Existence Result for the One-Dimensional Full Model of Phase Transitions*

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Abstract

This note deals with a nonlinear system of PDEs accounting for phase transition phenomena. The existence of solutions to a Cauchy-Neumann problem is established in the one dimensional space setting, using a regularization - a priori estimates - passage to limit procedure.

Key words: phase transitions, microscopic movements, existence result.

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1 Introduction

The present analysis is concerned with the evolution of two unknown fields θ and χ. In particular, we aim to consider the following pair of relations

$$\partial_t\theta + \theta\partial_t\chi - \partial_{xx}\theta = (\partial_t\chi)^2, \tag{1.1}$$

$$\partial_t\chi - \partial_{xx}\chi + \beta(\chi) \ni \theta - \theta_c, \tag{1.2}$$

to be fulfilled for almost every $(x,t) \in Q := (0,\ell) \times (0,T)$, for some $\ell, T > 0$. Here θ_c is a positive parameter and β stands for a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$.

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The latter system may rise in connection with a model for phase transition recently proposed by M. Frémond. We shall recall the framework of such model, referring indeed to the paper [4] for a full discussion about its derivation. First of all, let us consider a two-phase material located in a domain Ω in $\mathbb{R}^3$ and let us focus on the state variables θ (absolute temperature) and χ (local proportion of one of the two phases). We aim to describe the thermal evolution of this substance with the help of an energy balance equation coupled with a proper phase relation. From the physical point of view, the main novelty of this model relies on the assumption that the phase changes at the macroscopic level are affected by the microscopic movements of the molecules as well. Hence, a microscopic contribution has to be taken into account at the macroscopic level. Referring to [4] and [5] for the details, we are led to consider a system of the following kind

$$c_s\partial_t\theta + L\partial_t\chi - h\Delta\theta = -\frac{L}{\theta_c}(\theta-\theta_c)\partial_t\chi + \xi\partial_t\chi + \mu(\partial_t\chi)^2 + \kappa|\nabla\partial_t\chi|^2, \tag{1.3}$$

$$\mu\partial_t\chi + \xi - \kappa\Delta\partial_t\chi - \nu\Delta\chi + \partial I_{[0,1]}(\chi) \ni \frac{L}{\theta_c}(\theta-\theta_c), \qquad \xi\in\alpha(\partial_t\chi). \tag{1.4}$$

The graph $\alpha\subset\mathbb{R}\times\mathbb{R}$ is maximal monotone while the graph $\partial I_{[0,1]}$ turns out to be the subdifferential of the indicator function $I_{[0,1]}$ of the set $[0,1]$. In particular, we recall that

$$y\in\partial I_{[0,1]}(x) \quad \text{if and only if} \quad x\in[0,1] \ \text{ and } \ y(x-z)\geq 0 \ \ \forall z\in[0,1],$$

or, equivalently, $\partial I_{[0,1]}(x)=\emptyset$ if $x<0$ or $x>1$, $\partial I_{[0,1]}(0)=(-\infty,0]$, $\partial I_{[0,1]}(x)=\{0\}$ if $x\in(0,1)$, and $\partial I_{[0,1]}(1)=[0,+\infty)$. Moreover, the quantities c_s, L, h, k, and θ_c are positive physical parameters, while μ, κ, and ν are nonnegative. The reader is referred to [4, 5] for a full discussion on their physical meaning.

From the physical point of view, the presence of the nonlinear graph $\partial I_{[0,1]}$ has to be interpreted as the constraint $0\leq\chi\leq 1$ (recall that χ is a proportion) and the possible natural choices for the graph α are $\alpha\equiv 0$ and $\alpha=\partial I_{[0,+\infty)}$. Indeed, this latter position is specified from the variational inequality

$$y\in\partial I_{[0,+\infty)}(x) \quad \text{if and only if} \quad x\in[0,+\infty) \ \text{ and } \ y(x-z)\geq 0 \ \ \forall z\in[0,+\infty),$$

and accounts for possible *irreversibility* on the phase transition since $\partial_t\chi$ is forced to attain solely nonnegative values. On the contrary, this paper investigates exclusively the case when $\alpha\equiv 0$ (see (1.2)) i.e. a purely *reversible* phase transition is considered.

For the sake of introducing our results, we shall briefly recall some literature on the model. Many efforts have been recently directed to the analysis of Cauchy-Neumann problems related to different possible simplified version of the system (1.3)-(1.4). In particular, most of the former contributions address the irreversible case $\alpha=\partial I_{[0,+\infty)}$, assume $\kappa=0$ and perform some partial simplification of the energy balance equation (1.3) [4, 10]. The paper [11] proves the existence of a solution to the $\kappa=0$ problem (1.3)-(1.4) in the special case of a graph $\alpha=\partial I_{[0,\lambda]}$ where $\lambda>0$ stands for some limit speed for the phase transition. The full problem with $\kappa=\nu=0$ have been solved in [7] while, whenever $\nu>0$, the irreversible system (1.3)-(1.4) turns out to have a global strong solution in the one-dimensional setting due to [9]. As for the viscous problem (1.3)-(1.4) with $\kappa\neq 0$, we shall refer to the paper [5] that establishes the local in time existence of a weak solution to the problem under the assumption that the term $\kappa|\nabla\partial_t\chi|^2$ in (1.3) is negligible.

The aim of this paper is to investigate the purely reversible case in the 1-D setting for $\kappa = 0$. Normalizing most of the physical quantities, the system (1.3)-(1.4) turns out to be of the type of (1.1)-(1.2). The novelty of this contribution is the proof of the existence of a global strong solution in the one dimensional setting. The latter existence result stands as a reversible counterpart to the result of the paper [9].

Our work is organized as follows. First of all we set some useful notation in Section 2. Then, Section 3 is devoted to the precise statement of our existence result. The approximation of the system (1.1)- (1.2) in performed in Section 4, and Sections 5, 6, and 7 detail the study of the approximated problem. Finally, we establish some a priori estimates in Section 8 and the passage to the limit is obtained in Section 9.

2 Notations

We set

$$\Omega = (0,\ell), \quad Q_t = (0,\ell)\times(0,t) \quad \forall t \in (0,T].$$

Next, we let

$$H := L^2(0,\ell), \quad V := H^1(0,\ell), \quad \text{and} \quad W := H^2(0,\ell) \tag{2.1}$$

and identify H with its dual space H', so that

$$V \subset H \subset V',$$

with dense, compact, and continuous embeddings. Besides, let the symbol $\|\cdot\|$ denote the standard norm of H, while $\|\cdot\|_E$ stands for the norm of the general normed space E.

Moreover, we denote by $<\cdot,\cdot>$ the duality pairing between V' and V, by $(\cdot,\cdot)$ the scalar product in H, and by $J : V \to V'$ the Riesz isomorphism of V onto V'.

Note that, thanks to the one dimensional framework of our problems, we have the continuous injections

$$L^1(\Omega) \subset V', \qquad V \subset L^\infty(\Omega) \tag{2.2}$$

Hence, there exist two positive constants C_1 and C_2 such that the following relations hold

$$\begin{aligned} \|u\|_{V'} &\le C_1\|u\|_{L^1(\Omega)}, \quad \forall u \in L^1(\Omega), \\ \|u\|_{L^\infty(\Omega)} &\le C_2\|u\|_V, \quad \forall u \in V. \end{aligned} \tag{2.3}$$

Now, we recall an elementary inequality which will be useful in the sequel

$$ab \le (\delta/2)a^2 + (2\delta)^{-1}b^2 \quad \forall a,b \in \mathbb{R}, \quad \delta > 0\,. \tag{2.4}$$

Finally, we also remark that there exists a positive constant C_3 depending only on T such that the following holds for any $u \in H^1(0,T;H)$

$$\|u\|^2_{L^2(0,t;H)} \le C_3\left(\|u(0)\|^2 + \int_0^t \|\partial_t u\|^2_{L^2(0,s;H)} ds\right) \quad \forall t \in (0,T]. \tag{2.5}$$

3 Main result

We give here the precise statement of our problem, introducing the following assumptions on the data.

$$\theta_c, \theta^* > 0 \text{ are assigned constant,} \tag{3.1}$$

$$\begin{aligned} &\varphi : \mathbb{R} \to [0, +\infty] \quad \text{is proper, convex, and lower semicontinuous} \\ &\text{and there exists } \ C_4,\ C_5 > 0 \ \text{ such that} \\ &\varphi(r) \geq C_4 r^2 - C_5 \quad \forall r \in D(\varphi) \ \text{ and} \\ &\beta := \partial\varphi \text{ and } \ \varphi(0) = 0, \end{aligned} \tag{3.2}$$

$$\theta_0 \in V \text{ and } \ \theta_0 > 0 \quad \text{in } \ \overline{\Omega}, \tag{3.3}$$

$$\chi_0 \in H^3(\Omega), \tag{3.4}$$

$$\chi_0 \in D(\beta) \text{ a.e. in Q, and } \ \beta^0(\chi_0) \in H, \tag{3.5}$$

where $D(\varphi)$ and $D(\beta)$ denote the effective domain of φ and β, respectively, and $\beta^0(\chi_0)$ stands for the element of minimal norm of the set $\beta(\chi_0)$.

Problem 1. Find a triplet (θ, χ, η) such that

$$\theta \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{3.6}$$

$$\chi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^2(0,T;W), \tag{3.7}$$

$$\eta \in L^2(0,T;H), \tag{3.8}$$

$$\partial_t\theta + \theta\,\partial_t\chi - \partial_{xx}\theta = (\partial_t\chi)^2 \qquad \text{a.e. in } \ Q_T, \tag{3.9}$$

$$\partial_t\chi - \partial_{xx}\chi + \eta = \theta - \theta_c \qquad \text{a.e. in } \ Q_T, \tag{3.10}$$

$$\eta \in \beta(\chi) \qquad \text{a.e. in } \ Q_T, \tag{3.11}$$

$$\theta \geq 0 \qquad \text{a.e. in } \ Q_T, \tag{3.12}$$

$$\partial_x\theta(0,\cdot) = \partial_x\theta(\ell,\cdot) = 0, \quad \partial_x\chi(0,\cdot) = \partial_x\chi(\ell,\cdot) = 0 \quad \text{a.e. in } \ (0,T), \tag{3.13}$$

$$\theta(\cdot,0) = \theta_0, \quad \chi(\cdot,0) = \chi_0 \quad \text{a.e. in } \ \Omega. \tag{3.14}$$

Remark 3.1. Let us stress that the coercivity assumption on φ in (3.2) is perfectly motivated in our framework since $I_{[0,1]}(r) \geq r^2 - 1$ for all $r \in [0,1]$. Moreover, the hypothesis $\varphi(0) = 0$ may be replaced by the weaker $0 \in D(\varphi)$ without any particular intricacy.

Remark 3.2. We point out that the regularity assumption (3.4) is motivated by the sake of simplicity. Indeed, we are actually able to deal with data $\chi_0 \in W$ by exploiting a suitable regularization procedure.

Now, we are able to state the main result of the paper.

Theorem 3.3 (Existence). *Let assumptions (3.1)-(3.5) hold. Then Problem 1 admits at least a solution.*

The proof of this result will be carried out throughout the remainder of the paper by exploiting an approximation procedure. Indeed, we replace β with its Yosida approximation β_ε and solve the regularized problem by the means of fixed point techniques. Then, proper a priori estimates independent of ε are established and the passage to the limit is obtained via compactness and monotonicity arguments.

4 Approximation

For the sake of proving Theorem **3.3**, we apply a regularization procedure to the maximal monotone graph β. Namely, let β_ε be the Yosida approximation of β (we refer to [6] for details) and, consequently, denote by φ_ε the unique primitive of β_ε verifying $\varphi_\varepsilon(0)=0$. Note that (see [6, page 28]) one has

$$|\beta_\varepsilon(r)| \le |\beta^0(r)| \quad \text{for all } \varepsilon>0 \ \text{ and } \quad r\in D(\beta). \tag{4.1}$$

Moreover, it is straightforward to check that

$$\varphi_\varepsilon(r) = \min_{s\in D(\varphi)} \left(\frac{1}{2\varepsilon}|r-s|^2 + \varphi(s) \right). \tag{4.2}$$

Thus, we readily have that

$$\varphi_\varepsilon(r) \le \varphi(r) \quad \forall r\in D(\varphi). \tag{4.3}$$

Moreover, the function φ_ε is defined in all $\mathbb{R}$ and, taking into account the coercivity assumption in (3.2), it turns out to be coercive as well. Namely, we have that

$$\varphi_\varepsilon(r) \ge \frac{C_4}{2}r^2 - C_5 \quad \forall r\in\mathbb{R},\ \forall\varepsilon\in(0,(2C_4)^{-1}). \tag{4.4}$$

Indeed, let us consider $r\in\mathbb{R}$, $s\in D(\varphi)$, and $\varepsilon\in(0,(2C_4)^{-1})$. Then

$$\frac{C_4}{2}r^2 \le C_4|r-s|^2 + C_4 s^2$$
$$\le \frac{1}{2\varepsilon}|r-s|^2 + C_4 s^2 - C_5 + C_5 \le \frac{1}{2\varepsilon}|r-s|^2 + \varphi(s) + C_5,$$

and the assertion (4.4) is proved.

Let us introduce the approximating problems (the regularization parameter $\varepsilon>0$ being fixed)

Problem 1ε. Find a pair $(\theta_\varepsilon, \chi_\varepsilon)$ such that

$$\theta_\varepsilon \in H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W), \tag{4.5}$$
$$\chi_\varepsilon \in H^2(0,T;H)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;W), \tag{4.6}$$
$$\partial_t\theta_\varepsilon + \theta_\varepsilon\,\partial_t\chi_\varepsilon - \partial_{xx}\theta_\varepsilon = (\partial_t\chi_\varepsilon)^2 \quad \text{a.e. in } Q_T, \tag{4.7}$$
$$\partial_t\chi_\varepsilon - \partial_{xx}\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon) = \theta_\varepsilon - \theta_c \quad \text{a.e. in } Q_T, \tag{4.8}$$
$$\partial_x\theta_\varepsilon(0,\cdot) = \partial_x\theta_\varepsilon(\ell,\cdot) = 0, \quad \partial_x\chi_\varepsilon(0,\cdot) = \partial_x\chi_\varepsilon(\ell,\cdot) = 0 \quad \text{a.e. in } (0,T), \tag{4.9}$$
$$\theta_\varepsilon(\cdot,0) = \theta_0, \quad \chi_\varepsilon(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{4.10}$$

Remark 4.1. We stress that the approximation of relation (3.11) does not come into play in the formulation of Problem 1ε above. Indeed, the regularized graph β_ε is actually single-valued.

5 Uniqueness for the approximating problem

As regards the uniqueness of solutions to Problem 1ε, we are able to state and prove a result which also applies to more general situations.

Proposition 5.1 (Uniqueness to approximating problem). *Assume we are given $\theta_0, \chi_0 : \Omega \longrightarrow \mathbb{R}$. Then, there exists at most one solution to Problem 1ε.*

Proof. We proceed by contradiction. Let (θ_1, χ_1), (θ_2, χ_2) be two solutions to Problem 1ε and set $\tilde\theta = \theta_1 - \theta_2$ and $\tilde\chi = \chi_1 - \chi_2$. Moreover, let us consider the difference between the corresponding equations (4.8), multiply by $\partial_t\tilde\chi$ and integrate over Q_t. Thanks to the Lipschitz continuity of β_ε and to (2.4)-(2.5), we obtain

$$\begin{aligned} &\|\partial_t\tilde\chi\|^2_{L^2(0,t;V)} + \frac{1}{2}\|\partial_x\tilde\chi(t)\|^2 \\ &\leq \frac{1}{4}\|\partial_t\tilde\chi\|^2_{L^2(0,t;H)} + C\left(\|\tilde\theta\|^2_{L^2(0,t;H)} + \int_0^t \|\partial_t\tilde\chi\|_{L^2(0,s;H)}\,ds\right), \end{aligned} \tag{5.1}$$

where C depends on C_3 and ε. Next, we consider the difference between equation (4.7) written for (θ_1, χ_1) and the same equation written for (θ_2, χ_2), multiply by $\tilde\theta$ and integrate over Q_t. Using Hölder inequality we get

$$\begin{aligned} &\frac{1}{2}\|\tilde\theta(t)\|^2 + \|\partial_x\tilde\theta\|^2_{L^2(0,t;H)} \\ &\leq \iint_{Q_t} \big(|\partial_t\chi_1||\tilde\theta| + |\theta_2||\partial_t\tilde\chi| + |\partial_t\chi_1 + \partial_t\chi_2||\partial_t\tilde\chi|\big)|\tilde\theta| \\ &\leq \int_0^t \big(\|\partial_t\chi_1\|_{L^\infty(\Omega)}\|\tilde\theta\| + \|\theta_2\|_{L^\infty(\Omega)}\|\partial_t\tilde\chi\| + \|\partial_t\chi_1 + \partial_t\chi_2\|_{L^\infty(\Omega)}\|\partial_t\tilde\chi\|\big)\|\tilde\theta\| \\ &\leq \int_0^t \|\partial_t\chi_1\|_{L^\infty(\Omega)}\|\tilde\theta\|^2 + \frac{1}{2}\int_0^t \|\theta_2\|^2_{L^\infty(\Omega)}\|\partial_t\tilde\chi\|^2 + \frac{1}{2}\int_0^t \|\tilde\theta\|^2 \\ &+ \frac{1}{2}\int_0^t \|\partial_t\chi_1 + \partial_t\chi_2\|^2_{L^\infty(\Omega)}\|\tilde\theta\|^2 + \frac{1}{2}\|\partial_t\tilde\chi\|^2_{L^2(0,t;H)}. \end{aligned} \tag{5.2}$$

Now we add (5.1) and (5.2) and we apply the Gronwall lemma (see, e.g., [2]), noting that $\|\partial_t\chi_1\|_{L^\infty(\Omega)}$, $\|\theta_2\|^2_{L^\infty(\Omega)}$, and $\|\partial_t\chi_1 + \partial_t\chi_2\|^2_{L^\infty(\Omega)}$ belong to $L^\infty(0,T)$. We deduce that $\tilde\theta = \partial_t\tilde\chi = 0$ a.e. in Q and the assertion is proved. □

Remark 5.2 (on uniqueness). Let us stress that the uniqueness result still holds for any Lipschitz continuous function β_ε. Indeed, our proof does not rely on the particular properties of the Yosida approximation of β. Moreover, we didn't use any information on $\partial_{tt}\chi_\varepsilon$. Hence, in the case of β Lipschitz continuous, no regularization is needed and the solution to Problem 1, given by Theorem **3.3** is also unique.

6 Existence for the approximating problem

We conclude the well-posedness proof for Problem 1ε stating and proving the existence part.

Proposition 6.1 (Existence for the approximating problem). *Let (3.1)-(3.5) hold. Than, there exists at least a solution to Problem* 1ε.

Proof. We are going to apply Schauder fixed point theorem. To this end, we introduce

$$Y := \{ f \in H^1(0,T;H) \cap L^\infty(0,T;V) \} .$$

Standard arguments ensure that the following problem admits a unique solution u

Problem 1εa. Let $f \in Y$ be given. Find u such that

$$u \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W), \tag{6.1}$$

$$\partial_t u - \partial_{xx} u + \beta_\varepsilon(u) = f - \theta_c \quad \text{a.e. in } Q_T, \tag{6.2}$$

$$\partial_x u(0,\cdot) = \partial_x u(\ell,\cdot) = 0, \quad \text{a.e. in } (0,T), \tag{6.3}$$

$$u(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{6.4}$$

Now, given such u, it is easy to find the unique solution v of the following problem.

Problem 1εb. Let $u \in L^\infty(Q_T)$ with $\partial_t u \in L^\infty(Q_T)$ be given. Find v such that

$$v \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{6.5}$$

$$\partial_t v + v\partial_t u - \partial_{xx} v = (\partial_t u)^2 \quad \text{a.e. in } Q_T, \tag{6.6}$$

$$\partial_x v(0,\cdot) = \partial_x v(\ell,\cdot) = 0, \quad \text{a.e. in } (0,T), \tag{6.7}$$

$$v(\cdot,0) = \theta_0 \quad \text{a.e. in } \Omega. \tag{6.8}$$

We have implicitly defined a mapping $S: \ Y \to \ Y$, with $S(f) = v$. The continuity of S may be proved arguing as in the derivation of (6.19)-(6.21) below and we omit its direct check for the sake of simplicity. Since we aim to deduce the existence of fixed points, we need to prove that the S is compact in Y, i.e. it maps bounded sets into compact ones. Hence, let $\{f_n\}$ be a bounded set in Y, namely.

$$\|f_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le C_7, \tag{6.9}$$

for some positive constant C_7. By a classical compactness result (see, e.g., [8]), there exists f such that, at least for subsequences (not relabeled),

$$f_n \to f \quad \text{strongly in} \quad L^2(0,T;H), \tag{6.10}$$

as $n \to +\infty$.

Denote by u_n the solution of Problem 1εa corresponding to the datum f_n and by v_n the solution of Problem 1εb corresponding to u_n. As for to prove the strong convergence of $\{v_n\}$ in Y, we need some *a priori* bounds on $\partial_t u_n$ and v_n in order to deal with the nonlinearities. Henceforth C will denote any constant, possibly depending on data, ε, and C_7 but not on n. Of course, C may vary from line to line.

First of all, we multiply (6.2) (with f_n and u_n instead of f and u, respectively) by $\partial_t u_n$ and we integrate over Q_t. Using (2.4)-(2.5) we find that

$$\|\partial_t u_n\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_x u_n(t)\|^2 \le \frac{1}{2}\|\partial_x \chi_{0\varepsilon}\|^2 + \frac{1}{2}\|\partial_t u_n\|^2_{L^2(0,t;H)} + C\left(\|\chi_0\|^2 + \|f_n\|^2_{L^2(0,t;H)} + \int_0^t \|\partial_t u_n\|_{L^2(0,s;H)}\,ds\right). \tag{6.11}$$

Applying the Gronwall lemma and thanks to (6.9), we deduce that

$$\|u_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le C. \tag{6.12}$$

Next, we differentiate (6.2) (with f_n and u_n instead of f and u, respectively) with respect to t, multiply the resulting equation by $\partial_{tt} u_n$, and integrate over Q_t. One obtains

$$\|\partial_{tt} u_n\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_{xt} u_n(t)\|^2 \le \frac{1}{2}\|\partial_{xt} u_n(0)\|^2 + \int_0^t \beta'_\varepsilon(u_n)\|\partial_t u_n\|\,\|\partial_{tt} u_n\| + \int_0^t \|\partial_t f_n\|\,\|\partial_{tt} u_n\|. \tag{6.13}$$

Let us stress that, setting $t = 0$ in (6.2) (with f_n and u_n instead of f and u, respectively), we have

$$\|\partial_t u_n(0)\|_V \le \|\chi_0\|_{H^3(\Omega)} + \|\beta_\varepsilon(\chi_0)\|_V + \|\theta_0\|_V + \|\theta_c\|_V\,. \tag{6.14}$$

Using the Lipschitz continuity of β_ε (for ε fixed) (3.3), (6.9), and (6.12), we deduce that

$$\|u_n\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)} \le C\,. \tag{6.15}$$

Next, we multiply (6.6) (with $\partial_t u_n$ and v_n instead of $\partial_t u$ and v, respectively) by $\partial_t v_n$ and we integrate over Q_t obtaining

$$\|\partial_t v_n\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_x v_n(t)\|^2 \le \frac{1}{2}\|\partial_x \theta_0\|^2 + \int_0^t |v_n \partial_t u_n + (\partial_t u_n)^2||\partial_t v_n|. \tag{6.16}$$

Thanks to Hölder inequality, the integral in the above right hand side is bounded by

$$\int_0^t \Big(\|v_n\|_{L^\infty(\Omega)}\|\partial_t u_n\| + \|\partial_t u_n\|_{L^\infty(\Omega)}\|\partial_t u_n\|\Big)\|\partial_t v_n\|. \tag{6.17}$$

Using now (2.3)-(2.4), (6.15), and the Gronwall lemma, we deduce that

$$\|v_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le C. \tag{6.18}$$

Recall that f is the strong limit of f_n (see (6.10)) and denote by u the corresponding solution of Problem 1εa. We set $\tilde f = f_n - f$ and $\tilde u = u_n - u$; we consider the difference between the related equations (6.2), multiply by $\partial_t \tilde{u}$ and integrate over Q_t. Arguing as in the derivation of (5.1), Gronwall lemma and (6.10) enable us to deduce that

$$\partial_t u_n \to \partial_t u \quad \text{strongly in} \quad L^2(0,T;H), \tag{6.19}$$

as $n \to +\infty$. Our aim is to prove

$$v_n \to v \quad \text{strongly in} \quad H^1(0,T;H) \cap L^\infty(0,T;V), \tag{6.20}$$

as $n \to +\infty$, where v is the solution of Problem 1εb corresponding to the datum u.

We set $\tilde{v} = v_n - v$; we consider the difference between equation (6.6) written for v_n and the same relation written for v, multiply by $\partial_t \tilde{v}$, and integrate over Q_t. Arguing as above, we have

$$\begin{aligned} &\|\partial_t \tilde{v}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_x \tilde{v}(t)\|^2 \\ &\le \int_0^t \|\partial_t u_n\|^2_{L^\infty(\Omega)} \|\tilde{v}\|^2 + \int_0^t \|v\|^2_{L^\infty(\Omega)} \|\partial_t \tilde{u}\|^2 \\ &+ \int_0^t \|\partial_t u_n + \partial_t u\|^2_{L^\infty(\Omega)} \|\partial_t \tilde{u}\|^2 \, ds + \frac{3}{4}\|\partial_t \tilde{v}\|^2_{L^2(0,t;H)}. \end{aligned} \tag{6.21}$$

Thanks to (2.3), (6.15), and (6.18)-(6.19), we achieve (6.20). This completes the proof of Proposition **6.1** and hence our approximating Problem 1ε is well posed. □

7 Positivity for the approximating problem

Finally, we aim to establish a crucial lower bound for θ_ε in Q. Indeed, we will prove the following.

Lemma 7.1 (Positivity for the approximating problem). *Let $(\theta_\varepsilon, \chi_\varepsilon)$ be a solution to Problem 1ε. Then, there exists a constant $\theta^*_\varepsilon > 0$ such that*

$$\theta_\varepsilon \ge \theta^*_\varepsilon \quad \text{a.e. in} \;\; Q. \tag{7.1}$$

Proof. Let us take into account relation (4.7). One has that

$$\partial_t \theta_\varepsilon - \partial_{xx} \theta_\varepsilon = -\partial_t \chi_\varepsilon \theta_\varepsilon + (\partial_t \chi_\varepsilon)^2 =: a\theta_\varepsilon + b. \tag{7.2}$$

Now, owing to the regularity (4.6), we easily deduce that

$$a, b \in L^\infty(Q) \quad \text{and} \quad b \ge 0 \;\; \text{a.e. in} \;\; Q.$$

Let us set $\theta^* := \min \theta_0$ (recall that $\theta_0 \in V$) and $C_{5,\varepsilon} := \|a\|_{L^\infty(Q)}$. It suffices to multiply (4.7) by the function

$$\vartheta := (\theta_\varepsilon - \theta^* e^{-C_{5,\varepsilon} t})^- \in L^2(Q),$$

and take the integral over Q_t for $t \in (0,T)$. We obtain

$$\int_0^t\int_\Omega \big(\partial_t(\theta^* e^{-C_{5,\varepsilon} s} - \vartheta)\vartheta - \vartheta_x^2 - a(\theta^* e^{-C_{5,\varepsilon} s} - \vartheta)\vartheta\big)dx\,ds \geq 0,$$

hence

$$\frac{1}{2}\|\vartheta(t)\|^2 + \int_0^t\int_\Omega \big(|\partial_x\vartheta(x,s)|^2 + \theta^*(C_{5,\varepsilon} + a(x,s))e^{-C_{5,\varepsilon} s}\vartheta\big)\,dx\,ds \leq \frac{1}{2}\|\vartheta(0)\|^2 + C_{5,\varepsilon}\int_0^t \|\vartheta(s)\|^2 ds.$$

Since $\vartheta(0) = 0$, an application of Gronwall lemma implies that $\vartheta = 0$ a.e. on Q. Hence relation (7.1) is proved with $\theta_\varepsilon^* := \theta^* e^{-C_{5,\varepsilon} T}$.

□

8 A priori estimates

We are now interested in deducing some estimates for $(\theta_\varepsilon, \chi_\varepsilon)$, independent of the parameter ε. Henceforth, let C denote any constant, possibly depending on the data, but not on ε. Of course, C may vary from line to line.

8.1 First estimate

Let us multiply (4.7) by 1 and integrate over Q_t. Moreover, we multiply (4.8) by $\partial_t\chi_\varepsilon$ and integrate over Q_t. Taking the sum of the resulting expressions and performing some cancellations, we obtain that

$$\begin{aligned}
&\int_\Omega \theta_\varepsilon(t) + \frac{1}{2}\|\partial_x\chi_\varepsilon(t)\|^2 + \int_\Omega \varphi_\varepsilon(\chi_\varepsilon(t)) \\
&\leq \int_\Omega \theta_0 + \frac{1}{2}\|\partial_x\chi_0\|^2 + \int_\Omega \varphi_\varepsilon(\chi_0) + \theta_c\left|\int_\Omega \big(\chi_\varepsilon(t) - \chi_0\big)\right| \\
&\leq \int_\Omega \varphi(\chi_0) + \theta_c\|\chi_\varepsilon(t)\|_{L^1(\Omega)} + C,
\end{aligned} \tag{8.1}$$

where we also used assumptions (3.3)-(3.4) and relation (4.3). In order to control the right hand side of (8.1) it suffices to recall (3.5) and observe that (see (4.4))

$$\theta_c\|\chi_\varepsilon(t)\|_{L^1(\Omega)} \leq \frac{C_4}{4}\|\chi_\varepsilon(t)\|^2 + C \leq \frac{1}{2}\int_\Omega \varphi_\varepsilon(\chi_\varepsilon(t)) + C,$$

whenever ε is small enough.

Hence, moving from (7.1), we readily deduce that

$$\|\theta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \tag{8.2}$$

$$\|\chi_\varepsilon\|_{L^\infty(0,T;V)} \leq C, \tag{8.3}$$

$$\|\varphi_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \tag{8.4}$$

at least for sufficiently small ε.

8.2 Second estimate

Let us multiply equation (4.7) by the function $-\theta_\varepsilon^{-1}$. The latter choice turns out to be admissible due to (7.1) since $-\theta_\varepsilon^{-1} \in L^\infty(Q_T)$. Moreover, we integrate on Q_t, and exploit (3.3) and (8.2) in order to get

$$\begin{aligned} -\int_\Omega \ln\left(\theta_\varepsilon(t)\right) + \iint_{Q_t} \left(\frac{(\partial_x\theta_\varepsilon)^2}{\theta_\varepsilon^2} + \frac{\partial_t\chi_\varepsilon^2}{\theta_\varepsilon}\right) &\leq -\int_\Omega \ln(\theta_0) + \iint_{Q_t} \partial_t\chi_\varepsilon \\ \leq -\int_\Omega \ln(\theta_0) + \frac{1}{2}\iint_{Q_t} \left(\frac{\partial_t\chi_\varepsilon^2}{\theta_\varepsilon} + \theta_\varepsilon\right) &\leq C + \frac{1}{2}\iint_{Q_t} \frac{\partial_t\chi_\varepsilon^2}{\theta_\varepsilon}. \end{aligned}$$

Of course the first term in the latter left hand side is bounded by means of (8.2). Then, the bound (8.2) and the continuity of the inclusion $W^{1,1}(\Omega) \subset L^\infty(\Omega)$ entail in particular that

$$\begin{aligned} \int_0^T \|\theta\|_{L^\infty(\Omega)} &= \int_0^T \|\theta^{1/2}\|^2_{L^\infty(\Omega)} \leq C\int_0^T \left(\|\partial_x(\theta^{1/2})\|^2_{L^1(\Omega)} + \|\theta\|^2_{L^1(\Omega)}\right) \\ &\leq C\left(1 + \int_0^T \left(\int_\Omega \frac{\partial_x\theta}{\theta^{1/2}}\right)^2\right) \leq C\left(1 + \int_0^T \left(\|\partial_x\theta/\theta\|\,\|\theta^{1/2}\|\right)^2\right) \\ &\leq C\left(1 + \int_0^T \|\partial_x\theta/\theta\|^2\right) \leq C. \end{aligned} \tag{8.5}$$

Hence

$$\|\theta_\varepsilon\|_{L^1(0,T;L^\infty(\Omega))} \leq C, \tag{8.6}$$

and finally, by interpolation with (8.2),

$$\|\theta_\varepsilon\|_{L^2(0,T;H)} \leq C. \tag{8.7}$$

8.3 Third estimate

Taking into account (8.7), it is now a standard matter to choose $v = \partial_t\chi_\varepsilon$ in (4.8), integrate on Q_t, exploit the monotonicity of β_ε, and obtain the bound

$$\|\chi_\varepsilon\|_{H^1(0,T;H)} \leq C. \tag{8.8}$$

8.4 Fourth estimate

We multiply (4.7) by $J^{-1}\partial_t\theta_\varepsilon$ in the duality pairing between V' and V and we take the integral with respect to time. Moreover, let us differentiate (4.8) with respect to t, multiply it by $\partial_t\chi_\varepsilon$, and integrate over Q_t. Finally, we add the resulting expressions and, thanks to the monotonicity of β_ε, we obtain

$$\begin{aligned}
&\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;V')} + \frac{1}{2}\|\theta_\varepsilon(t)\|^2 + \frac{1}{2}\|\partial_t\chi_\varepsilon(t)\|^2 + \|\partial_{xt}\chi_\varepsilon\|^2_{L^2(0,t;H)} \\
&\le \frac{1}{2}\|\theta_0\|^2 + \frac{1}{2}\|\partial_t\chi_\varepsilon(0)\|^2 + \int_0^t |<\theta_\varepsilon, J^{-1}\partial_t\theta_\varepsilon>| + \int_0^t |<\theta_\varepsilon\partial_t\chi_\varepsilon, J^{-1}\partial_t\theta_\varepsilon>| \\
&\quad + \int_0^t |<(\partial_t\chi_\varepsilon)^2, J^{-1}\partial_t\theta_\varepsilon>| + \int_0^t |(\partial_t\theta_\varepsilon, \partial_t\chi_\varepsilon)|. \qquad (8.9)
\end{aligned}$$

We now estimate the integrals in the right hand side of (8.9). Using (2.3), the definition of J, and (2.4), we get

$$\begin{aligned}
\int_0^t |<\theta_\varepsilon, J^{-1}\partial_t\theta_\varepsilon>| &\le \int_0^t \|\theta_\varepsilon\|_{V'}\|J^{-1}\partial_t\theta_\varepsilon\|_V \\
&\le C\int_0^t \|\theta_\varepsilon\|^2 + \frac{1}{8}\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;V')}, \qquad (8.10)
\end{aligned}$$

$$\begin{aligned}
&\int_0^t |<\theta_\varepsilon\partial_t\chi_\varepsilon, J^{-1}\partial_t\theta_\varepsilon>| \le \int_0^t \|\theta_\varepsilon\partial_t\chi_\varepsilon\|_{V'}\|J^{-1}\partial_t\theta_\varepsilon\|_V \\
&\le C_1\int_0^t \|\theta_\varepsilon\partial_t\chi_\varepsilon\|_{L^1(\Omega)}\|\partial_t\theta_\varepsilon\|_{V'} \le 2C_1^2\int_0^t \|\theta_\varepsilon\|^2\|\partial_t\chi_\varepsilon\|^2 + \frac{1}{8}\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;V')}, \qquad (8.11)
\end{aligned}$$

$$\begin{aligned}
&\int_0^t |<(\partial_t\chi_\varepsilon)^2, J^{-1}\partial_t\theta_\varepsilon>| \le \int_0^t \|(\partial_t\chi_\varepsilon)^2\|_{V'}\|J^{-1}\partial_t\theta_\varepsilon\|_V \\
&\le C_1\int_0^t \|(\partial_t\chi_\varepsilon)^2\|_{L^1(\Omega)}\|\partial_t\theta_\varepsilon\|_{V'} \le 2C_1^2\int_0^t \|\partial_t\chi_\varepsilon\|^4 + \frac{1}{8}\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;V')}, \qquad (8.12)
\end{aligned}$$

$$\begin{aligned}
&\int_0^t |(\partial_t\theta_\varepsilon, \partial_t\chi_\varepsilon)| \\
&\le \int_0^t \|\partial_t\theta_\varepsilon\|_{V'}\|\partial_t\chi_\varepsilon\|_V \le \frac{1}{2}\|\partial_t\chi_\varepsilon\|^2_{L^2(0,t;V)} + \frac{1}{2}\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;V')}. \qquad (8.13)
\end{aligned}$$

Taking (8.10)-(8.13) into account, relation (8.9) becomes

$$\frac{1}{8}\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;V')}+\frac{1}{2}\|\theta_\varepsilon(t)\|^2+\frac{1}{2}\|\partial_t\chi_\varepsilon(t)\|^2+\frac{1}{2}\|\partial_{xt}\chi_\varepsilon\|^2_{L^2(0,t;H)}$$
$$\leq\frac{1}{2}\|\theta_0\|^2+\frac{1}{2}\|\partial_{xx}\chi_0-\beta_\varepsilon(\chi_0)+\theta_0-\theta_c\|^2+\frac{1}{2}\int_0^t\|\partial_t\chi_\varepsilon\|^2$$
$$+C\int_0^t\|\theta_\varepsilon\|^2+C_1^2\int_0^t\|\theta_\varepsilon\|^4+3C_1^2\int_0^t\|\partial_t\chi_\varepsilon\|^4. \tag{8.14}$$

We remark that, thanks to (3.5) and (4.1), we have

$$\|\beta_\varepsilon(\chi_0)\|\leq C. \tag{8.15}$$

Owing to (8.7) and (8.8), we are now in the position of applying the Gronwall lemma in order to conclude that the following upper bounds hold

$$\|\theta_\varepsilon\|_{H^1(0,T;V')\cap L^\infty(0,T;H)}\leq C, \tag{8.16}$$
$$\|\chi_\varepsilon\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)}\leq C. \tag{8.17}$$

8.5 Fifth estimate

We multiply (4.8) by $-\partial_{xx}\chi_\varepsilon$ and we integrate over Q_t obtaining

$$\frac{1}{2}\|\partial_x\chi_\varepsilon(t)\|^2+\|\partial_{xx}\chi_\varepsilon\|^2_{L^2(0,t;H)}+\iint_{Q_t}\beta'_\varepsilon(\chi_\varepsilon)|\partial_x\chi_\varepsilon|^2$$
$$\leq\frac{1}{2}\|\partial_x\chi_0\|^2+\|\theta_\varepsilon\|^2_{L^2(0,t;H)}+\theta_c^2\ell T+\frac{1}{2}\|\partial_{xx}\chi_\varepsilon\|^2_{L^2(0,t;H)}. \tag{8.18}$$

Thus, recalling (3.4) and the monotonicity of β_ε and taking into account standard elliptic estimates, we deduce the bound

$$\|\chi_\varepsilon\|_{L^2(0,T;W)\cap L^\infty(0,T;V)}\leq C. \tag{8.19}$$

8.6 Sixth estimate

We multiply (4.7) by $\partial_t\theta_\varepsilon$ and we integrate over Q_t getting

$$\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;H)}+\frac{1}{2}\|\partial_x\theta_\varepsilon(t)\|^2\leq\frac{1}{2}\|\partial_x\theta_0\|^2$$
$$+\frac{1}{4}\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;H)}+\|\partial_t\chi_\varepsilon\|^2_{L^\infty(0,t;H)}\|\partial_t\chi_\varepsilon\|^2_{L^2(0,t;L^\infty(\Omega))}$$
$$+\frac{1}{4}\|\partial_t\theta_\varepsilon\|^2_{L^2(0,t;H)}+\int_0^t\|\partial_t\chi_\varepsilon\|^2\|\theta_\varepsilon\|^2_{L^\infty(\Omega)}, \tag{8.20}$$

Hence, thanks to (2.3), (3.3), and (8.17), the Gronwall lemma allows us to deduce the bound

$$\|\theta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}\leq C. \tag{8.21}$$

8.7 Further estimates

By comparison in (4.7) and owing to (8.17), (8.21), and standard elliptic estimates, we obtain the bound

$$\|\theta_\varepsilon\|_{L^2(0,T;W)} \leq C, \tag{8.22}$$

while, from (4.8) (thanks to (8.16)-(8.17) and (8.19)), the information is

$$\|\beta_\varepsilon(\chi_\varepsilon)\|_{L^2(0,T;H)} \leq C. \tag{8.23}$$

9 Passage to the limit

Taking into account well-known compactness results, the bounds (8.17), (8.19), (8.21)-(8.23) allow us to deduce the existence of a triplet of functions (θ, χ, η) such that (possibly passing to not relabeled subsequences) the following convergences hold

$$\theta_\varepsilon \longrightarrow \theta \quad \text{weakly in} \quad H^1(0,T;H) \cap L^2(0,T;W), \tag{9.1}$$
$$\theta_\varepsilon \longrightarrow \theta \quad \text{weakly star in} \quad L^\infty(0,T;V), \tag{9.2}$$
$$\chi_\varepsilon \longrightarrow \chi \quad \text{weakly in} \quad H^1(0,T;V) \cap L^2(0,T;W), \tag{9.3}$$
$$\chi_\varepsilon \longrightarrow \chi \quad \text{weakly star in} \quad W^{1,\infty}(0,T;H), \tag{9.4}$$
$$\beta_\varepsilon(\chi_\varepsilon) \longrightarrow \eta \quad \text{weakly in} \quad L^2(0,T;H). \tag{9.5}$$

Moreover, from (8.17), (8.19), (8.21), and the generalized Ascoli theorem (see, e.g., [12, Cor. 4]), we may also infer the convergences

$$\theta_\varepsilon \longrightarrow \theta \quad \text{strongly in} \quad C^0([0,T];H) \cap L^2(0,T;V), \tag{9.6}$$
$$\chi_\varepsilon \longrightarrow \chi \quad \text{strongly in} \quad C^0([0,T];V). \tag{9.7}$$

Let us stress that, owing to (7.1), the convergence (9.6) entail that (3.12) holds.

Hence, we can pass to the limit in (4.8), and see that the properties (3.6)-(3.8) along with (3.10) are fulfilled by the triplet (θ, χ, η).

As for the interpretation of η, we observe that relations (9.5) and (9.7) allow us to use [3, Prop 1.1, p. 42], which readily yields (3.11).

Our next goal is to pass to the limit in (4.7). We remark that from (9.1), (9.3)-(9.4), and (9.6), we achieve that

$$\theta_\varepsilon\, \partial_t \chi_\varepsilon \;\longrightarrow\; \theta\, \partial_t \chi \quad \text{weakly star in} \quad L^\infty(0,T;H).$$

The critical term is $(\partial_t \chi_\varepsilon)^2$. We will prove that

$$\lim_{\varepsilon \to 0} \iint_{Q_T} (\partial_t \chi_\varepsilon)^2 = \iint_{Q_T} (\partial_t \chi)^2. \tag{9.8}$$

Let us multiply (4.8) by $\partial_t \chi_\varepsilon$, integrate over Q_T, and obtain

$$\iint_{Q_T} (\partial_t \chi_\varepsilon)^2 = -\frac{1}{2}\|\partial_x \chi_\varepsilon(T)\|^2 + \frac{1}{2}\|\partial_x \chi_0\|^2$$
$$+ \iint_{Q_T} (\theta_\varepsilon - \theta_c)\partial_t \chi_\varepsilon - \int_\Omega \varphi_\varepsilon(\chi_\varepsilon(T)) + \int_\Omega \varphi_\varepsilon(\chi_0). \tag{9.9}$$

We are now forced to discuss a technical argument about the convergence of convex functionals. Indeed, the functional induced by φ_ε on H, namely

$$\Phi_\varepsilon(v) := \int_\Omega \varphi_\varepsilon(v)\,dx \quad \text{for } v \in H, \tag{9.10}$$

turns out to converge in the sense of Mosco [1, Prop. 3.56, p. 354] in H to the functional

$$\Phi(v) := \begin{cases} \int_\Omega \varphi(v)\,dx & \text{if } v \in H \text{ and } \varphi(v) \in L^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{9.11}$$

In particular, owing to (9.7), one has that

$$\int_\Omega \varphi(\chi(T)) \le \limsup_{\varepsilon \to 0} \int_\Omega \varphi_\varepsilon(\chi_\varepsilon(T)).$$

Hence, taking into account the latter relation, (4.3), (9.3), (9.6)-(9.7) and passing to the limsup as $\varepsilon \to 0$ of both sides of (9.9) we infer that

$$\limsup_{\varepsilon \longrightarrow 0} \iint_{Q_T} (\partial_t \chi_\varepsilon)^2 \le -\frac{1}{2}\|\partial_x \chi(T)\|^2 + \frac{1}{2}\|\partial_x \chi_0\|^2$$
$$+ \iint_{Q_T} (\theta - \theta_c)\partial_t \chi - \int_\Omega \varphi(\chi(T)) + \int_\Omega \varphi(\chi_0) = \iint_{Q_T} (\partial_t \chi)^2, \tag{9.12}$$

and relation (9.8) is a straightforward consequence of the lower semicontinuity of the norm. Finally, owing to (9.8), we readily get that

$$\partial_t \chi_\varepsilon \longrightarrow \partial_t \chi \quad \text{strongly in} \quad L^2(0,T;H).$$

Hence, we easily pass to the limit in equation (4.7) obtaining (3.9). The proof of Theorem 3.3 is complete.

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