP. Throughout this section C stands for any positive constant, possibly depending on data but independent of $\mathcal{P}$. Of course C may vary from line to line.

7.1 Lower bound

First of all we shall prove by induction some uniform and positive lower bound on θ_i. In particular, let $\theta_{i-1} \geq \theta_*/2$ on $\overline{\Omega}$ where θ_* is exactly the parameter in the definition of φ. Then, we multiply (5.3) by the function $-\tau_i(\theta_i - \theta_*/2)^- := \tau_i \min\{\theta - \theta_*/2,\, 0\} \in V$, integrate over Ω, and obtain

$$c_s|(\theta_i - \theta_*/2)^-|^2 - \tau_i h \sum_{j=0}^{1} (\theta_i(j) - \theta_e)(\theta_i(j) - \theta_*/2)^- \leq \alpha c \tau_i \int_\Omega \varphi(\theta_i)\delta\varepsilon_i(\theta_i - \theta_*/2)^-.$$

The right-hand side above vanishes due to the definition of φ and the assertion follows.

7.2 First estimate

Let us sum (5.3) to (5.4) multiplied by $\delta\varepsilon_i$. We multiply the result by τ_i, integrate over Ω, and take the sum for $i = 1, \ldots, m$, obtaining

$$c_s \int_\Omega \theta_m + \frac{c}{2}|\varepsilon_m|^2 \leq \sum_{i=1}^{m} \tau_i\sigma_i\delta\varepsilon_i - \alpha c\theta_c \int_\Omega (\varepsilon_m - \varepsilon^0_{\mathcal{P}}) + c_s \int_\Omega \theta^0_{\mathcal{P}} + \frac{c}{2}|\varepsilon^0_{\mathcal{P}}|^2 + 2hT\theta_e.$$

The first term in the above right-hand side shall be controlled by means of a discrete integration by parts as follows

$$\sum_{i=1}^{m} \tau_i\sigma_i\delta\varepsilon_i \leq |\sigma_m|\,|\varepsilon_m| + |\sigma_0|\,|\varepsilon^0_{\mathcal{P}}| + \sum_{i=1}^{m} \tau_i|\delta\sigma_i|\,|\varepsilon_{i-1}|.$$

In particular, it is a standard matter to deduce the bounds

$$\|\overline{\theta}_{\mathcal{P}}\|_{L^\infty(0,T;L^1(\Omega))} + \|\overline{\varepsilon}_{\mathcal{P}}\|_{L^\infty(0,T;H)} \leq C. \tag{7.1}$$

For later convenience, let us observe that (5.4) yields in particular

$$\nu_*|\delta\varepsilon_i| \leq C\left(|\sigma_i| + \alpha c(|\theta_i| + \theta_c) + |\varepsilon^0_{\mathcal{P}}|\right) \quad \text{a.e. in } \Omega, \tag{7.2}$$

by taking τ small enough and exploiting the discrete Gronwall lemma. Hence, (5.1) and (7.1) ensure that

$$\|\overline{e}_{\mathcal{P}}\|_{L^\infty(0,T;L^1(\Omega))} \leq C. \tag{7.3}$$

7.3 Second estimate

We multiply (5.3) by $-\tau_i/\theta_i$ (recall the lower bound on θ_i), integrate over Ω, and take the sum for $i = 1, \ldots, m$. Owing to the concavity of the logarithm we get

$$-c_s \int_\Omega \ln\theta_i + k\sum_{i=1}^{m} \tau_i|\theta_{i,x}/\theta_i|^2 \leq -c_s \int_\Omega \ln\theta^0_{\mathcal{P}} + \alpha c \sum_{i=1}^{m} \tau^i \int_\Omega |\delta\varepsilon_i| + 2hT.$$

Thus, also using (7.1)-(7.3), the latter relation yields

$$\|(\ln\overline{\theta}_{\mathcal{P}})_x\|_{L^2(0,T;H)} \leq C. \tag{7.4}$$

Let us now recall the continuity of the injection of $W^{1,1}(\Omega)$ into $L^\infty(\Omega)$ and perform the standard computation

$$\begin{aligned}\|\theta_i\|_{L^\infty(\Omega)} &= \|\theta^{1/2}\|^2_{L^\infty(\Omega)} \le 2\|\theta^{1/2}\|^2_{L^1(\Omega)} + 2\|(\theta^{1/2})_x\|^2_{L^1(\Omega)} \\ &\le 2|\theta^{1/2}|^2 + \frac{1}{2}\left(\int_\Omega \frac{(\theta_i)_x}{\theta_i}\theta_i^{1/2}\right)^2 \le 2\|\theta_i\|_{L^1(\Omega)} + \frac{1}{2}|(\ln\theta_i)_x|^2|\theta^{1/2}|^2 \\ &= 2\|\theta_i\|_{L^1(\Omega)} + \frac{1}{2}|(\ln\theta_i)_x|^2\|\theta_i\|_{L^1(\Omega)},\end{aligned}$$

and exploit (7.1) and (7.4) in order to deduce that

$$\|\overline{\theta}_{\mathcal{P}}\|_{L^1(0,T;L^\infty(\Omega))} \le C. \tag{7.5}$$

Finally, by elementary interpolation and (7.1) we conclude for

$$\|\overline{\theta}_{\mathcal{P}}\|_{L^2(0,T;H)} \le C. \tag{7.6}$$

7.4 Third estimate

This is nothing but the energy estimate. Indeed, we multiply (5.3) by $\tau_i\theta_i$, integrate over Ω, and take the sum for $i = 1, \ldots, m$. Owing to (5.1) and (7.1)-(7.2) we readily get that

$$\frac{c_s}{2}|\theta_m|^2 + k\sum_{i=1}^m \tau_i|\theta_{i,x}|^2 + h\sum_{\substack{i=1,\ldots,m\\ j=0,1}} \tau_i\theta_i^2(j) \le C\sum_{i=1}^m \tau_i\int_\Omega \theta_i^3 + h\sum_{\substack{i=1,\ldots,m\\ j=0,1}} \tau_i\theta_e\theta_i(j) + C.$$

In order to control the cubic term above we just exploit (7.1) and the compactness injection of V into L^∞. In particular, for all $\delta > 0$ there exists a positive constant C_δ such that

$$\begin{aligned}&\|\overline{\theta}_{\mathcal{P}}\|^3_{L^3(0,T;L^3(\Omega))} \le \|\overline{\theta}_{\mathcal{P}}\|_{L^\infty(0,T;L^1(\Omega))}\|\overline{\theta}^2_{\mathcal{P}}\|_{L^1(0,T;L^\infty(\Omega))} \\ &\le C\|\overline{\theta}_{\mathcal{P}}\|^2_{L^2(0,T;L^\infty(\Omega))} \le \delta\|\overline{\theta}_{\mathcal{P},x}\|^2_{L^2(0,T;H)} + C_\delta\|\overline{\theta}_{\mathcal{P}}\|^2_{L^2(0,T;H)}.\end{aligned}$$

Hence, by choosing $\delta < k$ and τ small enough, and applying the discrete Gronwall lemma we deduce that

$$\|\overline{\theta}_{\mathcal{P}}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C. \tag{7.7}$$

7.5 Fourth estimate

We multiply (5.3) by $\tau_i\delta\theta_i$, integrate over Ω, and take the sum for $i = 1, \ldots, m$, getting

$$\begin{aligned}c_s\sum_{i=1}^m \tau_i|\delta\theta_i|^2 + \frac{k}{2}|\theta_{m,x}|^2 + \frac{h}{2}\sum_{j=0}^1(\theta_m(j) - \theta_e)^2 &\le \frac{k}{2}|\theta^0_{\mathcal{P},x}|^2 + \frac{h}{2}\sum_{j=0}^1(\theta^0_{\mathcal{P}}(j) - \theta_e)^2 \\ &+ \sum_{i=1}^m \tau_i\big(\alpha c\|\theta_i\|_{L^4(\Omega)}\|\delta\varepsilon_i\|_{L^4(\Omega)} + \nu^*|(\delta\varepsilon_i)^2|\big)|\delta\theta_i| \\ &\le C + C\sum_{i=1}^m \tau_i\|\theta_i\|^4_{L^4(\Omega)} + \frac{c_s}{2}\sum_{i=1}^m \tau_i|\delta\theta_i|^2,\end{aligned}$$

where we also exploited (5.1) and (7.2). By observing that, thanks to (7.7) and some standard interpolation, one has that

$$\sum_{i=1}^{N} \tau_i \|\theta_i\|^4_{L^4(\Omega)} \leq \|\overline{\theta}_{\mathcal{P}}\|^2_{L^\infty(0,T;H)} \|\overline{\theta}_{\mathcal{P}}\|^2_{L^2(0,T;L^\infty(\Omega))} \leq C,$$

and we readily obtain the bound

$$\|\dot{\theta}_{\mathcal{P}}\|_{L^2(0,T;H)} + \|\overline{\theta}_{\mathcal{P}}\|_{L^\infty(0,T;V)} \leq C. \tag{7.8}$$

In particular, again using (5.1) and (7.2),

$$\|\overline{\theta}_{\mathcal{P}}\|_{L^\infty(Q)} + \|\dot{\varepsilon}_{\mathcal{P}}\|_{L^\infty(Q)} \leq C, \tag{7.9}$$

and, by comparison in (5.3) and standard elliptic estimates,

$$\|\overline{\theta}_{\mathcal{P}}\|_{L^2(0,T;W)} \leq C, \tag{7.10}$$

(actually much more is true).

7.6 Fifth estimate

We shall define

$$\delta\varepsilon_0 = e_0 := \big(\sigma_0 + \alpha c(\theta_0 - \theta_c) - c\varepsilon_0\big) / \nu(\theta_0),$$

in such a way that relation (5.4) is fulfilled for $i = 0$ as well. Next, let us take the difference between (5.4) and the same relation written for the index $i-1$, multiply by δe_i, integrate over Ω, and take the sum for $i = 1, \ldots, m$. One has

$$\nu_* \sum_{i=1}^{m} \tau_i |\delta e_i|^2 \leq \sum_{i-1}^{m} \tau_i \int_\Omega \left(\delta\sigma_i \delta e_i - \frac{\nu(\theta_i) - \nu(\theta_{i-1})}{\tau_i} e_{i-1}\delta e_i - c e_i \delta e_i + 2\alpha c |\delta\theta_i|\, |\delta e_i| \right)$$
$$\leq \frac{\nu_*}{2} \sum_{i=1}^{m} \tau_i |\delta e_i|^2 + C,$$

where we just made use of (5.1), (7.8)-(7.9). In particular, we are entitled to deduce the bound

$$\|\dot{e}_{\mathcal{P}}\|_{L^2(0,T;H)} \leq C. \tag{7.11}$$

7.7 Sixth estimate

By exploiting (5.1), (7.8)-(7.9), and performing a comparison in (5.4) we readily get that

$$\|\overline{e}_{\mathcal{P},x}\|_{L^\infty(0,T;H)} \leq C(1 + \|\overline{\varepsilon}_{\mathcal{P},x}\|_{L^\infty(0,T;H)}).$$

Hence, at least for sufficiently small diameters τ, it suffices to apply the discrete Gronwall lemma in order to conclude for the bound

$$\|\overline{e}_{\mathcal{P},x}\|_{L^\infty(0,T;H)} \leq C. \tag{7.12}$$

8 Convergence

Let us turn to the proof of Lemma 5.3. For the sake of later convenience, let us we rewrite (5.3)-(5.4) in the more compact form

$$c_s\dot{\theta}_{\mathcal{P}} - k\overline{\theta}_{\mathcal{P},x} = -\alpha c\varphi(\overline{\theta}_{\mathcal{P}})\overline{e}_{\mathcal{P}} + \nu(\overline{\theta}_{\mathcal{P}})(\overline{e}_{\mathcal{P}})^2, \tag{8.1}$$

$$\nu(\overline{\theta}_{\mathcal{P}})\overline{e}_{\mathcal{P}} + c\overline{\varepsilon}_{\mathcal{P}} = \overline{\sigma}_{\mathcal{P}} + \alpha c\big(\varphi(\overline{\theta}_{\mathcal{P}}) - \theta_c\big). \tag{8.2}$$

Next, we let the diameter τ of partition $\mathcal{P}$ go to zero. Taking advantage of (5.7) and well-known compactness results [22], we are actually in the position of finding (θ, ε) such that (possibly taking not relabeled subsequences) the following convergences hold

$$\begin{aligned}
\theta_{\mathcal{P}} &\to \theta \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \\
&\qquad \text{and strongly in } C([0,T];H) \cap L^2(0,T;V), && (8.3)\\
\overline{\theta}_{\mathcal{P}} &\to \theta \quad \text{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;W) \\
&\qquad \text{and strongly in } L^\infty(0,T;H) \cap L^2(0,T;V), && (8.4)\\
\varepsilon_{\mathcal{P}} &\to \varepsilon \quad \text{weakly star in } W^{1,\infty}(0,T;V), && (8.5)\\
e_{\mathcal{P}} &\to \dot{\varepsilon} \quad \text{weakly in } H^1(0,T;H) \cap L^2(0,T;V) \\
&\qquad \text{and strongly in } C([0,T];H). && (8.6)\\
\overline{e}_{\mathcal{P}} &\to \dot{\varepsilon} \quad \text{weakly in } L^2(0,T;V) \\
&\qquad \text{and strongly in } L^\infty(0,T;H). && (8.7)
\end{aligned}$$

Hence, also owing to (5.2), we readily pass to the limit in (8.1)-(8.2) and obtain that indeed (θ, ε) solve

$$c_s\dot{\theta} - k\theta_{xx} = -\alpha c\varphi(\theta)\dot{\varepsilon} + \nu(\theta)\dot{\varepsilon}^2, \tag{8.8}$$

$$\nu(\theta)\dot{\varepsilon} + c\varepsilon = \sigma + \alpha c\big(\varphi(\theta) - \theta_c\big), \tag{8.9}$$

as well as (1.4)-(1.5) at least almost everywhere. The latter problem may be proved to have unique solutions by arguing exactly as in the proof of Lemma 3.2. Hence, the above stated convergences turn out to hold for the whole sequence and not only for some extracted subsequence.

We shall conclude the proof of Lemma 5.3 by providing a suitable θ_* such that any solution (θ, ε) of the problem (8.8)-(8.9) fulfills $\theta \geq \theta_*$ everywhere. Indeed, owing to the definition of φ, in this case (1.1)-(1.2) and (8.8)-(8.9) coincide. Namely, we need to establish an a priori positive lower bound for θ. This actually follows along the very same lines of Section 3 upon suitably choosing the test function ζ as

$$\zeta(t) := -\left(\theta(t) - \underline{\theta}\exp\left(-\frac{\alpha c}{c_s}\int_0^t \frac{\varphi(\theta(s))}{\theta(s)}\|\dot{\varepsilon}(s)\|_{L^\infty(\Omega)}\,ds\right)\right)^- .$$

Omitting the details for the sake of brevity, it suffices to let

$$\theta_* := \underline{\theta}\,\exp\left(-\frac{\alpha c}{c_s}\int_0^T \frac{\varphi(\theta(s))}{\theta(s)}\|\dot{\varepsilon}(s)\|_{L^\infty(\Omega)}\,ds\right)$$

in order to prove that the above established solution to problem (8.8)-(8.9) fulfills $\theta \geq \theta_*$ thus solving problem CP as well.

9 Error control

Let us now turn to the proof of Lemma 5.4. Henceforth, C will stand for any positive constant depending on data and, in particular, on $(\sigma, \theta^0, \varepsilon^0)$ and $(\overline{\sigma}_{\mathcal{P}}, \theta^0_{\mathcal{P}}, \varepsilon^0_{\mathcal{P}})$. To be more precise, let $C_3 > 0$ be such that

$$\|\sigma\|_{L^2(0,T;V)} + \|\overline{\sigma}_{\mathcal{P}}\|_{L^2(0,T;V)} + |\theta^0| + |\theta^0_{\mathcal{P}}| + \|\varepsilon^0\|_V + \|\varepsilon^0_{\mathcal{P}}\|_V \le C_3.$$

Then all the forthcoming constants indicated with C will possibly depend on $c_s, k, h, \alpha, c, \nu, T$, and C_3 but not on $\mathcal{P}$.

First of all, we shall observe that Lemma 5.2 entails in particular that

$$\|\theta_{\mathcal{P}} - \overline{\theta}_{\mathcal{P}}\|_{L^2(0,T;H)} + \|\varepsilon_{\mathcal{P}} - \overline{\varepsilon}_{\mathcal{P}}\|_{L^2(0,T;V)} \le C\tau. \tag{9.1}$$

Let us now take the difference of equations (1.1) and (8.1), take the integral over $(0,s)$ for $s \in (0,T]$, multiply by the function $\theta - \overline{\theta}_{\mathcal{P}}$, and take the integral again over $\Omega \times (0,t)$ for $t \in (0,T]$. One has that

$$c_s \int_0^t |\theta - \theta_{\mathcal{P}}|^2 + \frac{k}{2} \left| \int_0^t (\theta - \overline{\theta}_{\mathcal{P}})_x \right|^2 + \frac{h}{2} \sum_{j=0}^{1} \left(\int_0^t (\theta - \overline{\theta}_{\mathcal{P}})(j) \right)^2 = \sum_{i=1}^{6} I_i(t), \tag{9.2}$$

where, letting $e := \dot{\varepsilon}$,

$$\begin{aligned}
I_1(t) &:= c_s \int_0^t (\theta^0 - \theta^0_{\mathcal{P}}, \theta - \overline{\theta}_{\mathcal{P}}), \\
I_2(t) &:= c_s \int_0^t (\theta - \theta_{\mathcal{P}}, \overline{\theta}_{\mathcal{P}} - \theta_{\mathcal{P}}), \\
I_3(t) &:= -\alpha c \int_0^t \left(\int_0^s (\theta - \overline{\theta}_{\mathcal{P}}) e, \theta - \overline{\theta}_{\mathcal{P}} \right), \\
I_4(t) &:= -\alpha c \int_0^t \left(\int_0^s (e - \overline{e}_{\mathcal{P}}) \overline{\theta}_{\mathcal{P}}, \theta - \overline{\theta}_{\mathcal{P}} \right), \\
I_5(t) &:= \int_0^t \left(\int_0^s (\nu(\theta) - \nu(\overline{\theta}_{\mathcal{P}})) e^2, \theta - \overline{\theta}_{\mathcal{P}} \right), \\
I_6(t) &:= \int_0^t \left(\int_0^s \nu(\overline{\theta}_{\mathcal{P}})(e + \overline{e}_{\mathcal{P}})(e - \overline{e}_{\mathcal{P}}), \theta - \overline{\theta}_{\mathcal{P}} \right).
\end{aligned}$$

Next, also owing to (5.7) and (9.1), it is a standard matter to check that

$$\begin{aligned}
I_1(t) &\le \frac{c_s}{5} \int_0^t |\theta - \theta_{\mathcal{P}}|^2 + C \left(|\theta^0 - \theta^0_{\mathcal{P}}|^2 + \tau^2 \right), \\
I_2(t) &\le \frac{c_s}{5} \int_0^t |\theta - \theta_{\mathcal{P}}|^2 + C\tau^2, \\
I_3(t) + I_5(t) &\le \frac{c_s}{5} \int_0^t |\theta - \theta_{\mathcal{P}}|^2 + C \left(\int_0^t \|\theta - \theta_{\mathcal{P}}\|^2_{L^2(0,s;H)} + \tau^2 \right), \\
I_4(t) + I_6(t) &\le \frac{c_s}{5} \int_0^t |\theta - \theta_{\mathcal{P}}|^2 + C \left(\int_0^t \|e - \overline{e}_{\mathcal{P}}\|^2_{L^2(0,s;H)} + \tau^2 \right),
\end{aligned}$$

so that (9.2) reduces to

$$\int_0^t |\theta - \theta_{\mathcal{P}}|^2 + \left| \int_0^t (\theta - \overline{\theta}_{\mathcal{P}})_x \right|^2 + \sum_{j=0}^{1} \left(\int_0^t (\theta - \overline{\theta}_{\mathcal{P}})(j) \right)^2$$
$$\leq C \left(|\theta^0 - \theta^0_{\mathcal{P}}|^2 + \int_0^t \|\theta - \theta_{\mathcal{P}}\|^2_{L^2(0,s;H)} + \int_0^t \|e - \overline{e}_{\mathcal{P}}\|^2_{L^2(0,s;H)} + \tau^2 \right). \tag{9.3}$$

On the other hand, one easily computes that, for all $t \in (0, T]$,

$$|(\varepsilon - \overline{\varepsilon}_{\mathcal{P}})(t)| \leq C \left(|(\varepsilon - \varepsilon_{\mathcal{P}})(t)| + |\varepsilon^0 - \varepsilon^0_{\mathcal{P}}| + \int_0^t |e - \overline{e}_{\mathcal{P}}| \right).$$

Hence, using (9.1), we get that

$$\int_0^t |\varepsilon - \overline{\varepsilon}_{\mathcal{P}}|^2 \leq \left(|\varepsilon^0 - \varepsilon^0_{\mathcal{P}}|^2 + \int_0^t \|e - \overline{e}_{\mathcal{P}}\|_{L^2(0,s;H)} + \tau^2 \right) \tag{9.4}$$

Let us now take the difference between (1.2) and (8.2), multiply it by $e - \overline{e}_{\mathcal{P}}$, and take the integral over $\Omega \times (0, t)$ for $t \in (0, T]$. We obtain

$$\int_0^t (\nu(\theta)(e - \overline{e}_{\mathcal{P}}), e - \overline{e}_{\mathcal{P}}) = \int_0^t (\sigma - \overline{\sigma}_{\mathcal{P}}, e - \overline{e}_{\mathcal{P}}) + \alpha c \int_0^t (\theta - \overline{\theta}_{\mathcal{P}}, e - \overline{e}_{\mathcal{P}})$$
$$-c \int_0^t (\varepsilon - \overline{\varepsilon}, e - \overline{e}_{\mathcal{P}}) + \int_0^t ((\nu(\overline{\theta}_{\mathcal{P}}) - \nu(\theta))\overline{e}_{\mathcal{P}}, e - \overline{e}_{\mathcal{P}}).$$

Hence, again owing to (5.7), (9.1), and (9.4), one can handle the latter right hand side in order to get that

$$\frac{\nu_*}{2} \int_0^t |e - \overline{e}_{\mathcal{P}}|^2 \leq C \left(\int_0^t \left(|\sigma - \overline{\sigma}_{\mathcal{P}}|^2 + |\theta - \theta_{\mathcal{P}}|^2 + |\varepsilon - \overline{\varepsilon}_{\mathcal{P}}|^2 \right) + \tau^2 \right)$$
$$\leq C \left(\int_0^t \left(|\sigma - \overline{\sigma}_{\mathcal{P}}|^2 + |\theta - \theta_{\mathcal{P}}|^2 + \|e - \overline{e}_{\mathcal{P}}\|^2_{L^2(0,s;H)} \right) + |\varepsilon^0 - \varepsilon^0_{\mathcal{P}}|^2 + \tau^2 \right). \tag{9.5}$$

Combining now (9.3) and (9.5) we deduce that

$$\int_0^t |\theta - \theta_{\mathcal{P}}|^2 + \left| \int_0^t (\theta - \overline{\theta}_{\mathcal{P}})_x \right|^2 + \sum_{j=0}^{1} \left(\int_0^t (\theta - \overline{\theta}_{\mathcal{P}})(j) \right)^2 + \int_0^t |e - \overline{e}_{\mathcal{P}}|^2$$
$$\leq C \left(|\theta^0 - \theta^0_{\mathcal{P}}|^2 + |\varepsilon^0 - \varepsilon^0_{\mathcal{P}}|^2 + \int_0^t |\sigma - \overline{\sigma}_{\mathcal{P}}|^2 \right)$$
$$+C \left(\int_0^t \|\theta - \theta_{\mathcal{P}}\|^2_{L^2(0,s;H)} + \int_0^t \|e - \overline{e}_{\mathcal{P}}\|^2_{L^2(0,s;H)} + \tau^2 \right),$$

and (5.11) follows from an application of Gronwall's lemma.

As for (5.12) we simply take the difference of equations (1.1) and (8.1), multiply it by $\theta - \overline{\theta}$, and take the integral over $\Omega \times (0, t)$ for $t \in (0, T]$ obtaining

$$\frac{c_s}{2} |(\theta - \theta_{\mathcal{P}})(t)|^2 + k \int_0^t |(\theta - \overline{\theta}_{\mathcal{P}})_x|^2 + h \sum_{j=0}^{1} (\theta - \overline{\theta}_{\mathcal{P}})^2(j) = \frac{c_s}{2} |\theta^0 - \theta^0_{\mathcal{P}}|^2 + \sum_{i=7}^{11} I_i(t), \tag{9.6}$$

where

$$\begin{aligned} I_7(t) &:= c_s \int_0^t (\dot\theta - \dot\theta_{\mathcal{P}}, \overline{\theta}_{\mathcal{P}} - \theta_{\mathcal{P}}), \\ I_8(t) &:= -\alpha c \int_0^t ((\theta - \overline{\theta}_{\mathcal{P}})e, \theta - \overline{\theta}_{\mathcal{P}}), \\ I_9(t) &:= -\alpha c \int_0^t (\overline{\theta}_{\mathcal{P}}(e - \overline{e}_{\mathcal{P}}), \theta - \overline{\theta}_{\mathcal{P}}), \\ I_{10}(t) &:= \int_0^t ((\nu(\theta) - \nu(\overline{\theta}_{\mathcal{P}}))e^2, \theta - \overline{\theta}_{\mathcal{P}}), \\ I_{11}(t) &:= \int_0^t (\nu(\overline{\theta}_{\mathcal{P}})(e + \overline{e}_{\mathcal{P}})(e - \overline{e}_{\mathcal{P}}), \theta - \overline{\theta}_{\mathcal{P}}). \end{aligned}$$

Next, taking into account (5.7), (9.1), and the established (5.11), we control

$$\begin{aligned} I_7(t) &\leq C \left(\int_0^t |\theta_{\mathcal{P}} - \overline{\theta}_{\mathcal{P}}|^2 \right)^{1/2} \leq C\tau, \\ I_8(t) + I_{10}(t) &\leq C \int_0^t |\theta - \overline{\theta}_{\mathcal{P}}|^2, \\ I_9(t) + I_{11}(t) &\leq C \int_0^t \left(|\theta - \overline{\theta}_{\mathcal{P}}|^2 + |e - \overline{e}_{\mathcal{P}}|^2 \right). \end{aligned}$$

Finally, relation (9.6) yields

$$\begin{aligned} &|(\theta - \theta_{\mathcal{P}})(t)|^2 + \int_0^t |(\theta - \overline{\theta}_{\mathcal{P}})_x|^2 + \sum_{j=0}^{1} (\theta - \overline{\theta}_{\mathcal{P}})^2(j) \\ &\leq C \left(|\theta^0 - \theta^0_{\mathcal{P}}|^2 + |\varepsilon^0 - \varepsilon^0_{\mathcal{P}}|^2 + \int_0^t |\sigma - \overline{\sigma}_{\mathcal{P}}|^2 + \tau^2 \right) + C\tau, \end{aligned}$$

and estimate (5.12) follows.

Let us now check for relation (5.13). To this aim we take the difference between (1.2) and (8.2), differentiate with respect to space, multiply by $(e - \overline{e}_{\mathcal{P}})_x$, and take the integral over $\Omega \times (0,t)$ for $t \in (0,T]$ obtaining

$$\int_0^t (\nu(\theta)(e - \overline{e}_{\mathcal{P}})_x, (e - \overline{e}_{\mathcal{P}})_x) = \sum_{i=12}^{18} I_i(t), \tag{9.7}$$

where

$$
\begin{aligned}
I_{12} &:= \int_0^t ((\sigma - \overline{\sigma}_{\mathcal{P}})_x, (e - \overline{e}_{\mathcal{P}})_x), \\
I_{13} &:= \alpha c \int_0^t ((\theta - \overline{\theta}_{\mathcal{P}})_x, (e - \overline{e}_{\mathcal{P}})_x), \\
I_{14} &:= -c \int_0^t ((\varepsilon - \overline{\varepsilon}_{\mathcal{P}})_x, (e - \overline{e}_{\mathcal{P}})_x), \\
I_{15} &:= \int_0^t ((\nu(\overline{\theta}_{\mathcal{P}}) - \nu(\theta))\overline{e}_{\mathcal{P},x}, (e - \overline{e}_{\mathcal{P}})_x), \\
I_{16} &:= \int_0^t ((\nu'(\overline{\theta}_{\mathcal{P}}) - \nu'(\theta))\overline{\theta}_{\mathcal{P},x}\overline{e}_{\mathcal{P}}, (e - \overline{e}_{\mathcal{P}})_x), \\
I_{17} &:= \int_0^t (\nu'(\theta)(\overline{\theta}_{\mathcal{P}} - \theta)_x\overline{e}_{\mathcal{P}}, (e - \overline{e}_{\mathcal{P}})_x), \\
I_{18} &:= \int_0^t (\nu'(\theta)\theta_x(\overline{e}_{\mathcal{P}} - e), (e - \overline{e}_{\mathcal{P}})_x).
\end{aligned}
$$

Once more owing to (5.7) and assuming $\nu \in W^{2,\infty}(\mathbb{R})$ one has that

$$
\begin{aligned}
I_{12} &\leq \frac{\nu_*}{7}\int_0^t |(e - \overline{e}_{\mathcal{P}})_x|^2 + C\int_0^t |(\sigma - \overline{\sigma}_{\mathcal{P}})_x|^2, \\
I_{13} + I_{17}(t) &\leq \frac{\nu_*}{7}\int_0^t |(e - \overline{e}_{\mathcal{P}})_x|^2 + C\int_0^t |(\theta - \overline{\theta}_{\mathcal{P}})_x|^2, \\
I_{14}(t) &\leq \frac{\nu_*}{7}\int_0^t |(e - \overline{e}_{\mathcal{P}})_x|^2 + C\int_0^t |(\varepsilon - \overline{\varepsilon}_{\mathcal{P}})_x|^2, \\
I_{15}(t) &\leq \frac{\nu_*}{7}\int_0^t |(e - \overline{e}_{\mathcal{P}})_x|^2 + C\|\theta - \overline{\theta}_{\mathcal{P}}\|^2_{L^2(0,T;L^\infty(\Omega))}, \\
I_{16}(t) &\leq \frac{\nu_*}{7}\int_0^t |(e - \overline{e}_{\mathcal{P}})_x|^2 + C\|\theta - \overline{\theta}_{\mathcal{P}}\|^2_{L^\infty(0,T;H)}, \\
I_{18}(t) &\leq \frac{\nu_*}{7}\int_0^t |(e - \overline{e}_{\mathcal{P}})_x|^2 + C\int_0^t |e - \overline{e}_{\mathcal{P}}|^2.
\end{aligned}
$$

Finally, looking back to (9.7) and exploiting (5.11)-(5.12), one readily concludes for (5.13).

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References

[1] D. Blanchard and O. Guibé. Existence of a solution for a nonlinear system in thermoviscoelasticity. *Adv. Differential Equations*, 5(10-12):1221–1252, 2000.

[2] E. Bonetti and G. Bonfanti. Existence and uniqueness of the solution to a 3D thermo-viscoelastic system. *Electron. J. Differential Equations*, pages No. 50, 15 pp. (electronic), 2003.

[3] M. Brokate and J. Sprekels. *Hysteresis and Phase Transitions.* Number 121 in Appl. Math. Sci. Springer, Berlin, 1996.

[4] C. M. Dafermos. Global smooth solutions to the initial-boundary value problem for the equations of one-dimensional nonlinear thermoviscoelasticity. *SIAM J. Math. Anal.*, 13(3):397–408, 1982.

[5] C. M. Dafermos and L. Hsiao. Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity. *Nonlinear Anal.*, 6(5):435–454, 1982.

[6] A. D. Drozdov. A constitutive model in linear thermoviscoelasticity of polymers based on the concept of cooperative relaxation. *Contin. Mech. Thermodyn.*, 11(3):193–216, 1999.

[7] G. A. Francfort and P.-M. Suquet. Homogenization and mechanical dissipation in thermo-viscoelasticity. *Arch. Rational Mech. Anal.*, 96(3):265–293, 1986.

[8] M. Frémond. *Non-smooth Thermomechanics.* Springer-Verlag, Berlin, 2002.

[9] P. Germain. *Mécanique des milieux continus.* Masson, Paris, 1973.

[10] K. H. Hoffmann, M. Niezgódka, and S. Zheng. Existence and uniqueness to an extended model of the dynamical developments in shape memory alloys. *Nonlinear Anal.*, 15:977–990, 1990.

[11] K.-H. Hoffmann and S. Zheng. Uniqueness for structural phase transitions in shape memory alloys. *Math. Methods Appl. Sci.*, 10(2):145–151, 1988.

[12] K. H. Hoffmann and A. Żochowski. Existence of solutions of some nonlinear thermoelastic systems with viscosity. *Math. Methods Appl. Sci.*, 15(3):187–204, 1992.

[13] S. Jiang. On a finite element method for the equations of one-dimensional nonlinear thermoviscoelasticity. *Computing*, 40(2):111–124, 1988.

[14] S. Jiang. A C^1 finite element collocation method for the equations of one-dimensional nonlinear thermoviscoelasticity. *Math. Comput. Simulation*, 31(3):227–239, 1989.

[15] P. Krejčí, J. Sprekels, and U. Stefanelli. Phase-field models with hysteresis in one dimensional thermo-visco-plasticity. *SIAM J. Math. Anal.*, 34(2):409–434, 2002.

[16] P. Krejčí, J. Sprekels, and U. Stefanelli. One-dimensional thermo-visco-plastic processes with hysteresis and phase transitions. *Adv. Math. Sci. Appl.*, 13(2):695–712, 2003.

[17] J.-L. Lions and E. Magenes. *Non-homogeneus boundary value problems and applications*, volume 1. Springer-Verlag, New York - Heidelberg, 1972.

[18] F. Meissonnier, R. Peyroux, and A. Chrysochoos. Finite element analysis of strain localization induced by thermomechanical couplings in a kelvin-voigt behaviour model. In R. Ohayon, A. Faghri, and A. Chrysochoos, editors, *ESDA'96, ASME International Conference, Montpellier, 1-4.7.96*, pages 207–213, 1996.

[19] J. T. Oden and W. H. Armstrong. Analysis of nonlinear, dynamic coupled thermoviscoelasticity problems by the finite element method. *Computers & Structures*, 1:603–621, 1971.

[20] R. Racke and S. Zheng. Global existence and asymptotic behavior in nonlinear thermoviscoelasticity. *J. Differential Equations*, 134(1):46–67, 1997.

[21] Y. Shibata. Global in time existence of small solutions of nonlinear thermoviscoelastic equations. *Math. Methods Appl. Sci.*, 18(11):871–895, 1995.

[22] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[23] S. J. Watson. Unique global solvability for initial-boundary value problems in one-dimensional nonlinear thermoviscoelasticity. *Arch. Ration. Mech. Anal.*, 153(1):1–37, 2000.

[24] B. Wattrisse, J. M. Muracciole, and A. Chrysochoos. Thermomechanical effects accompanying the localized necking of semi-crystalline polymers. *Int. J. Therm. Sci.*, 41:422–427, 2002.

[25] J. Zimmer. Global existence for a nonlinear system in thermoviscoelasticity with nonconvex energy. *J. Math. Anal. Appl.*, 292(2):589–604, 2004.