A variational principle in non-smooth Mechanics

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Let Y be a Banach space with dual Y^*, $\psi : Y \to (-\infty, \infty]$ be convex, proper, and lower semicontinuous dissipation potential, $\phi \in C^1(Y;\mathbb{R})$ be an energy, and the load $\ell \in L^\infty(0,T;Y^*)$ and the initial state $y_0 \in Y$ be given. A variety of non-smooth (thermo-)mechanical evolution models can be represented by the relation

$$\partial\psi(\dot{y}) + D\phi(y) \ni \ell \quad \text{a.e. in } \ (0,T), \quad y(0) = y_0, \tag{1}$$

where ∂ stands for the subdifferential in the sense of Convex Analysis and $t \mapsto y(t)$ is meant to be in $W^{1,1}(0,T;Y)$. Here $\partial\psi$ represents the system of dissipative forces whereas $D\phi - \ell$ stands for the conservative forces instead. Relation (1) appears in connection with many different applicative situations among which plasticity, visco-elasticity, friction [6], heat conduction, and phase change.

Fenchel's inequality $\psi(u) + \psi^*(v) \geq \langle v, u\rangle$ holds for all $u \in Y$ and $v \in Y^*$ where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between Y^* and Y and ψ^* is the conjugate of ψ. Moreover, the latter inequality reduces to an equality iff $v \in \partial\psi(u)$. Hence, by defining the *Lagrangian* $L : (0,T) \times Y \times Y \to \mathbb{R}$ as

$$L(t,y,p) := \psi(p) + \psi^*(\ell(t) - D\phi(y)) - \langle \ell(t) - D\phi(y), p\rangle$$

for almost every $t \in (0,T)$, one readily checks that $L(t,y,p) \geq 0$ and

$$L(t,y,p) = 0 \ \text{ iff } \ \partial\phi(p) \ni \ell(t) - D\phi(y).$$

Let now the functional $F : W^{1,1}(0,T;Y) \to [0,\infty]$ be defined as

$$F(y) := \int_0^T L(t, y(t), \dot{y}(t))\, dt + \phi(y(0) - y_0).$$

Then, minimizers of F and solutions to (1) coincide. Indeed, as y solves (1), we have that $L(t,y(t),\dot{y}(t)) = 0$ for almost every $t \in (0,T)$ and $y(0) = y_0$. That is $F(y) = 0$. On the contrary, if $F(y) = 0$, then $L(t,y(t),\dot{y}(t)) = 0$ for almost every $t \in (0,T)$ and $\phi(y(0) - y_0) = 0$ and (1) follows. Hence, we have the following.

Theorem 1. *y solves* (1) *iff $F(y) = 0 = \min F$.*

The characterization of solutions of differential problems driven by convex potentials as minimizers of functionals via the Fenchel approach is rather classical and has to be traced back to Brezis & Ekeland [1, 2]. In the referred papers, the Authors provide some similar variational characterization for gradient flows of convex functionals in Hilbert spaces. I have recently extended these ideas to the case of doubly nonlinear equations [7].

In order to extract some information from the variational characterization of Theorem 1, it is convenient to focus on some more specific situation. I shall in particular consider the case of linearized hardening elasto-plasticity [4] by additionally requiring

$$\psi \text{ positively 1-homogeneous and } \phi \text{ quadratic and coercive on } D(\psi),$$

namely the domain of ψ. In this specific situation $D\phi$ is linear and Theorem 1 is providing a useful tool in order to deal with limiting procedures. Indeed, since one reinterprets the equation as a minimum problem, it is clear that the natural concept to be considered is that of Γ-convergence [3]. More precisely, as Theorem 1 directly quantifies the value of the minimum to be 0, what is actually needed are so-called Γ-$\liminf$ inequalities only.

The variational approach by Theorem 1 provides the possibility of recovering convergence in a variety of approximated situations.

Space approximation. Conformal finite elements and data approximations can be considered. In particular, one can easily handle by means of this variational approach the case where Y is replaced by a nested sequence of finite dimensional subspaces Y_h such that $\cup_h Y_h$ is dense in Y, ϕ_h is the restriction of ϕ to Y_h, and $\psi_h \to \psi$ in the sense of Mosco in Y [3].

Time discretization. One can provide a discrete analogue to Theorem 1. Namely, it is possible to variationally characterize solutions of the so-called θ-scheme

$$y^0 = y_0, \quad \psi(y^i - y^{i-1}) + A\big(\theta y^i + (1-\theta)y^{i-1}\big) \ni \ell_\theta^i \quad \text{for} \quad i = 1, \ldots, N,$$

where $\ell_\theta^i = \ell(\theta t^i + (1-\theta)t^{i-1})$ for $0 = t^0 < t^1 < \cdots < t^N = T$, as minimizers of a suitable discrete functional tailored on F. This approach can be exploited in order to prove the unconditional stability of the scheme in $W^{1,\infty}(0,T;Y)$ and its convergence with respect to the corresponding weak-star topology as the diameter of the partition goes to 0. Finally, the discrete functional itself can be used in order to control the discretization error.

Fully discrete schemes. The above-mentioned techniques for space and time discretization can be combined in order to provide the weak-star convergence in $W^{1,\infty}(0,T;Y)$ of some fully discretized solutions as well. This result has to be compared with the former analysis in [5], where a stronger convergence in proved by means of a somehow more complicated argument.

References

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