An order approach to a class of quasivariational sweeping processes

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Abstract

This note deals with the solvability of a class of quasivariational evolution problems in Hilbert spaces. In particular, we address the so-called quasivariational sweeping process and prove suitable well-posedness results by means of an order method.

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1 Introduction

Assume we are given a separable Hilbert space H and a positive reference time T. Moreover, let $C : [0,T] \to 2^H$ be a set-valued mapping with non-empty, convex, and closed values for all times. In his pioneering papers [17, 18], J.-J. MOREAU has proved under suitable assumptions on C the existence and uniqueness of a function $u : [0,T] \to H$ fulfilling

$$u'(t) + \partial I_{C(t)}(u(t)) \ni 0 \quad \text{for } t \in (0,T), \tag{1.1}$$

$$u(0) = u_0. \tag{1.2}$$

In the above relations the prime stands for differentiation with respect to time, while $\partial I_{C(t)}(u(t))$ denotes the *normal cone* to the set $C(t)$ at the point $u(t)$ (see below), and $u_0 \in C(0)$ is an initial datum.

The problem (1.1)-(1.2) describes the evolution of a point $u(t)$ which is forced to belong to the moving convex set $C(t)$. In particular, relation (1.1) entails that the point does not move as long as it belongs to the *interior* of $C(t)$, and is swept around by $C(t)$ as soon as it touches the boundary of the convex set. Owing to this heuristics, the above relations are usually referred to as a *sweeping process*; we will term (1.1)-(1.2) as Problem SP.

Let us point out that the latter sweeping process stems from a variety of applications, ranging indeed from non-smooth mechanics to convex optimization and mathematical economics, among others (see [14] and the references therein). Moreover, it formally includes as a special case the *evolution variational inequality* for $v : [0,T] \to H$

$$v(t) \in C', \quad \langle v'(t), v(t) - w \rangle \leq \langle f(t), v(t) - w \rangle \quad \forall w \in C', \quad v(0) = u_0, \tag{1.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in H, C' is a non-empty, convex, and closed subset of H, and $f \in L^1(0,T;H)$. We can in fact rephrase (1.3) in the form (1.1)-(1.2) through the positions $u(t) := v(t) - (1 * f)(t)$ and $C(t) := C' - (1 * f)(t)$, with the standard notation $(1 * f)(t) := \int_0^t f(s)\, ds$ for $t \in [0,T]$.

The present analysis focuses on a generalization of Problem SP. Indeed, we are going to address the case of a set-valued driving function which depends on the solution u as well. In particular, assume we are given

$$K : [0,T] \times H \to 2^H \quad \text{with non-empty, convex, and closed values,} \tag{1.4}$$

and a point $u_0 \in K(0, u_0)$. Hence, we look for a solution to the quasivariational problem

$$u'(t) + \partial I_{K(t,u(t))}(u(t)) \ni 0 \quad \text{for } t \in (0,T), \tag{1.5}$$
$$u(0) = u_0. \tag{1.6}$$

The latter is often referred to as a *state-dependent* or *quasivariational* sweeping process and we will hereafter refer to (1.5)-(1.6) as Problem QSP. We mention that the interest in the study of Problem QSP arises in connection with the treatment of quasistatical evolution problems with friction, micro-mechanical damage models (see [11] and the references therein), and the evolution of shape memory alloys [1, 2]. Moreover, Problem QSP formally includes the *quasivariational evolution inequality*

$$v(t) \in K'(v(t)), \quad \langle v'(t), v(t) - w \rangle \leq \langle f(t), v(t) - w \rangle \quad \forall w \in K'(v(t)),$$
$$v(0) = u_0,$$

(where now $K' : H \to 2^H$ with non-empty, convex, and closed values), by means of the positions $u(t) := v(t) - (1 * f)(t)$, $K(t,u) := K'(u + (1 * f)(t)) - (1 * f)(t)$ for $t \in (0,T)$.

A first result in the direction of the existence of a global strong solution to Problem QSP was obtained by M. Kunze & M. D. P. Monteiro Marques in [11] under some compactness assumptions and the condition that K is Lipschitz continuous with respect to the Hausdorff metric $d_{\mathcal{H}}$ (see Section 2.2 later on). In particular, the function K is required to fulfill

$$d_{\mathcal{H}}(K(t,u), K(s,v)) \leq \lambda_1 |t - s| + \lambda_2 |u - v|$$

for all $t, s \in [0,T]$ and $u, v \in H$, where the symbol $|\cdot|$ denotes both the modulus in $\mathbb{R}$ and the norm in H, and $\lambda_1, \lambda_2 \geq 0$ are given constants. Actually, the analysis of [11] is applicable to the case in which

$$\lambda_2 < 1. \tag{1.7}$$

The reader is referred to [3, 8, 9] for some other related results in the same direction.

The restriction (1.7) has of course a clear mathematical drawback and is motivated in [11] by a suitable counter-example. Indeed, the authors show that the existence of a global strong solution to Problem QSP may fail [11, Example 3.1] in a situation in which (1.7) does not hold. We recast here this counter-example in a slightly modified form. To this aim, let $H := \mathbb{R}$ and define $K'(w) := [\psi(w), +\infty)$, with $\psi(w) := (2w - 1/2)^+$ for $w \in \mathbb{R}$. It is straightforward to check that the problem

$$\text{find } w : [0,1] \to H \text{ such that } \quad w'(t) + \partial I_{K'(w(t))}(w(t)) \ni 1, \quad t \in (0,1], \quad w(0) = 0,$$

has the unique strong solution $w(t) := t$ up to $t = 1/2$ (note that $\psi(1/2) = 1/2$). On the other hand, there is no absolutely continuous solution to the latter problem outside the interval $[0, 1/2]$. Indeed, for $t > 1/2$ the variational inequality entails $w' \geq 1$, but the region $\{1/2 < w < 1\}$ is not accessible for w, since $1/2 < w < 1$ implies $w < \psi(w)$ and $w \notin K'(w)$. Starting from the latter example, one easily checks that the choice $K(t,u) := K'(u+t) - t$ is uniformly Lipschitz continuous with constant $\lambda_2 = 2$ and the corresponding Problem QSP admits no absolutely continuous solutions on $[0,1]$.

As a matter of fact, this counter-example is just based on a *viability obstruction*, namely the fact that the region $\{1/2 < w < 1\}$ is not accessible for the solution. In particular, the regularity of K essentially plays no role there.

Moving from these considerations, the main novelty of this paper is that we will investigate Problem QSP without any compactness or Lipschitz continuity assumptions for K (in particular, we will drop (1.7)). On the other hand, we shall replace them with a suitable *monotonicity assumption* and some structural restriction on the values of K. Referring to the sequel of the paper for precise definitions and statements, we just mention that we will provide H with an order and require the set-valued function K to take *interval* values with non-increasing bounds. Let us clarify this picture in the simplest situation of $H = \mathbb{R}$ by assuming that we are given two continuous and non-increasing functions $k_*, k^* : \mathbb{R} \to \mathbb{R}$ (possibly non-smooth), such that $k_* \leq k^*$, and let us define $K(w) := [k_*(w), k^*(w)]$ for all $w \in \mathbb{R}$. Hence, it is easy to check that the viability obstruction of the above counter-example cannot occur, independently of the regularity of K (namely, of k_* and k^*).

In this paper, we are concerned with the global well-posedness of strong and weak formulations of Problem QSP in suitable classes of Hilbert spaces. In particular, we will extend the existence result of [11] and the existence and uniqueness result of [8] to *some* class of irregular, yet *monotone* functions K. We just point out that the choice $K(t,u) := K'(u+t) - t$ in the above counter-example is of course still not admissible in our setting. Although we are addressing here an abstract problem, we stress that the present monotonicity framework may be of some interest within applications. In particular, we may mention the results of [1, 2], where a model for the mechanical evolution of shape memory alloys, complying with the current setting, is discussed.

The idea of exploiting order methods for solving quasivariational problems is quite classical (the reader can check the seminal papers [5, 13, 23]). On the other hand, its application to the quasivariational sweeping process seems new. In particular, let us mention the related contributions [21, 22], where the second author addresses the problem of the weak solvability of Problem QSP by means of an order approach. More precisely, the latter papers deal with the even more general problem in which the normal cone $\partial I_{K(t,u(t))}(u(t))$ is replaced by the (suitably generalized) gradient of a proper, convex, and lower semicontinuous potential [21], and the functional dependence of K on u may be nonlocal with respect to time [21, 22].

Before going on, let us briefly outline our solution procedure for Problem QSP. Hereafter, we will assume that the Hilbert space H is provided with a suitable order structure. We will be mainly concerned with discussing some general assumptions in order to define a map $C : [0,T] \to 2^H$ with non-empty, convex, and closed values such that

$$C(t) = \{u \in H \ : u \in K(t,u)\} \quad \forall t \in (0,T),$$
$$\text{and} \quad C(t) \subset K(t,u) \quad \forall u \in K(t,u), \ \forall t \in (0,T).$$

Then, making use of the set-valued map C, we shall focus on Problem SP and we will show that (see Proposition 3.6) if Problem QSP and Problem SP, supplemented with the same initial condition ($v_0 = u_0$), admit solutions u and v, respectively, then the functions u and v

coincide on $(0,T)$. This approach entitles us to relate the investigation of the well-posedness of the *state-dependent* evolution inclusion (1.5)-(1.6), to the analysis of the *state-independent* Problem SP, which is considerably simpler.

In particular, we will prove that, under suitable regularity assumptions for $t \mapsto K(t,\cdot)$, suitable weak and strong notions of solutions to Problem QSP may be introduced, and we will discuss some monotonicity assumptions entailing in particular that these solutions are unique. Then, we will restrict our analysis to some relevant classes of Hilbert spaces, and develop existence results for both weak and strong solutions of Problem QSP. In particular, we will also obtain a converging time discretization scheme as a by-product of the existence analysis.

This is the plan of the paper. We first give some introductory material on orders in Hilbert spaces and recall Moreau's result in Section 2. Than, Section 3 contains the precise strong and weak formulations of Problem QSP, and the related uniqueness results. Finally, Section 4 is devoted to our existence result for a suitable class of Hilbert spaces.

2 Preliminary results

2.1 Orders in Hilbert spaces

Assume we are given a non-empty subset $P \subset H$ having the property that

$$P = \{u \in H : \langle u, v \rangle \geq 0 \quad \forall v \in P\}$$

(i.e., P coincides with its *dual* set). Hence, P is a *strict cone* (namely, a cone such that for all $x \in P \setminus \{0\}$, $-x \notin P$), with vertex at the origin; it is closed, and the relation $\leq$ defined by

$$u \leq v \;\text{ iff }\; v - u \in P \quad \forall u, v \in H,$$

turns out to be an *order* on H. The pair (H, P) is usually referred to as a *Hilbert pseudo-lattice* (see [4, Sec. 19.5, p. 399]).

Example 2.1. Examples of Hilbert pseudo-lattices are of course $(\mathbb{R}, [0, +\infty))$ and $(\mathbb{R}^m, Q_m)$, where the cone $Q_m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : \; x_i \geq 0, \; i = 1, \ldots, m\}$ induces the order relation $x \leq x'$ iff $x_i \leq x_i{}'$ for all $i = 1, \ldots, m$.
Further, let $(X, \mathcal{S}, m)$ be a measure space, m a positive measure, and $L^2(X, m)$ be the space of the (real-valued) square integrable functions on X, with the scalar product

$$\langle f, g \rangle := \int_X f(x) g(x) \; dm(x) \quad \forall f, g \in L^2(X, m).$$

Then, we can endow $L^2(X, m)$ with the *essential pointwise order*, induced by the cone $\mathcal{P} = \{f \in L^2(X, m) : \; f(x) \geq 0 \;$ for $m-$almost every $x \in \Omega\}$. Namely,

$$f \leq g \quad \text{iff} \quad f(x) \leq g(x) \quad \text{for } m\text{-a.e } \; x \in X.$$

For the reader's convenience and in order to make this paper more readable and self-contained, we will recall some notions and properties of Hilbert pseudo-lattices. In fact, several of them could be discussed in the framework of fully general ordered spaces, for which we refer to the monograph [4]. Let us introduce the elements

$$u^+ := \mathrm{proj}(u, P), \quad u^- := \mathrm{proj}(-u, P) = (-u)^+ \quad \forall u \in H, \tag{2.1}$$

$$u \vee v := u + (v - u)^+, \quad u \wedge v := u - (u - v)^+ \quad \forall u, v \in H, \tag{2.2}$$

where of course $\operatorname{proj}(\cdot, P)$ stands for the well-defined projection on P. Note that, at this stage, $u \vee v$ and $u \wedge v$ need not be the *supremum* and the *infimum* of the pair $\{u, v\}$, which may actually fail to exist. It is a standard matter to check that

$$u = u^+ - u^-, \quad \langle u^+, u^- \rangle = 0, \quad \langle u, u^+ \rangle = |u^+|^2. \tag{2.3}$$

Let us now recall some definitions which will play a crucial role in the sequel.

Definition 2.2. *Let (H, P) be a Hilbert pseudo-lattice and $F : H \to 2^H$. We say that F is* monotone *iff*

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0 \quad \forall u_1, u_2,\ v_i \in F(u_i),\ i = 1, 2.$$

Moreover a monotone operator F is maximal *if it is maximal in the sense of inclusion of graphs.*

We say that F is T-monotone [7] *iff*

$$\langle v_1 - v_2, (u_1 - u_2)^+ \rangle \geq 0 \quad \forall u_1, u_2,\ v_i \in F(u_i),\ i = 1, 2.$$

We say that F is non-decreasing *iff it is single-valued and*

$$u_1 \leq u_2 \Rightarrow F(u_1) \leq F(u_2) \quad \forall u_1, u_2 \in D(F),$$

where $D(F) := \{u \in H : F(u) \neq \emptyset\}$ stands for the domain of F.

It is straightforward to check that any T-monotone operator turns out to be monotone as well. More generally, let us comment on the relations between the latter notions by means of some examples.

Example 2.3. Obviously, in the simplest setting of $(H, P) := (\mathbb{R}, [0, +\infty))$, the monotonicity properties we have just introduced are strictly related. Indeed, in the special case of single-valued functions they are completely equivalent.

Example 2.4. We now consider the case $H := L^2(X, m)$, where $(X, \mathcal{S}, m)$ is a measure space and m is a positive measure. Assume we are given a Carathéodory function $f : X \times \mathbb{R} \to \mathbb{R}$ (i.e. for all $u \in \mathbb{R}$ the mapping $x \mapsto f(x, u)$ is $m-$measurable and for $m-$almost every $x \in X$ the mapping $u \mapsto f(x, u)$ is continuous), a function $g \in L^2(X, m)$, and $c \geq 0$ such that

$$|f(x, u)| \leq g(x) + c|u| \quad \text{for } m\text{-a.e. } x \in X \quad \text{and} \quad \forall u \in \mathbb{R}. \tag{2.4}$$

Hence, we can define an operator $F : L^2(X, m) \to L^2(X, m)$ by

$$F(u)(x) := f(x, u(x)) \quad \text{for } m\text{-a.e. } x \in X \quad \text{and} \quad \forall u \in L^2(X, m).$$

Whenever

$$f(x, \cdot) \quad \text{is non-decreasing for } m - \text{a.e. } x \in X,$$

the corresponding operator F turns out to be T-monotone and non-decreasing as well. Note that (2.4) is assumed just in order to ensure that $F(u) \in L^2(X, m)$ for all $u \in L^2(X, m)$, and could be weakened. Alternatively, we could suppose that the function $f : X \times \mathbb{R} \to \mathbb{R}$ is such that the mapping $x \mapsto f(x, u)$ is $m-$measurable for all $u \in \mathbb{R}$,

$$f(x, \cdot) \quad \text{is maximal monotone for } m - \text{a.e. } x \in X \tag{2.5}$$

and f fulfills (2.4), for instance. In this case as well, F is again well defined on $L^2(X, m)$. In particular $x \mapsto f(x, u(x))$ is $m-$measurable for all $u \in L^2(X, m)$. We mention [19] for further reference on measurability issues. In the latter situation, the operator F turns out to be T-monotone (hence monotone). Let us point out that F is maximal as well. To check this, by [6, Prop. 2.2, p. 23] it is sufficient to note that for every $u \in L^2(X,m)$ the function $v(x) := (\mathrm{Id} + f(x,\cdot))^{-1}u(x)$ for $m-$almost every $x \in X$ (where Id denotes the identity operator), fulfills by construction

$$(\mathrm{Id} + F)(v(x)) = v(x) + f(x, v(x)) = u(x) \quad \text{for } m\text{-a.e. } x \in X,$$

and that $v \in L^2(X,m)$. This fact follows from the previous discussion, since the operator $(\mathrm{Id} + f(x,\cdot))^{-1}$ complies with (2.4) (with the choice $g(x) := 2f(x,0)$ and $c := 2$).

Example 2.5. As soon as we turn to the Hilbert pseudo-lattice $(H,P) := (\mathbb{R}^2, Q_2)$, the three monotonicity notions are indeed independent one from another, as it is shown by the elementary example of a linear operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad a,b,c,d \in \mathbb{R}.$$

Indeed, F is non-decreasing iff, setting $(y_1,y_2) := F((x_1,x_2))$ for $(x_1,x_2) \in \mathbb{R}^2$, one has that $y_i \geq 0$, $i = 1,2$ as soon as $x_i \geq 0$, $i = 1,2$. It is easy to see that this is equivalent to the property

$$a,b,c,d \geq 0. \tag{2.6}$$

On the other hand, the operator F is monotone iff $\langle (y_1,y_2),(x_1,x_2)\rangle \geq 0$, i.e. $(ax_1 + bx_2)x_1 + (cx_1 + dx_2)x_2 \geq 0$ for every $(x_1,x_2) \in \mathbb{R}^2$: a *necessary and sufficient* condition for this inequality to hold for all $(x_1,x_2) \in \mathbb{R}^2$ is that

$$a,d \geq 0 \quad \text{and} \quad |b+c| \leq 2\sqrt{ad}. \tag{2.7}$$

Finally, F is T-monotone iff $\langle (y_1,y_2),(x_1,x_2)^+\rangle \geq 0$ for all $(x_1,x_2) \in \mathbb{R}^2$. In particular, the previous condition reduces to $(cx_1 + dx_2)x_2 \geq 0$ for every (x_1,x_2) with $x_1 \leq 0$ and $x_2 \geq 0$ (in this case, $(x_1,x_2)^+ = (0,x_2)$). It is easy to check that this forces $c \leq 0$ and $d \geq 0$. Arguing likewise for the elements (x_1,x_2) with $x_1 \geq 0$ and $x_2 \leq 0$, we deduce that if F is T-monotone, then necessarily $a \geq 0$ and $b \leq 0$. Actually, some straightforward computations yield that a *necessary and sufficient* condition for F to be T-monotone is

$$a,d \geq 0, \quad b,c \leq 0, \quad \text{and} \quad b + c + 2\sqrt{ad} \geq 0. \tag{2.8}$$

Moving from these preliminaries, we can construct several examples. For instance, in view of (2.6) and (2.8), a linear operator F is *both* T-monotone *and* non-decreasing iff it is associated to a diagonal matrix with non-negative eigenvalues. In contrast, the operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by means of the matrix

$$M_1 := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

complies with (2.6)-(2.7), and is therefore monotone and non-decreasing, but it is not T-monotone (indeed, take $(-3,1)$: then $(-3,1)^+ = (0,1)$ and $F((-3,1)) = (-5,-1)$, but $\langle F((-3,1)),(-3,1)^+\rangle = -1 < 0$). On the other hand, the anticlockwise rotation

$$M_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is monotone, but neither T-monotone nor non-decreasing, in view of the characterization (2.6). A slight modification of the latter example, namely

$$M_3 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

yields a non-decreasing operator, which is not monotone (nor T-monotone, of course) since we have that $\langle F((-1,1)),(-1,1)\rangle = \langle (1,0),(-1,1)\rangle = -1 < 0$.

2.2 Existence results for state-independent sweeping processes

We recall some results of the theory of sweeping processes, originally developed in Moreau's seminal papers [15, 16, 17, 18], see also [12].

Let $A, B \subset H$ be two non-empty subsets of H; we define the *Hausdorff semidistance* $e(A,B)$ between A and B as $e(A,B) := \sup_{a\in A}\inf_{b\in B}|a-b|$. Note that $e(A,B)=0$ iff $A\subset \mathrm{cl}(B)$, $\mathrm{cl}(B)$ denoting the closure of B, and $e(A,B)\neq e(B,A)$, in general. Moreover, we define the Hausdorff distance $d_{\mathcal{H}}$ by $d_{\mathcal{H}}(A,B) := \max\{e(A,B), e(B,A)\}$.

Let $C : [0,T] \to 2^H$ be a set-valued function with non-empty values (a multifunction, for short). We say that C *has a finite retraction on* $[0,T]$ if

$$\mathrm{ret}(C;s,t) := \sup\left\{ \sum_{i=1}^{n} e(C(t_{i-1}),C(t_i)) \ :\ \{t_0,\dots,t_n\} \text{ partition of } [s,t] \right\} < +\infty$$

for every $[s,t]\subset[0,T]$. Note that this is equivalent to the existence of a non-decreasing real function r on $[0,T]$ such that $\mathrm{ret}(C;s,t) = r(t)-r(s)$ for every $s\le t$.

Hereafter, we will assume that the set-valued function $C:[0,T]\to 2^H$ fulfills

$$C(t) \quad \text{is non-empty, closed, and convex for all } t\in[0,T]. \tag{2.9}$$

We recall that for every non-empty, closed, and convex subset $C\subset H$ one can define the proper, lower semicontinuous, and convex *indicator function* $I_C : H \to [0,+\infty]$ by setting $I_C(x) := 0$ if $x\in C$ and $I_C(x) := +\infty$ otherwise. Then, we introduce the *subdifferential* operator $\partial I_C : H \to 2^H$, given by

$$y\in\partial I_C(x) \iff x\in C \ \text{ and } \ \langle y, w-x\rangle \le 0 \quad \forall w\in C.$$

Note that $\partial I_C(x)$ is the normal cone to C at the point x; we recall that ∂I_C is a maximal monotone operator [6].

We will address the Cauchy problem

$$v'(t) + \partial I_{C(t)}(v(t)) \ni 0, \quad t\in[0,T] \tag{2.10}$$

$$v(0) = v_0, \tag{2.11}$$

for the sweeping process associated to the moving set $C(t)$.

Weak and strong solutions to the sweeping process. In this setting, we can give two different formulations of (2.10)-(2.11), which need a few preliminaries on BV functions. We recall that $BV([0,T];H)$ is the space of all functions $v\in L^1(0,T;H)$ such that the distributional derivative Dv is a ($H-$valued) measure with finite total variation on $[0,T]$. It is well known that all functions in $BV([0,T];H)$ are continuous, except for an at most countable number of points, at which the left and right limits $v^-(t)$ and $v^+(t)$ exist, and

$$v^+(t) - v^-(s) = \int_{[s,t]} Dv \quad \forall s,t\in[0,T]. \tag{2.12}$$

We can also introduce the subspace $BV_R([0,T];H)$ of *right-continuous* functions on $[0,T]$ with bounded variation.

Definition 2.6. *Given $v_0 \in C(0)$, we say that a function $v \in BV_R([0,T];H)$ is a* solution in the sense of differential measures *to Problem* SP *if $v(0) = v_0$,*

$$v(t) \in C(t) \quad \forall t \in [0,T], \tag{2.13}$$

and there exists a non-negative measure ν such that Dv is absolutely continuous with respect to ν and admits the representation $Dv = v_\nu d\nu$, for some function $v_\nu \in L^1(0,T,\nu;H)$ fulfilling

$$v_\nu(t) + \partial I_{C(t)}(v(t)) \ni 0 \quad \textit{for } \nu\textit{-a.e. } t \in (0,T). \tag{2.14}$$

We say that $v \in W^{1,1}(0,T;H)$ is a strong solution *to Problem* SP *if the initial condition holds and*

$$v'(t) + \partial I_{C(t)}(v(t)) \ni 0 \quad \textit{for a.e. } t \in (0,T). \tag{2.15}$$

Remark 2.7 (Compatibility I). Note that the two definitions we have just introduced are compatible: namely, any $v \in W^{1,1}(0,T;H)$ solving (2.10) in the sense of differential measures is also a strong solution.

Indeed, when v is absolutely continuous on $[0,T]$, the measure ν factorizing Dv is absolutely continuous with respect to the Lebesgue measure on $[0,T]$, so that $d\nu = \bar{\nu}dt$ for a *non-negative* function $\bar{\nu} \in L^1(0,T)$. Then, $Dv = v_\nu \bar{\nu} dt$, $v_\nu \bar{\nu}$ being now the density of Dv with respect to the Lebesgue measure. Therefore, there exists a subset $L' \subset (0,T)$ with $|L'| = 0$ (where $|\cdot|$ denotes the Lebesgue measure), and

$$v'(t) = v_\nu(t)\bar{\nu}(t) \quad \text{for } t \in (0,T) \setminus L'.$$

In the same way, let $L'' \subset (0,T)$ be the ν-negligible set such that (2.14) holds on $(0,T) \setminus L''$. Note that $\nu(L' \cup L'') = 0$, so that we have

$$|L'''| = 0, \quad \text{with } L''' := \{t \in L' \cup L'' \ : \ \bar{\nu}(t) > 0\}$$

and, by the conical property of the subdifferential $\partial I_{C(t)}(v(t))$, we easily deduce from (2.14) that

$$v'(t) + \partial I_{C(t)}(v(t)) \ni 0 \quad \text{for } t \in (0,T) \setminus L''',$$

whence (2.15).

Continuous dependence for the sweeping process. We recall the following uniqueness result for the sweeping process (2.10), due to Moreau [18, Prop. 3a].

Proposition 2.8 (Moreau). *Assume* (2.9), *and let u_1, $u_2 \in BV_R([0,T];H)$ be two solutions of* (2.10) *in the sense of differential measures. Then, the function*

$$t \in [0,T] \mapsto |u_1(t) - u_2(t)| \quad \textit{is non-increasing.}$$

In particular, the Cauchy problem (2.10)-(2.11) *has a unique solution in the sense of differential measures.*

In fact, this result will play a crucial role in establishing the continuous dependence for our target quasivariational problem (1.5)-(1.6). We directly refer to [18] for the proof of Proposition 2.8, which features a technique we will exploit later on in the proof of Proposition 3.6.
In turn, for the reader's convenience we briefly outline the (by now standard) approximation scheme for (2.10)-(2.11) proposed in [18].

Existence for the sweeping process: the catching-up algorithm. Let us fix $N \in \mathbb{N}$ and divide $(0,T)$ into N subintervals (t_{n-1}, t_n), $n = 1, \dots, N$, of size $\tau = T/N$; we define recursively $\{V^n\}_{n=0}^N$ by

$$V^0 = u_0, \quad V^{n+1} := \mathrm{proj}(V^n, C(t_{n+1})), \tag{2.16}$$

and introduce the piecewise constant interpolating function $\overline{V}_\tau$ of the V^n defined by

$$\overline{V}_\tau(t) = V^n, \quad t_{n-1} < t \le t_n, \quad n = 1, \dots, N.$$

The sequence $\{\overline{V}_\tau\}_{\tau>0}$ will provide the approximate solutions to our problem.

It is straightforward to obtain some a priori estimates on $\{\overline{V}_\tau\}$. First of all, we note that for every $\tau > 0$ and $t, s \in (0,T)$, with $t_{j-1} < s \le t_j \le t_{i-1} < t \le t_i$ for some $1 \le j < i \le N$, and for every partition $s =: s_0 < s_1 < \dots < s_M := t$ of the interval $[s,t]$, we have

$$\sum_{k=1}^{M} |\overline{V}_\tau(s_k) - \overline{V}_\tau(s_{k-1})| = \sum_{k=j+1}^{i} |V^k - V^{k-1}| = \sum_{k=j+1}^{i} |\overline{V}_\tau(t_k) - \overline{V}_\tau(t_{k-1})|. \tag{2.17}$$

Of course, an analogous computation holds both for $s = 0$ and in the case in which $s, t \in (t_{i-1}, t_i]$ for some $i = 1, \dots, N$, and we have focused on the above situation just for the sake of notational simplicity. Since the partition of $[s,t]$ we have considered is arbitrary, recalling (2.16) we conclude that

$$\mathrm{Var}_{[s,t]}(\overline{V}_\tau) \le \sum_{k=j+1}^{i} e(C(t_{k-1}), C(t_k)), \tag{2.18}$$

$\mathrm{Var}_{[s,t]}$ denoting the total variation over the interval $[s,t]$. Therefore, if the multifunction $t \mapsto C(t)$ has finite retraction on $[0,T]$, we infer from (2.18) that

$$\mathrm{Var}_{[s,t]}(\overline{V}_\tau) \le r(t_i) - r(s), \tag{2.19}$$

for some bounded and non-decreasing retraction function r. Then, it follows that there exists a positive constant c, depending on u_0 and r, but not on τ, such that

$$\|\overline{V}_\tau\|_{L^\infty(0,T);H} + \mathrm{Var}_{[0,T]}(\overline{V}_\tau) \le c \quad \forall \tau > 0. \tag{2.20}$$

Now, recalling a compactness theorem contained, e.g., in [14, p. 10], we deduce from the estimate (2.20) that there exist a subsequence $\{\overline{V}_{\tau_k}\}$ and a function $v \in BV([0,T];H)$ such that $\overline{V}_{\tau_k}(t) \to v(t)$ weakly in H as $k \uparrow \infty$ for every $t \in [0,T]$. In particular, $v(0) = v_0$. Moreover, by exploiting (2.17)-(2.19) and assuming that

$$r \text{ is right-continuous on } [0,T], \tag{2.21}$$

we obtain that

$$\mathrm{Var}_{[s,t]}(v) \le r(t) - r(s) \quad \forall\ [s,t] \subset [0,T],$$

and $v \in BV_R([0,T];H)$ as well. Further, we can check that

$$v(t) \in C(t) \quad \forall t \in [0,T]. \tag{2.22}$$

Indeed, it is easy to see that for every $t \in [0,T]$ we can fix a sequence of partitions with diameter $\tau_k \downarrow 0$ as $k \uparrow \infty$ and a point t_k in each partition such that $t_k - \tau_k < t \le t_k$. Therefore, $\overline{V}_{\tau_k}(t_k) = \overline{V}_{\tau_k}(t) \to v(t)$ weakly. On the other hand, the convexity of the set $C(t)$ easily yields that the functional defined by $x \mapsto \operatorname{dist}(x, C(t)) = \inf_{y\in C(t)} |y - x|$, $x \in H$, is convex, hence (sequentially) weakly lower semicontinuous. Collecting these facts and recalling the catching-up algorithm (2.16), we conclude that

$$\operatorname{dist}(v(t), C(t)) \le \liminf_{k\uparrow\infty} \operatorname{dist}(\overline{V}_{\tau_k}(t_k), C(t)) \le \\ \liminf_{k\uparrow\infty} e(C(t_k), C(t)) \le \liminf_{k\uparrow\infty} (r(t_k) - r(t)).$$

Therefore, in view of (2.21), the right-hand side of the previous inequality is zero, and (2.22) ensues from the closure of $C(t)$.

In [18], it is in fact proved that the function v here obtained is indeed a solution to (2.10)-(2.11), namely we have the following

Proposition 2.9 (Moreau). *Let $C : [0,T] \to 2^H$ be a multifunction with finite retraction fulfilling* (2.9) *and* (2.21). *Then there exists a* unique *solution v in the* sense of differential measures *to Problem* SP. *Moreover, v is the pointwise limit of the sequence of the discrete solutions constructed by means of the catching-up algorithm. Further, if the retraction function*

$$r \text{ is absolutely continuous on } [0,T], \tag{2.23}$$

then v is a strong *solution to Problem* SP.

Remark 2.10 (Compatibility II). Within the regularity framework for the multifunction C here considered (cf. (2.21)), it may be easily checked that any strong solution to Problem SP is also a solution in the sense of differential measures.
Indeed, since $v \in W^{1,1}(0,T;H)$, we have $Dv = v'dt$, with $v' \in L^1(0,T;H)$, so that Definition 2.6 is fulfilled choosing as ν the standard Lebesgue measure and $v_\nu \equiv v'$. Taking into account these positions, (2.14) is nothing but (2.15). To check (2.13), we fix $t \in [0,T]$ and note that, by (2.15), there exists a decreasing sequence $\{t_k\} \in [0,T]$ such that $v(t_k) \in C(t_k)$ $\forall k \in \mathbb{N}$, and $t_k \downarrow t$ as $k \to +\infty$. Then, one readily computes that

$$|v(t) - \operatorname{proj}(v(t), C(t))| = \lim_{k\to+\infty} |v(t_k) - \operatorname{proj}(v(t_k), C(t))| \\ \le \liminf_{k\to+\infty} e(C(t_k), C(t)) \le \liminf_{k\to+\infty} (r(t_k) - r(t)) = 0,$$

whence (2.13), so that v is a solution in the sense of differential measures. Let us stress that the only feature of v which was exploited in the latter computation is its right-continuity and the fact that $v(t) \in C(t)$ for almost every $t \in (0,T)$.

3 Problem formulation and continuous dependence results

In accordance with the twofold formulation of Definition 2.6, we can now give two different notions of solutions for Problem QSP.

Definition 3.1. *Given $u_0 \in K(0,u_0)$, we say that a function $u \in BV_R([0,T];H)$ is a* solution in the sense of differential measures *to Problem* QSP *if $u(0) = u_0$,*

$$u(t) \in K(t, u(t)) \quad \forall t \in [0,T], \tag{3.1}$$

and there exists a non-negative measure μ *such that* $Du = u_\mu d\mu$, *for some function* $u_\mu \in L^1(0,T,\mu;H)$ *fulfilling*

$$u_\mu(t) + \partial I_{K(t,u(t))}(u(t)) \ni 0 \quad \textit{for } \mu\textit{-a.e. } t \in (0,T). \tag{3.2}$$

Moreover, we say that $u \in W^{1,1}(0,T;H)$ *is a* strong solution *to Problem* QSP *if the initial condition holds and*

$$u'(t) + \partial I_{K(t,u(t))}(u(t)) \ni 0 \quad \textit{for a.e. } t \in (0,T). \tag{3.3}$$

Remark 3.2. Arguing along the same lines as in Remark 2.7, it is not difficult to check that any $u \in W^{1,1}(0,T;H)$ solving Problem QSP in the sense of differential measures is also a strong solution. Conversely, we will prove that under the forthcoming regularity and monotonicity assumptions on K, strong solutions to Problem QSP are also solutions in the sense of differential measures, cf. Remark 4.4.

3.1 Monotonicity assumptions and key lemma

We can now enlist the main assumptions under which we will tackle the well-posedness of Problem QSP in a general Hilbert pseudo-lattice (H,P).

First of all, we will suppose that the map $K : [0,T] \times H \to 2^H$ has the structure

$$K(t,u) = [K_*(t,u), K^*(t,u)] \quad \text{for some} \quad K_*(t,u), K^*(t,u) \in H, \tag{3.4}$$

for every $(t,u) \in [0,T] \times H$: namely, $K(t,u)$ is a closed subinterval of H. We will also assume that for every $t \in (0,T)$ the operators $-K^*(t,\cdot)$, $-K_*(t,\cdot)$ are (see Definition 2.2)

$$\begin{aligned}
&\textit{maximal} \text{ (for graph inclusion within } \textit{monotone} \text{ operators)}, && (3.5)\\
&\textit{T-monotone}, && (3.6)\\
&\textit{non-decreasing}. && (3.7)
\end{aligned}$$

The *monotonicity* assumptions (3.6)-(3.7) play a key role in the proof of the following

Lemma 3.3. *Assume that* (3.4)-(3.7) *hold. Then*
i) for every $t \in [0,T]$ *there exists a unique pair* $(c^*(t), c_*(t)) \in H \times H$ *fulfilling*

$$c^*(t) = K^*(t, c^*(t)) \quad \textit{in } H, \tag{3.8}$$

$$c_*(t) = K_*(t, c_*(t)) \quad \textit{in } H, \tag{3.9}$$

and $c_*(t) \le c^*(t)$.
ii) The multifunction $C : [0,T] \to 2^H$ *given by*

$$C(t) := [c_*(t), c^*(t)] \quad \forall\, t \in [0,T], \tag{3.10}$$

fulfills

$$C(t) = [c_*(t), c^*(t)] = \{u \in H \ : u \in K(t,u)\} \quad \forall\, t \in [0,T], \tag{3.11}$$

$$C(t) \subset K(t,u) \quad \textit{for every } u \textit{ such that } u \in K(t,u) \quad \forall\, t \in [0,T]. \tag{3.12}$$

Proof. i) It follows from our assumptions that the operators $-K^*(t,\cdot), -K_*(t,\cdot)$ are maximal monotone for every $t \in (0,T)$. Then, the general theory of maximal monotone operators (see [6, Prop. 2.2 p.23]) ensures that equation (3.8) ((3.9), resp.) has a unique solution $c^*(t)$

($c_*(t)$, resp.), for every $t \in [0,T]$. Note that, by construction, we have $c_*(t) = K_*(t, c_*(t)) \leq K^*(t, c_*(t))$. Therefore,

$$
\begin{aligned}
0 \geq \langle (c_*(t) - c^*(t))^+, c_*(t) - K^*(t, c_*(t)) \rangle = \\
\langle (c_*(t) - c^*(t))^+, c_*(t) - c^*(t) + c^*(t) - K^*(t, c_*(t)) \rangle = \\
\langle (c_*(t) - c^*(t))^+, c_*(t) - c^*(t) \rangle + \\
\langle (c_*(t) - c^*(t))^+, K^*(t, c^*(t)) - K^*(t, c_*(t)) \rangle \geq \\
|(c_*(t) - c^*(t))^+|^2,
\end{aligned} \tag{3.13}
$$

the last inequality following from (2.3) and (3.6). Hence, $c_*(t) \leq c^*(t)$.
ii) First of all, we will show that

$$
\xi \leq c^*(t) \;\Leftrightarrow \xi \leq K^*(t, \xi) \tag{3.14}
$$

for every $t \in [0,T]$. Assume that $\xi \leq c^*(t)$: since $-K^*$ is non-decreasing, we have $K^*(t,\xi) \geq K^*(t, c^*(t)) = c^*(t) \geq \xi$ and the left-to-right implication in (3.14) follows. As for the converse implication, we can carry out the same computations as in (3.13) by T-monotonicity, obtaining that

$$
\begin{aligned}
0 \geq \langle \xi - K^*(t,\xi), (\xi - c^*(t))^+ \rangle = |(\xi - c^*(t))^+|^2 + \\
\langle K^*(t, c^*(t)) - K^*(t,\xi), (\xi - c^*(t))^+ \rangle \geq |(\xi - c^*(t))^+|^2,
\end{aligned} \tag{3.15}
$$

whence $\xi \leq c^*(t)$. Of course, the analogue of (3.14) holds for c_* and K_*, so that, recalling the structural assumption (3.4), we may conclude (3.11).

As for (3.12), let us note that for every $v \in C(t)$ and $u \in K(t,u)$

$$
\Big(v \leq c^*, \;\; u \leq K^*(t,u)\Big) \;\Rightarrow\; \Big(v \leq K^*(t, c^*(t)), \;\; u \leq c^*(t)\Big). \tag{3.16}
$$

In fact, one inequality is trivial, whereas the second one can be shown by making use of the T-monotonicity of $-K^*$, again in the same way as in (3.13) and (3.15). Then, in view of (3.7), the last inequality in the above line yields that $v \leq K^*(t, c^*(t)) \leq K^*(t,u)$. Arguing for c_* and K_* in the same way, we obtain the desired conclusion. □

Remark 3.4. Let us point out that assumption (3.7) could be replaced by the *weaker* assumptions

$$
v \leq (\mathrm{Id} - K^*(t,\cdot))^{-1}(0) \;\Rightarrow\; K^*(t,v) \geq (\mathrm{Id} - K^*(t,\cdot))^{-1}(0), \tag{3.17}
$$

$$
v \geq (\mathrm{Id} - K_*(t,\cdot))^{-1}(0) \;\Rightarrow\; K_*(t,v) \leq (\mathrm{Id} - K_*(t,\cdot))^{-1}(0), \tag{3.18}
$$

for every $v \in H$ and $t \in [0,T]$. For instance, the left-to-right implication in (3.14) is indeed established by noting that

$$
\Big\{\xi \leq c^*(t) \text{ and } c^*(t) - K^*(t, c^*(t)) = 0\Big\} \;\Rightarrow\; \xi - K^*(t,\xi) \leq 0,
$$

which is a straightforward consequence of (3.17).

Remark 3.5. The assumptions (3.5)-(3.7) could indeed be replaced by the following set of hypotheses (which we just state for $K^*(t,\cdot)$, the assumptions for $K_*(t,\cdot)$ being perfectly analogous). Namely, we could alternatively suppose that for every $t \in [0,T]$

the operator $-K^*(t,\cdot)$ is *non-decreasing* and *monotone*, (3.19)

there exists an operator $L : H \to H$, *T-monotone* and *maximal* such that (3.20)

$L + K^*(t,\cdot)$ is *non-decreasing*. (3.21)

On the one hand, we allow $-K^*(t,\cdot)$ to be neither maximal nor T-monotone, while, on the other hand, we require $-K^*(t,\cdot)$ to be in some sense controlled (cf. (3.21)), by a maximal T-monotone operator. Of course, as soon as (3.5)-(3.7) hold true, it suffices to choose $L = -K^*(t,\cdot)$ in order to comply with (3.20)-(3.21).

In the framework of (3.19)-(3.21), we are still able to associate to our quasivariational sweeping process a multifunction C complying with the crucial properties (3.11)-(3.12) by means of a slightly different method from the one so far developed. This new approach relies on a specific fixed point device, which is in the same spirit of the well known fixed point result for non-decreasing maps in general ordered sets due to I.I. KOLODNER, see [4, 10].

We prefer not to give here the details of this alternative construction for the sake of simplicity.

3.2 Continuous dependence for Problem QSP.

Lemma 3.3 enables us to show that the quasivariational evolution (1.5) is in fact encoded in the evolution of the moving set $C(t)$, as the following result states.

Proposition 3.6. *Assume that* (3.4)-(3.7) *hold. Let u be a solution in the sense of differential measures (a strong solution, resp.) to Problem* QSP, *supplemented with the initial condition $u(0) = u_0$, where $u_0 \in K(0, u_0)$. Then, u is also a solution in the sense of differential measures (a strong solution, resp.) to Problem* SP *for the set-valued function (3.10), with the initial datum u_0.*

Proof. First of all, we shall observe that indeed $u_0 \in C(0)$ as a consequence of (3.11). Then, let us start by considering the case in which u is a solution in the sense of differential measures to Problem QSP. Then, u fulfils (3.1), which entails that $u(t) \in C(t)$ in view of (3.11). Furthermore, let μ be a non-negative measure such that the function $u_\mu \in L^1(0,T,\mu;H)$ decomposes the finite measure Du according to the Definition 3.1, i.e., $Du = u_\mu d\mu$. Owing to (3.2) we get that,

$$\langle u(t) - w, u_\mu(t)\rangle \le 0 \quad \forall w \in K(t,u(t)), \quad \text{for } \mu-\text{a.e. } t \in [0,T].$$

In particular, in view of (3.12),

$$\langle u(t) - w, u_\mu(t)\rangle \le 0 \quad \forall w \in C(t), \quad \text{for } \mu-\text{a.e. } t \in [0,T],$$

whence (2.14) (for the measure μ). We have thus proved that u complies with the Definition 2.6 of solution to SP in the sense of differential measures.

The argument in the case in which u is a strong solution to Problem QSP is analogous. □

It follows from the above Proposition 3.6 and from Proposition 2.8 that, given a solution u (strong or in the sense of differential measures) to Problem QSP and a solution v (strong or in the sense of differential measures, respectively) to Problem SP, supplemented with the initial conditions $u(0) = u_0$ and $v(0) = v_0$, then

$$|u(t) - v(t)| \le |u_0 - v_0| \quad \text{for every } t \in [0,T]. \tag{3.22}$$

Of course, a first consequence of the above inequality is that the solution u of Problem QSP depends continuously on the initial data as well. In particular, we have the following uniqueness result.

Corollary 3.7. *Under the assumptions* (3.4)-(3.7), *let u_1, u_2 be two solutions in the sense of differential measures (strong solutions, resp.) of Problem* QSP. *Then, $u_1(t) = u_2(t)$ for every $t \in [0,T]$.*

4 Existence results for Problem QSP.

4.1 The case $H = \mathbb{R}$.

Let us now tackle the case in which the multifunction K is defined on $[0,T] \times \mathbb{R}$ and takes values in $2^{\mathbb{R}}$. In this one-dimensional framework, (3.4) is trivially implied by (1.4). Further, the assumptions (3.5)-(3.7), yielding the construction of a multifunction C related to K by (3.11)-(3.12), considerably simplify. In fact, (3.5)-(3.7) are equivalent to

$$\begin{gathered}\text{the maps } u \mapsto K^*(t,u),\ u \mapsto K_*(t,u) \\ \text{are continuous and non-increasing for every } t \in [0,T]. \end{gathered} \tag{4.1}$$

Indeed, let us note that, in this one-dimensional setting (cf. Example 2.3), both (3.6) and (3.7) are yielded by our monotonicity assumption on the *characteristics* K_* and K^*. On the other hand, by continuity the operators $\mathrm{Id} - K^*(t,\cdot)$ and $\mathrm{Id} - K_*(t,\cdot)$ are onto, which entails (3.5), in view of [6, Prop. 2.2., p. 23].
Therefore, we may still introduce the set-valued function C (3.10), fulfilling the key properties (3.11)-(3.12), and our uniqueness result Corollary 3.7 holds under the sole (4.1). Turning now to the issue of existence for Problem QSP, we will additionally suppose that there exist two functions R^*, $R_* \in BV_R([0,T])$ such that

$$|K^*(t,u) - K^*(s,u)| \le |R^*(t) - R^*(s)|, \tag{4.2}$$
$$|K_*(t,u) - K_*(s,u)| \le |R_*(t) - R_*(s)| \tag{4.3}$$

for all $s,t \in [0,T]$ and $u \in \mathbb{R}$. Note that (4.2) ((4.3), resp.) implies that the map $t \mapsto K^*(t,u)$ ($t \mapsto K_*(t,u)$, resp.) has uniformly bounded variation with respect to $u \in \mathbb{R}$.
Then, we have the following

Lemma 4.1. *Let* (3.4), (4.1), *and* (4.2)-(4.3) *hold. Then, the functions* c^* *and* c_* *are in* $BV_R([0,T])$, *and the multifunction* $C : [0,T] \to 2^{\mathbb{R}}$ *given by* (3.10) *has finite retraction on* $[0,T]$, *with a retraction function* $r \in BV_R([0,T])$.
Moreover, if the functions

$$R^* \text{ and } R_* \text{ are absolutely continuous on } [0,T], \tag{4.4}$$

then r *is absolutely continuous on* $[0,T]$ *as well.*

Proof. Firstly, let us note that for every $t,s \in [0,T]$

$$\begin{aligned} |c^*(t) - c^*(s)|^2 = \langle K^*(t,c^*(t)) - K^*(s,c^*(s)), c^*(t) - c^*(s)\rangle = \\ \langle K^*(t,c^*(t)) - K^*(s,c^*(t)), c^*(t) - c^*(s)\rangle + \\ \langle K^*(s,c^*(t)) - K^*(s,c^*(s)), c^*(t) - c^*(s)\rangle \le \\ |K^*(t,c^*(t)) - K^*(s,c^*(t))||c^*(t) - c^*(s)|, \end{aligned} \tag{4.5}$$

where we have used (4.1). Taking into account (4.2), we have that

$$|c^*(t) - c^*(s)| \le |K^*(t,c^*(t)) - K^*(s,c^*(t))| \le |R^*(t) - R^*(s)| \tag{4.6}$$

for all $t,s \in [0,T]$, whence we easily deduce that $c^* \in BV_R([0,T])$. On behalf of (4.3), we can repeat the same computations for c_*.

Moreover, we have that

$$d_{\mathcal{H}}(C(s), C(t)) \leq \max\{|c^*(t) - c^*(s)|, |c_*(t) - c_*(s)|\} \leq \\ \max\{|R^*(t) - R^*(s)|, |R_*(t) - R_*(s)|\} \quad \forall\, 0 \leq s \leq t \leq T, \tag{4.7}$$

the first inequality following from trivial computations and the second one from (4.6) and its analogue for c_* and R_*. Therefore, for every fixed partition $\{s_0, \ldots, s_M\}$ of the interval $[s,t]$, we have

$$\sum_{j=1}^{M} e(C(s_{j-1}), C(s_j)) \leq \sum_{j=1}^{M} d_{\mathcal{H}}(C(s_{j-1}), C(s_j)) \leq \\ \sum_{j=1}^{M} \max\{|R^*(s_{j-1}) - R^*(s_j)|, |R_*(s_{j-1}) - R_*(s_j)|\}.$$

Taking the sup over all partitions of $[s,t]$ and using that both R^* and R_* have bounded variation, we readily obtain that the multifunction C has finite retraction, and there holds

$$\mathrm{ret}(C; s, t) = r(t) - r(s) \leq \mathrm{Var}_{[s,t]}(R^*) + \mathrm{Var}_{[s,t]}(R_*). \tag{4.8}$$

Then, it is a standard matter to check that the function r itself has bounded variation over $[0,T]$. Moreover, we infer from (4.8) that r is also right-continuous at every $t_0 \in [0,T)$, since

$$\lim_{t \downarrow t_0} \mathrm{Var}_{[t_0,t]}(R^*) = |\lim_{t\downarrow t_0} R^*(t) - R^*(t_0)| = 0,$$

where we have used the right-continuity of R^* (analogously for R_*). Then, (4.8) entails that $\lim_{t\downarrow t_0} r(t) \leq r(t_0)$, and the right-continuity at t_0 follows from the fact that r is non-decreasing.

As for the last part of the statement, we may observe that if (4.4) holds, one deduces from (4.8) that

$$r(t) - r(s) \leq \int_s^t \big(|(R^*)'(\sigma)| + |(R_*)'(\sigma)|\big)\, d\sigma \quad \forall t \geq s.$$

In particular, the absolute continuity of r easily follows. □

In view of Lemma 4.1 and of Proposition 2.9, we can conclude that, under the present assumptions, Problem SP for the multifunction C has a (unique) solution $v \in BV_R([0,T])$ in the sense of differential measures; if we further assume (4.4), we find that v is indeed a strong solution to Problem SP. Indeed, v turns out to solve our target Problem QSP as well.

Proposition 4.2. *Assume* (3.4), (4.1), *and* (4.2)-(4.3), *and let* $v \in BV_R([0,T])$ *be the unique solution in the sense of differential measures to Problem* SP *for the multifunction* C (3.10). *Then,* v *solves Problem* QSP *in the sense of differential measures. Moreover, if* (4.4) *holds, the unique strong solution* $v \in W^{1,1}(0,T)$ *to Problem* SP *is also a strong solution for Problem* QSP.

Taking into account Corollary 3.7 as well, as a by-product of Proposition 4.2 we have our first existence and approximation theorem.

Theorem 4.3. *Under* (3.4), (4.1), *and* (4.2)-(4.3), *Problem* QSP *admits a unique solution* u *in the sense of differential measures. Moreover,* u *is a pointwise limit of the sequence of the discrete solutions constructed by means of the catching-up algorithm. Assuming further* (4.4), u *turns out to be the unique strong solution to Problem* QSP.

Proof of Proposition 4.2. As for the first part of the statement, let us recall that we can associate to the unique solution v to Problem SP in the sense of differential measures a function $v_\nu \in L^1(0,T,\nu;H)$ (cf. Definition 2.6), fulfilling

$$-v_\nu(t) \in \partial I_{C(t)}(v(t)) \quad \text{for } \nu\text{−a.e. } t \in [0,T] \tag{4.9}$$

as well as (2.13), from which we deduce that v complies with (3.1), in view of (3.11). Then, in order to conclude that v is a solution to Problem QSP in the sense of differential measures, it suffices to show that

$$-v_\nu(t) \in \partial I_{K(t,v(t))}(v(t)) \quad \text{for } \nu\text{−a.e. } t \in [0,T],$$

namely

$$v_\nu(t)(v(t)-w) \leq 0 \quad \forall w \in K(t,v(t)) \quad \text{for } \nu\text{−a.e. } t \in [0,T]. \tag{4.10}$$

On the other hand, given our one-dimensional framework, either $c_*(t) = c^*(t)$ (and, in this case, we readily check that $C(t) \equiv K(t,v(t))$), or $c_*(t) < c^*(t)$. In the latter case, we have that

$$v_\nu(t) \begin{cases} \in (-\infty, 0] & \text{if } v(t) = c^*(t), \\ = 0 & \text{if } c_*(t) < v(t) < c^*(t), \\ \in [0,+\infty) & \text{if } v(t) = c_*(t), \end{cases} \tag{4.11}$$

for ν−almost every $t \in [0,T]$. Therefore, let us fix $t \in [0,T]$ such that (4.9) holds and $c_*(t) \neq c^*(t)$. Taking into account (4.11), we see that (4.10) trivially holds in t as soon as $c_*(t) < v(t) < c^*(t)$. Let us then assume that $v(t) = c^*(t) = K^*(t,c^*(t)) = K^*(t,v(t))$. By construction, $w \leq v(t)$ for every $w \in K(t,v(t))$, whereas by (4.11) necessarily $v_\nu(t) \leq 0$, so that once again (4.10) follows. Of course, (4.10) may be verified in a perfectly analogous way in the case $v(t) = c_*(t)$.

As for the second part of the statement, let us first recall that, by Remark 2.10, the unique strong solution v to Problem SP is also a solution in the sense of differential measures. Thus, it follows from the first part of this proof that v is a solution to Problem QSP in the sense of differential measures with the regularity $v \in W^{1,1}(0,T)$. Then, Remark 3.2 yields that v is a strong solution to Problem QSP, which concludes the proof. □

Remark 4.4. Let $u \in W^{1,1}(0,T;H)$ be a strong solution to Problem QSP: it follows from Proposition 3.6 that u is also both a strong solution and a solution in the sense of differential measures to Problem SP (cf. Remark 2.10). Then, by Proposition 4.2 u is the unique solution in the sense of differential measures to Problem QSP.

4.2 The case $H = L^2(\Omega)$.

We will now focus on the case in which our ambient space H is $L^2(\Omega)$, Ω being now a non-empty and open subset of $\mathbb{R}^N$. We will keep to this setting for the sake of clarity and develop the extension of our results to more general measure spaces in the forthcoming subsection.

We consider two integrands f^*, $f_* : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ such that there exist two functions g_*, $g^* \in L^2(\Omega)$ fulfilling

$$\begin{gathered} g_*(x) \leq f_*(t,x,w) \leq f^*(t,x,w) \leq g^*(x) \\ \forall w \in \mathbb{R}, \ \forall t \in [0,T], \text{ and for a.e. } x \in \Omega. \end{gathered} \tag{4.12}$$

We also suppose that for every $t \in [0,T]$ and for almost every $x \in \Omega$ the real functions

$$w \mapsto f^*(t,x,w) \quad \text{and} \quad w \mapsto f_*(t,x,w) \quad \text{are continuous,} \tag{4.13}$$
$$w \mapsto f^*(t,x,w) \quad \text{and} \quad w \mapsto f_*(t,x,w) \quad \text{are non-increasing.} \tag{4.14}$$

We can thus define a set-valued function $K : [0,T] \times L^2(\Omega) \to 2^{L^2(\Omega)}$ by setting

$$K(t,w) := \{z \in L^2(\Omega) \ : \ f_*(t,x,w(x)) \leq z(x) \leq f^*(t,x,w(x)) \text{ for a.e. } x \in \Omega\}, \tag{4.15}$$

for every $t \in [0,T]$, for every $w \in L^2(\Omega)$. In particular $K(t,w)$ is a subinterval (with respect to the pointwise essential order induced on $L^2(\Omega)$ by the order of $\mathbb{R}$), of the interval $[g_*, g^*]$.

In the sequel, we will adapt the approach developed in Subsection 4.1 to the the Problem QSP driven by K, supplemented as usual with an initial datum

$$u_0 \in L^2(\Omega) \quad \text{such that} \quad u_0 \in K(0,u_0). \tag{4.16}$$

Following the notation introduced in the previous sections, we define K^* and K_* pointwise by

$$K^*(t,w)(x) := f^*(t,x,w(x)), \quad K_*(t,w)(x) := f_*(t,x,w(x)) \quad \text{for a.e. } x \in \Omega,$$

for every $t \in [0,T]$, and every $w \in L^2(\Omega)$. The following lemma collects the properties of the multifunction K.

Lemma 4.5. *i) Under the assumptions* (4.12)-(4.14)*, for all $t \in [0,T]$ the operators $-K^*(t,\cdot)$, $-K_*(t,\cdot) : L^2(\Omega) \to L^2(\Omega)$ comply with* (3.5)-(3.7)*. In particular, there exists a unique pair c^*, $c_* \in L^\infty(0,T;L^2(\Omega))$ fulfilling*

$$f^*(t,x,c^*(t,x)) = c^*(t,x), \quad f_*(t,x,c_*(t,x)) = c_*(t,x), \quad \textit{and}$$
$$c_*(t,x) \leq c^*(t,x) \quad \textit{for a.e. } x \in \Omega \quad \forall t \in [0,T].$$

Further, the multifunction $C : [0,T] \to 2^{L^2(\Omega)}$ given by

$$C(t)(x) := [c_*(t,x), c^*(t,x)] \quad \textit{for a.e. } x \in \Omega \quad \forall t \in [0,T] \tag{4.17}$$

complies with (3.11)-(3.12).
ii). In addition, suppose that there exist two functions R^*, $R_* \in BV_R([0,T])$ such that

$$|f^*(t,x,w) - f^*(s,x,w)| \leq |R^*(t) - R^*(s)|, \tag{4.18}$$
$$|f_*(t,x,w) - f_*(s,x,w)| \leq |R_*(t) - R_*(s)| \tag{4.19}$$

for all $s,t \in [0,T]$ and $w \in \mathbb{R}$, and almost every $x \in \Omega$. Then, the multifunction C has finite retraction on $[0,T]$, with retraction function r in $BV_R([0,T])$. Moreover, if

$$R^*,\ R_* \quad \textit{are absolutely continuous on } [0,T], \tag{4.20}$$

then $r \in W^{1,1}(0,T)$.

Outline of the proof. Taking into account Example 2.4, we easily infer from (4.12)-(4.14) that for every $t \in [0,T]$ the functionals $-K^*(t,\cdot)$ and $-K_*(t,\cdot)$ comply with the assumptions (3.5)-(3.7). Then, on behalf of the general result Lemma 3.3 we can construct the set-valued function C (4.17) fulfilling (3.11)-(3.12). Note that c^* and c_* are in $L^\infty(0,T;L^2(\Omega))$ thanks to (4.12). Finally, in order to prove the last part of the statement it is sufficient to adapt the argument developed in the proof of Lemma 4.1. □

Moreau's existence result of Proposition 2.9 then ensures that, under the set of assumptions (4.12)-(4.19), the Cauchy problem SP for the sweeping process associated to the multifunction C (4.17), with the initial condition u_0 (4.16), admits a unique solution v in the sense of differential measures. Of course, v is our candidate solution for Problem QSP, and we have the following result.

Proposition 4.6. *Let* (4.12)-(4.14) *and* (4.18)-(4.19) *hold, and let v be the unique solution in the sense of differential measures to Problem* SP *for the multifunction C, supplemented with the initial condition u_0. Then, v is also the unique solution to Problem* QSP *in the sense of differential measures. Moreover, if* (4.20) *holds, then v is also a strong solution for Problem* QSP.

Proof. In order to prove the first part of the statement (the argument for the second one being analogous), we shall argue as in the proof of Proposition 4.2 by suitably reducing to dense subsets of Ω. Indeed, let us stress from the very beginning that the only property of the measure space we are going to exploit is the density of the Lebesgue points of integrable functions (see the forthcoming subsection, namely the representation formula (4.28) for the Lebesgue measure).

We start from a solution v to Problem SP in the sense of differential measures. Hence, $v \in BV_R([0,T], L^2(\Omega))$ and $v(t) \in C(t)$ for all $t \in [0,T]$. Owing to (3.11), we readily check that

$$v(t) \in K(t, v(t)) \quad \forall\, t \in [0,T]. \tag{4.21}$$

Let us now fix $t \in [0,T]$ such that (2.14) holds. It is a standard matter to verify that, for almost every $x \in D := \{y \in \Omega \ : \ c_*(t,y) \neq c^*(t,y)\}$, one has that

$$v_\nu(t,x) \begin{cases} \in (-\infty, 0] & \text{if } v(t,x) = c^*(t,x), \\ = 0 & \text{if } c_*(t,x) < v(t,x) < c^*(t,x), \\ \in [0, +\infty) & \text{if } v(t,x) = c_*(t,x). \end{cases} \tag{4.22}$$

We prove the first claim, the proof for the other two ones analogous. Let $x \in D$ be a Lebesgue point for the function $y \mapsto v_\nu(t,y)(v(t,y) - c_*(t,y))$ (which belongs to $L^1(\Omega)$), fulfilling $v(t,x) = c^*(t,x)$. Then, we put

$$z(y) := \begin{cases} c_*(t,y) & \text{if } y \in B_\rho(x), \\ v(t,y) & \text{otherwise}, \end{cases}$$

where $B_\rho(x) := \{y \in \Omega \ : \ |x-y| < \rho\}$, and we note that $z \in C(t)$ by construction. Then, owing to (2.14) we have that

$$\int_{B_\rho(x)} v_\nu(t,y)(v(t,y) - c_*(t,y))dy = \int_\Omega v_\nu(t,y)(v(t,y) - z(y))dy \leq 0,$$

Letting now ρ go to zero, one obtains that

$$v_\nu(t,x)(v(t,x) - c_*(t,x)) \leq 0,$$

and the first of (4.22) follows.

Let us now fix $w \in K(t, v(t))$ and define

$$\bar{w}(y) := c_*(t,y) \vee (w(y) \wedge c^*(t,y)) \quad \text{for a.e. } y \in \Omega.$$

We shall prove that

$$\int_\Omega v_\nu(t,y)(v(t,y)-w(y))dy =$$
$$\int_\Omega v_\nu(t,y)(v(t,y)-\bar{w}(y))dy + \int_\Omega v_\nu(t,y)(\bar{w}(y)-w(y))dy \leq 0. \tag{4.23}$$

To this aim, noting that $\bar{w} \in C(t)$ and thus the first term on the right-hand side of (4.23) is non-positive by (2.14), it is sufficient to show that

$$\int_D v_\nu(t,y)(\bar{w}(y)-w(y))dy \leq 0, \tag{4.24}$$

since, by (3.11)-(3.12), for every $x \in \Omega \setminus D$

$$c_*(t,x) = f_*(t,x,v(t,x)) = f^*(t,x,v(t,x)) = c^*(t,x) \quad \text{and} \quad \bar{w}(x) = w(x).$$

Now, in view of (4.22), we have

$$\begin{aligned} &\int_D v_\nu(t,y)(\bar{w}(y)-w(y))dy \\ &= \int_{\{y\,:\,v(t,y)=c^*(t,y)\}} v_\nu(t,y)(\bar{w}(y)-w(y))dy \\ &+ \int_{\{y\,:\,v(t,y)=c_*(t,y)\}} v_\nu(t,y)(\bar{w}(y)-w(y))dy. \end{aligned} \tag{4.25}$$

It is straightforward to check that $\bar{w}-w \geq 0$ (≤ 0, respectively), almost everywhere on the set $\{v=c^*\}$ ($\{v=c_*\}$ respectively). Then, taking again into account (4.22), we deduce that the right-hand side of (4.25) is non-positive and (4.24) follows. □

For the sake of completeness, let us state precisely our existence and approximation result.

Theorem 4.7. *Under* (4.12)-(4.16) *and* (4.18)-(4.19), *Problem* QSP *for the multifunction* K (4.15) *admits a unique solution in the sense of differential measures. Moreover,* u *is the pointwise limit of the sequence of the discrete solutions constructed by means of the catching-up algorithm. Assuming further* (4.20), u *turns out to be the unique strong solution to Problem* QSP.

4.3 The case $H = L^2(X,m)$.

Let us point out that in the proof of Proposition 4.6 we have actually made no use of the special choice of the measure space $\Omega \subset \mathbb{R}^N$, endowed with the Lebesgue measure. Indeed, the proof of Proposition 4.6 just relies on the fact that, given any integrable function $f \in L^1(\Omega)$, almost every point in Ω is a Lebesgue point for f. Namely, it turns out that the existence results of the latter subsection still hold in quite more general Hilbert spaces of L^2 type, as soon as the above mentioned property of Lebesgue points is preserved. So, we assume that

$$(X,d) \quad \text{is a locally compact and separable metric space} \tag{4.26}$$

and we endow it with

$$\text{a Radon measure } m, \tag{4.27}$$

(namely a Borel measure which is finite on compact subsets of X), such that, for all $f \in L^1(X,m)$,

$$f(x) = \lim_{\rho \downarrow 0} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} f\, dm \quad \text{for } m\text{–a.e. } x \in X, \tag{4.28}$$

where $B_\rho(x) := \{y \in X \ : \ d(x,y) < \rho\}$. Then, by simply following the proof of Proposition 4.6 we can establish our existence and uniqueness result in the case of $H := L^2(X,m)$. Instead of repeating the existence argument, we prefer to develop a brief discussion on condition (4.28) for the reader's convenience. In particular, we will provide some sufficient conditions for (4.28). First of all, we mention that (4.28) is fulfilled for all measure spaces (X,m) complying with (4.26)-(4.27), and the so-called *symmetric Vitali property*, i.e., for every collection $\mathcal{B}$ of balls which covers its set of centers $A := \{x \in X \ : \ B_\rho(x) \in \mathcal{B} \text{ for some } \rho > 0\}$ *finely* (i.e., for each $x \in A$ we have $\inf\{\rho > 0 \ : \ B_\rho(x) \in \mathcal{B}\} = 0$), there exists a countable pairwise disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ such that

$$y \in \bigcup_{B' \in \mathcal{B}'} B' \quad \text{for } m\text{–a.e. } y \in A,$$

provided that $m(A) < +\infty$ (see [20, Thm. 4.7, p. 24]).

As a matter of fact (see [20, Cor. 3.4, p. 12]), whenever there exists a positive constant c such that

$$m(B_{2\rho}(x)) \ \leq \ c\, m(B_\rho(x))$$

for all $x \in X$, $\rho > 0$, then (X,m) fulfills the symmetric Vitali property. In particular, the case of Subsection 4.2 belongs to this class.

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