Some remarks on convergence and approximation for a class of hysteresis problems

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Abstract

This note addresses some convergence and approximation issues for hysteresis operators of stop and Prandtl-Ishlinskiĭ type as well as for some semilinear and quasilinear parabolic PDE's with hysteresis. By exploiting classic approximation results for evolutions equations with convex potentials, we discuss both some conditional weak continuity of the above-mentioned operators and their approximability by means of penalization, regularization etc. Moreover, we present some PDE examples where the latter conditional weak continuity naturally applies.

Key words: Mosco convergence, hysteresis operators, PDE's with hysteresis.

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Sunto. In questa nota ci occupiamo di alcune questioni di convergenza ed approssimazione per l'operatore di stop ed operatori di tipo Prandtl-Ishlinskiĭ e per alcuni problemi parabolici con isteresi. Utilizzando i classici risultati di convergenza per problemi di evoluzione con potenziali convessi, ci si occupa sia della debole continuità condizionata degli operatori sopra menzionati che della loro approssimabilità mediante metodi di penalizzazione, regolarizzazione, ecc. Si presentano quindi alcuni esempi di problemi alle derivate parziali di tipo semilineare e quasilineare che rientrano naturalmente nella teoria.

1 Introduction

In the last decades the mathematical treatment of hysteresis phenomena has attracted a good deal of attention and a considerable collection of differential models of hysteresis is nowadays available. The reader can find a comprehensive presentation on the mathematical theory of hysteresis in the monographs by Brokate & Sprekels [7], Krejčí [13], and Visintin [24]. Moving from the original interest on the properties of input-output hysteresis relations (see Krasnosel'skiĭ & Pokrovskiĭ [12]), the research focus has progressively shifted in the direction of the PDE problems with hysteresis effects. Form the applicative point of view, this is indeed a meaningful step. In fact, PDE problems with hysteresis are commonly arising in a variety of different modeling situations ranging from Thermo-Mechanics (plasticity, fatigue

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and damage, phase change, for instance), to ferroelectricity, ferromagnetism, filtration, and economic models (among others).

PDE problems with hysteresis are generally tackled by approximation. From the analytical point of view, one usually addresses some simpler problem (discretized, regularized, etc.) and passes to suitable limits in order to check for existence of solutions. This is often a delicate task since hysteresis relations are generally not continuous with respect to weak topologies. On the other hand, the latter weak topologies are usually suited with respect to generalized solvability. This issues will be discussed to some extent in the present note. Of course, the possibility of implementing some effective approximation procedure is the starting point both for the numerical assessment of the model and the necessary experimental validation (not discussed here).

The aim of this note is to provide some general discussion on convergence and approximation issues for a special class of hysteresis relations, namely the *stop* operator and *Prandtl-Ishlinskiĭ* operators of stop type. These are surely the simplest examples of hysteresis models arising within applications (plasticity, for instance). We shall not provide here an extensive discussion on this class of operators but rather refer to the aforementioned books for a collection of details and results. In particular, our framework is taken from Krejčí [14] where the reader can find also some detailed discussion on continuity issues in different functional settings (see also [24, Sec. III.7, p. 94]).

The crucial feature of the stop operator and of Prandtl-Ishlinskiĭ operators of stop type is that they can be characterized (within a suitable functional frame) by the solution of (possibly an infinite system of) evolution variational inequalities. This is also the case of the so-called *play* operator and Prandtl-Ishlinskiĭ operators of play type [24]. The latter are related to the corresponding stop operators by duality and are not directly considered in this note. Of course the present analysis can be reinterpreted for play-type operators with no particular intricacy.

Letting $\mathcal{H}$ be a separable Hilbert space endowed with the scalar product $(\cdot,\cdot)$, $T > 0$ be any fixed reference time, and $K \subset \mathcal{H}$ be a non-empty, convex, and closed set, we recall that the *stop operator* from $H^1(0,T;\mathcal{H})$ to itself is given by the solution of the evolution variational inequality

$$u' + \partial I_K(u) \ni v' \quad \text{a.e. in } (0,T) \text{ and } u(0) = u^0, \tag{1.1}$$

where $v \in H^1(0,T;\mathcal{H})$ and $u^0 \in K$ are given data, the prime denotes differentiation with respect to time, I_K is the indicator function of K ($I_K(u) = 0$ if $u \in K$ and $I_K(u) = +\infty$ otherwise), and ∂ is the subdifferential in the sense of Convex Analysis. By exploiting this variational character, the approximation of the stop operator can be recovered from the general approximation theory for abstract evolution equations with convex potentials (see, e.g., [1]). In particular, given $\phi_n : \mathcal{H} \to (-\infty, +\infty]$ convex, proper, and lower semicontinuous and initial values u_n^0 such that $\phi_n(u_n^0) < +\infty$, we shall consider possible limits of the unique solutions $u_n \in H^1(0,T;\mathcal{H})$ of the following

$$u_n' + \partial\phi_n(u_n) \ni v_n' \quad \text{a.e. in } (0,T) \text{ and } u(0) = u_n^0, \tag{1.2}$$

where now ϕ_n, v_n, and u_n^0 are approximating I_K, v, and u^0, respectively.

The first focus of this note is to remark that indeed the stop operator S is both strongly continuous and *conditionally* weakly continuous. Namely, letting $\phi_n := I_K$ and $u_n^0 := u^0$ in

(1.2), one has that (see Lemmas 4.2-4.3 below)

$$v_n' \to v' \quad \text{strongly in} \quad L^2(0,T;\mathcal{H}) \quad \Rightarrow \quad u_n \to u \quad \text{strongly in} \quad H^1(0,T;\mathcal{H}),$$

$$v_n' \to v' \quad \text{weakly in} \quad L^2(0,T;\mathcal{H}) \quad \text{and} \quad \limsup_{n\to+\infty} \int_0^T (v_n' - v', u_n) \le 0$$
$$\Rightarrow \quad u_n \to u \quad \text{weakly in} \quad H^1(0,T;\mathcal{H}),$$

Let us remark that the $\limsup$ condition above may be proved to be necessary in order to obtain weak convergence for the stop operator (see below). Analogous results will be provided for Prandtl-Ishlinskiĭ operators of stop type as well (Lemmas 5.3-5.4)

A second issue of this paper is that of considering the full approximation situation $v_n' \to v'$, $\phi_n \to I_K$, and $u_n^0 \to u^0$. In this direction, our main focus is on the role of Mosco convergence of the functionals ϕ_n. By fixing for simplicity $v_n := v$ and $u_n^0 := u^0$ (the full approximation situation being discussed below) we have that (Lemmas 4.4-4.5)

$$\phi_n \to I_K \quad \text{in the Mosco sense} \quad \Rightarrow \quad u_n \to u \quad \text{strongly in} \quad H^1(0,T;\mathcal{H}).$$

Indeed, the latter can be regarded as a unifying approach to penalization, regularization, and singular perturbation of the stop operator and we will briefly comment this perspective below. Moreover, we can as well consider the convex set approximation $\phi_n = I_{K_n}$ where the sets $K_n \in \mathcal{H}$ are non-empty, convex, and closed. Referring to this latter situation we possibly obtain an extension of former approximation results based on Hausdorff set-convergence (see, e.g., [13, Cor. 3.8, p. 32]). Let us stress that the singular perturbation of some extended stop-type relation has been already addressed by means of PDE's techniques in [15].

Our discussion on conditional weak continuity and approximation issues turn out to be especially well-tailored to the situation of semilinear and quasilinear parabolic equations. In particular, the last part of this paper is devoted to the study of relations

$$\partial_t v_n - \Delta v_n + H_n(v_n) = 0 \quad \text{or} \quad \partial_t(v_n + H_n(v_n)) - \Delta v_n = 0,$$

to be posed in a smooth domain $\Omega \in \mathbb{R}^d$ and complemented with homogeneous Dirichlet conditions (for simplicity) and suitable initial conditions. In the latter, H_n represents the datum of a nonlocal in time operator mimicking indeed a stop operator or a Prandtl-Ishlinskiĭ operator of stop type. The present analysis of the stop and the Prandtl-Ishlinskiĭ case suggests a natural notion of operator convergence $H_n \to H$. The latter, along with initial data convergence, entails $v_n \to v$ in suitable topologies where v is a solution to the limit problems

$$\partial_t v - \Delta v + H(v) = 0 \quad \text{or} \quad \partial_t(v + H(v)) - \Delta v = 0.$$

This is the plan of the paper. We collect in Section 2 some introductory material on convex sets, convex functionals, and their respective topologies. Then, Section 3 is devoted to recall the convergence results for evolution equations which are the starting point of our analysis. Continuity and approximation results for the stop operator are presented in Section 4 along with some comment on penalization, singular perturbation, and convex set approximation. The extension to some classes of Prandtl-Ishlinskiĭ operators is discussed in Section 5. Finally, Sections 6-7 deal with the application to semilinear and quasilinear parabolic problems above.

2 Preliminaries

Let us start by recalling some notation. In the following $\mathcal{H}$ will stand for a separable Hilbert space. Although most of the results may be extended with no particular intricacy to some reflexive Banach setting, we shall limit ourselves to the Hilbert case, for the sake of simplicity. As usual, we will denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and the corresponding norm of $\mathcal{H}$ and by $\|\cdot\|_E$ the norm in the generic normed space E. Finally $[\cdot,\cdot]$ stands for the generic pair and we will systematically identify relations and their respective graphs without changing notations and unless otherwise stated.

In the forthcoming analysis the notion of *subdifferential* will be used. Letting $\phi:\mathcal{H}\to(-\infty,+\infty]$ be a proper, convex, and lower semicontinuous functional, we define the subdifferential $\partial\phi$ of ϕ as the set

$$\partial\phi := \big\{[u,v]\in\mathcal{H}\times\mathcal{H} \ : \ u\in D(\phi) \ \text{ and } \ (v,w-u)\le\phi(w)-\phi(u) \ \ \forall w\in\mathcal{H}\big\},$$

where $D(\phi):=\{u\in\mathcal{H} \ : \ \phi(u)<+\infty\}$ stands for the effective domain of ϕ. It is a standard matter to prove that $\partial\phi$ turns out to be a maximal monotone operator in the usual sense [5].

We shall now recall some basic set-convergence notion. In particular, we limit ourselves in discuss two convergence notions for non-empty, closed, and convex subsets of $\mathcal{H}$. The interested reader is referred for instance to the monograph [4] for a comprehensive discussion and full generality. Given K, $K_n\subset\mathcal{H}$ non-empty, convex, and closed, we say that K_n *converges in the Mosco sense to* K (*M-converges*, for short) iff [18]

$$\forall u\in K \ \text{ there exists a sequence } \ u_n\in K_n \ \text{ such that } \ u_n\to u \ \text{ strongly in } \ \mathcal{H} \tag{2.1}$$

$$\text{and for all increasing subsequences } n_k \text{ one has that}$$

$$\Big(u_{n_k}\in K_{n_k} \ \text{ and } \ u_{n_k}\to u \ \text{ weakly in } \ \mathcal{H}\Big) \ \Rightarrow \ u\in K. \tag{2.2}$$

The latter is nothing but the Kuratowski set-convergence with respect to *both* weak and strong topologies of $\mathcal{H}$ [1, Cor. 1.35, p. 95].

Secondly, we say that K_n *converges to* K *in the Hausdorff sense* (briefly, *H-converges*) if

$$d(K,K_n) := \max\big\{e(K_n,K),e(K,K_n)\big\}\to 0 \ \text{ as } \ n\to+\infty,$$

where d is the usual Hausdorff metric and $e(K',K''):=\sup_{k'\in K'}\inf_{k''\in K''}\|k'-k''\|$ denotes the *excess* of K' with respect to K'', for all non-empty sets $K\ K''\subset\mathcal{H}$. Of course, the Hausdorff topology is finer than the Mosco one (see, e.g., [3, Thm 3.5.(b)]). On the other hand, we readily find examples of M-converging sequences which are not H-converging (for instance $\mathcal{H}=\mathbb{R}$, $K:=[0,+\infty)$, and $K_n=[0,n]$).

The notion of M-convergence is naturally related to a corresponding convergence notion for convex functionals. Namely, letting ϕ, $\phi_n:\mathcal{H}\to(-\infty.+\infty]$ be proper, convex, and lower semicontinuous, we say that ϕ_n *converges in the Mosco sense to* ϕ (briefly, *M-converges*) iff the respective epigraphs $\mathrm{epi}(\phi_n)$ M-converge to $\mathrm{epi}(\phi)$ in the Hilbert space $\mathcal{H}\times\mathbb{R}$ endowed with its natural scalar product [1, Thm. 1.39, p. 98] (although possibly misleading, the same name for both the set and the corresponding functional convergence is commonly used). In particular, one can equivalently ask that

$$\forall u\in\mathcal{H} \ \exists u_n \ \text{ such that } \ u_n\to u \ \text{ strongly in } \mathcal{H} \ \text{ and } \ \varphi(u)=\lim_{n\to+\infty}\varphi_n(u_n) \ \text{ and} \tag{2.3}$$

$$\forall u_n \ \text{ such that } \ u_n\to u \ \text{ weakly in } \mathcal{H} \ \text{ one has that } \ \varphi(u)\le\liminf_{n\to+\infty}\varphi_n(u_n). \tag{2.4}$$

Namely, it is straightforward to check that M-convergence is nothing but the well known Γ-convergence with respect to both the strong and weak topologies of $\mathcal{H}$ (see the seminal papers [10, 11] and the monograph [8]). In this regard, M-convergence for proper, convex, and lower semicontinuous functionals is related with the convergence of minimizers of (convex) optimization problems. In order to clarify this statement let us recall the following result (see, for instance, [1, Prop. 3.59, p. 361 and Thm. 3.66, p. 373]).

Lemma 2.1 (Identification of weak limits). *Let ϕ_n M-converge to ϕ, $[u_n, v_n] \in \partial\phi_n$, $[u_n, v_n] \to [u, v]$ weakly in $\mathcal{H} \times \mathcal{H}$, and $\liminf_{n\to+\infty}(u_n, v_n) \leq (u, v)$. Then $[u, v] \in \partial\phi$.*

Let us stress that indeed M-convergence arises in a variety of situations, ranging indeed from penalization to regularization. The reader is referred to the forthcoming Section 4 for some detail in this direction.

Moreover, the notion of M-convergence for proper, convex, and lower semicontinuous functionals corresponds to a useful set-convergence notion for their respective subdifferentials. In particular, let us denote by A, A_n non-empty subsets of $\mathcal{H} \times \mathcal{H}$ and say that A_n *converges in the graph sense to* A (briefly, *G-converges*) iff for any point $[a, b] \in A$ there exists a sequence $[a_n, b_n] \in A_n$ strongly converging to $[a, b]$ in $\mathcal{H} \times \mathcal{H}$. We have the following crucial result [1, Thm. 3.66, p. 373].

Lemma 2.2 (M versus G-convergence). *Let $\phi, \phi_n : \mathcal{H} \to (-\infty. + \infty]$ be proper, convex, and lower semicontinuous. Then, the following are equivalent*

i) ϕ_n *M-converges to* ϕ,

ii) $\partial\phi_n$ *G-converges to* $\partial\phi$ *and there exist* $[u, v] \in \partial\phi$, $[u_n, v_n] \in \partial\phi_n$ *such that* $[u_n, v_n] \to [u, v]$ *strongly in* $\mathcal{H} \times \mathcal{H}$ *and* $\phi_n(u_n) \to \phi(u)$.

3 Convergence results for evolution equations

We shall now be concerned with the abstract evolution problem of finding $u \in H^1(0, T; \mathcal{H})$ fulfilling

$$u' + \partial\phi(u) \ni f \quad \text{a.e. in} \quad (0, T), \quad u(0) = u^0, \tag{3.1}$$

where the prime denotes the derivative with respect to time, $\phi : \mathcal{H} \to (-\infty, +\infty]$ is a proper, convex, and lower semicontinuous functional, and $f \in L^2(0, T; \mathcal{H})$, $u^0 \in D(\phi)$ are given data. It is well known that the latter problem admits a unique solution. Moreover, the latter solution may be successfully recovered by means of some approximation procedures (time-discretization, regularizations etc.). In particular, one could consider some data approximation. Namely, we may look for a function $u_n \in H^1(0, T; \mathcal{H})$ $(n \in \mathbb{N})$ fulfilling

$$u_n' + \partial\phi_n(u_n) \ni f_n \quad \text{a.e. in} \quad (0, T), \quad u_n(0) = u_n^0, \tag{3.2}$$

where again all $\phi_n : \mathcal{H} \to (-\infty, +\infty]$ are proper, convex, and lower semicontinuous functionals, $f_n \in L^2(0, T; \mathcal{H})$ and $u_n^0 \in D(\phi_n)$ are given and converge to ϕ, f, and u^0, respectively, in some suitable sense.

From here on we shall tacitly assume that the functionals ϕ_n are equi-bounded from below. The latter choice is intended to simplify the statements and could be weakened. Let us recall the following result.

Lemma 3.1 (Approximation + strong convergence). *Let ϕ_n M-converge to ϕ, $f_n \to f$ strongly in $L^2(0,T;\mathcal{H})$, $u_n^0 \to u^0$ strongly in $\mathcal{H}$, and $\phi_n(u_n^0)$ be bounded. Then $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$. Moreover, if also $\phi_n(u_n^0) \to \phi(u^0)$, then $u_n \to u$ strongly in $H^1(0,T;\mathcal{H})$.*

Proof. This argument is essentially contained in [1, Thm. 3.74, p. 388]. First of all, let us take the scalar product of the inclusion in (3.2) with u_n' and take the integral on $(0,T)$. Thanks to the well known chain rule [5, Lemme 3.3, p. 73] one obtains that

$$\phi_n(u_n(T)) + \int_0^T \|u_n'\|^2 = \phi_n(u_n^0) + \int_0^T (f_n, u_n'). \tag{3.3}$$

Hence, it is straightforward to check that u_n is bounded in $H^1(0,T;\mathcal{H})$ independently of n. Moreover, we also have by comparison that $\xi_n = f_n - u_n'$ is bounded in $L^2(0,T;\mathcal{H})$. Namely, possibly extracting not relabeled subsequences, we may find two functions $\tilde{u}$ and ξ such that

$$u_n \to \tilde{u} \ \text{ weakly in } \ H^1(0,T;\mathcal{H}), \tag{3.4}$$

$$\xi_n \to \xi \ \text{ weakly in } \ L^2(0,T;\mathcal{H}). \tag{3.5}$$

Then, we are in the position of passing to the limit for $n \to +\infty$ in (3.2) and obtain that

$$\tilde{u}' + \xi = f \ \text{ a.e in } \ (0,T), \quad \tilde{u}(0) = u^0. \tag{3.6}$$

Let us now test the inclusion in (3.2) with u_n, take the integral on $(0,T)$, and pass to the lim sup for $n \to +\infty$. Thanks to (3.4) and (3.6) we obtain that

$$\limsup_{n\to+\infty} \int_0^T (\xi_n, u_n) \le -\frac{1}{2}\|\tilde{u}(T)\|^2 + \frac{1}{2}\|u^0\|^2 + \int_0^T (f, \tilde{u}) = \int_0^T (\xi, \tilde{u}).$$

Next, by exploiting Lemma 2.1, we have proved that $\tilde{u}$ solves (3.1). Moreover, since the solution to (3.1) is unique, we have that $\tilde{u} = u$ and the convergences (3.4)-(3.5) hold for the whole sequence.

Finally, if we assume that $\limsup_{n\to+\infty} \phi_n(u_n^0) \le \phi(u^0)$ and exploit the fact that

$$\phi(u(T)) \le \liminf_{n\to+\infty} \phi_n(u_n(T))$$

since $u_n(T)$ converges weakly to $u(T)$ and ϕ_n M-converges to ϕ, we can also pass to the lim sup for $n \to +\infty$ in (3.3) and check that

$$\limsup_{n\to+\infty} \int_0^T \|u_n'\|^2 \le \int_0^T \|u'\|^2,$$

and the assertion follows. ☐

We shall now prepare a result that takes into account some weakly converging data.

Lemma 3.2 (Approximation + weak convergence). *Let ϕ_n M-converge to ϕ, $f_n \to f$ weakly in $L^2(0,T;\mathcal{H})$, $u_n^0 \to u^0$ strongly in $\mathcal{H}$, $\phi_n(u_n^0)$ be bounded, and*

$$\limsup_{n\to+\infty} \int_0^T (f_n - f, u_n) \le 0. \tag{3.7}$$

Then $u_n \to u$ *weakly in* $H^1(0,T;\mathcal{H})$. *Moreover, if also*

$$\phi_n(u_n^0) \to \phi(u^0) \quad \text{and} \quad \limsup_{n\to+\infty} \int_0^T (f_n, u_n') \le \int_0^T (f, u'), \tag{3.8}$$

then $u_n \to u$ *strongly in* $H^1(0,T;\mathcal{H})$.

Proof. It suffices to follow the lines of the proof of Lemma 3.1 and use (3.7)-(3.8) instead of the strong convergence of f_n. □

4 Convergence results for the stop operator

Let us fix $K \subset \mathcal{H}$ non-empty, convex, and closed, and $u^0 \in K$. We are here interested in introducing the *stop operator* S associated to K and u^0 from $H^1(0,T;\mathcal{H})$ to itself. This approach basically follows from the well know theory on the vectorial stop operator and the reader is referred to [9, p. 47-110] for a comprehensive reference on the subject. We shall regard the *stop operator* S associated to K and u^0 from $H^1(0,T;\mathcal{H})$ to itself as a subset of the product space $(H^1(0,T;\mathcal{H}))^2$. In particular, let us state the following.

Definition 4.1 (Stop operator). *We say that* $[v,u] \in S \subset (H^1(0,T;\mathcal{H}))^2$ *iff*

$$u' + \partial I_K(u) \ni v' \quad \text{a.e. in} \quad (0,T), \quad u(0) = u^0.$$

Of course the latter relation S is non-empty, defined on the whole $H^1(0,T;\mathcal{H})$, and single-valued since, for all $v \in H^1(0,T;\mathcal{H})$ and $u^0 \in K$, there exists a unique u such that $[v,u] \in S$. We firstly focus on the situation of converging inputs. Indeed, one can prove the following results.

Lemma 4.2 (Strongly converging inputs). *Let* $[v_n,u_n] \in S$, $v_n' \to v'$ *strongly in* $L^2(0,T;\mathcal{H})$, *and* $u_n \to u$ *weakly in* $H^1(0,T;\mathcal{H})$. *Then* $[v,u] \in S$ *and* $u_n \to u$ *strongly in* $H^1(0,T;\mathcal{H})$.

Lemma 4.3 (Weakly converging inputs). *Let* $[v_n,u_n] \in S$, $v_n' \to v'$ *weakly in* $L^2(0,T;\mathcal{H})$, $u_n \to u$ *weakly in* $H^1(0,T;\mathcal{H})$, *and*

$$\limsup_{n\to+\infty} \int_0^T (v_n', u_n) \le \int_0^T (v', u). \tag{4.1}$$

Then $[v,u] \in S$. *Moreover, if also*

$$\limsup_{n\to+\infty} \int_0^T (v_n', u_n') \le \int_0^T (v', u'), \tag{4.2}$$

then $u_n \to u$ *strongly in* $H^1(0,T;\mathcal{H})$.

We shall not give proofs of the latter Lemmas since they can be easily obtained from those of Lemmas 3.1 and 3.2. It is however noteworthy to remark that the latter results are somehow of a different flavor. Indeed in Lemmas 4.2 and 4.3 we assume that the functions u_n do have some weak limit and we just identify it. That is to say that Lemmas 4.2 and 4.3 are actually *identification* results instead of *convergence* ones. On the other hand, in the current case this is merely a matter of notation since we could easily remove the weak convergence assumption on u_n from the statements by means of weak compactness of bounded sets in $H^1(0,T;\mathcal{H})$. Namely, Lemma 4.2 states the strong continuity of S from $H^1(0,T;\mathcal{H})$ endowed with the strong topology to itself, while Lemma 4.3 entails a *conditional continuity* of S from $H^1(0,T;\mathcal{H})$ endowed with the weak topology to itself.

Let us stress that, owing to the nonlinearity of S, weak continuity from $H^1(0,T;\mathcal{H})$ to itself is not to be expected. In order to give a proof of this fact we however cannot refer to finite dimensional spaces $\mathcal{H}$ where indeed (4.1) follows from the required convergences (stated differently: the embedding $H^1(0,T;\mathcal{H}) \subset C([0,T];\mathcal{H})$ is compact for finite dimensional spaces $\mathcal{H}$ and S is known to admit a unique strongly continuous extension to $(C([0,T];\mathcal{H}))^2$, see, e.g., [13, Thm. 3.7, p. 32]). We let $\mathcal{H} := L^2(0,2\pi)$, $T = 2$, $K = \{w \in L^2(0,2\pi) \, : \, w \in [-1,1]$ a.e. in $(0,2\pi)\}$ and define $v_n(t) := t\sin nx$ for $[x,t] \in (0,2\pi)\times(0,2)$, $n \in \mathbb{N}$ and $u^0 = 1$. Then, one easily computes that $[v_n,u_n] \in S$ implies that $u_n(x,t) = -t(\sin nx)^- + 1$ for $[x,t] \in (0,2\pi)\times(0,2)$. In particular $v_n \to 0$ weakly in $H^1(0,2;\mathcal{H})$ (actually $v_n'(t) \to 0$ weakly in $\mathcal{H}$ for all $t \in [0,2]$ as well). On the other hand, we directly compute that $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$ where $u(x,t) = 1 - t/\pi$. Hence $u \neq 1$. On the other hand $[0,1] \in S$. In fact, condition (4.1) is not fulfilled in the present example since

$$\int_0^2 \int_0^{2\pi} v_n' \, u_n = \pi > 0.$$

The latter computation shows that indeed condition (4.1) is needed for the weak convergence of the stop operator.

Let us now consider the possibility of approximating the stop operator S by means of a sequence of subsets $S_n \subset (H^1(0,T;\mathcal{H}))^2$. In particular, let $\phi_n : \mathcal{H} \to (-\infty,+\infty]$ be convex, proper, and lower semicontinuous, and $u_n^0 \in D(\phi_n)$. Hence, we say that $[v,u] \in S_n \subset (H^1(0,T;\mathcal{H}))^2$ iff

$$u' + \partial\phi_n(u) \ni v' \quad \text{a.e. in } \ (0,T), \quad u(0) = u_n^0.$$

Namely ϕ_n may be indicator functions of some non-empty, convex, and closed sets (in which case the relations S_n themselves are stop operators) or else (see below). We have the following.

Lemma 4.4 (Approximation + strong convergence). *Let ϕ_n M-converge to ϕ and $u_n^0 \to u^0$ weakly in $\mathcal{H}$. Moreover, let $[v_n,u_n] \in S_n$, $v_n' \to v'$ strongly in $L^2(0,T;\mathcal{H})$, and $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$. Then $[v,u] \in S$. Moreover, if also $\phi_n(u_n^0) \to \phi(u^0)$, then $u_n \to u$ strongly in $H^1(0,T;\mathcal{H})$.*

Lemma 4.5 (Approximation + weak convergence). *Let ϕ_n M-converge to ϕ and $u_n^0 \to u^0$ weakly in $\mathcal{H}$. Moreover, let $[v_n,u_n] \in S_n$, $v_n' \to v'$ weakly in $L^2(0,T;\mathcal{H})$, $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$, and*

$$\limsup_{n\to+\infty} \int_0^T (v_n', u_n) \leq \int_0^T (v', u). \tag{4.3}$$

Then $[v,u] \in S$. Moreover, if also

$$\phi_n(u_n^0) \to 0 \quad \text{and} \quad \limsup_{n\to+\infty} \int_0^T (v_n', u_n') \leq \int_0^T (v', u'), \tag{4.4}$$

then $u_n \to u$ strongly in $H^1(0,T;\mathcal{H})$.

Again, it suffices to apply in this context Lemmas 3.1 and 3.2 in order to check for the proofs. Of course Lemmas 4.2 and 4.3 may be easily obtained as corollaries of Lemmas 4.4 and 4.5 with the choices $\phi_n := I_K$ and $u_n^0 := u^0$. Let us however mention that, in order to reformulate the above statements as convergence results rather than identification ones, some uniform bound on $\phi_n(u_n^0)$ would be required.

Before going on, we shall complement the above abstract material by giving some examples of situations where our analysis applies.

4.1 Penalization

Assume we are given $K \subset \mathcal{H}$ non-empty, convex, and closed. Let now

$$\phi_n(u) := \inf_{v\in\mathcal{H}} \left(I_K(v) + \frac{n}{p}\|u-v\|^p \right) = \frac{n}{p} \inf_{k\in K} \|u-k\|^p, \ p \in [1,\infty).$$

In the case $p = 2$ this is nothing but the well known Moreau-Yosida approximation of I_K, and the evolution problem related to ϕ_n is indeed a *quadratically penalized* version of the stop operator. In particular, the subdifferential $\partial\phi_n$ turns out to be single-valued, Lipschitz continuous, and G-convergent to ∂I_K. The same convergence holds true for all $p \in [1,\infty)$ and, in general, for all increasing sequences ϕ_n such that $I_K = \sup_{n\in\mathbb{N}} \phi_n$ [1, Thm. 3.20.(i), p. 298].

4.2 M-converging convex sets

We are indeed in the position of considering the limit of stop operators related to M-converging non-empty, convex, and closed sets K_n. In particular, we can prove the following.

Corollary 4.6. *Let $K, K_n \subset \mathcal{H}$ be non-empty, convex, and closed, K_n M-converge to K, $\phi_n := I_{K_n}$, $u_n^0 \in K_n$, $u^0 \in K$, and $u_n^0 \to u^0$ weakly in $\mathcal{H}$. Moreover, let $[v_n, u_n] \in S_n$, $v_n' \to v'$ strongly in $L^2(0,T;\mathcal{H})$, and $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$. Then $[v,u] \in S$ and $u_n \to u$ strongly in $H^1(0,T;\mathcal{H})$.*

The interest of this corollary is twofold. From one side, we are able to identify limits under mild and natural assumptions on the convergence of the convex sets K_n. Indeed, let us remark that some similar identification results have been proved just in the case of Hausdorff converging sets K_n (see, e.g., [13, Thm. 3.12, p. 34] for some related result). On the other hand, we ensure the possibility of approximating stop operators with unbounded convex sets by sequences of stops with bounded convex set. As we commented above, H-convergence is not the right tool in order to consider unbounded convex sets.

4.3 Singular perturbations

Assume we are given a second Hilbert space $\mathcal{V} \subset \mathcal{H}$ such that the latter embedding is continuous and dense. Moreover, let $\phi : \mathcal{H} \to (-\infty, +\infty]$ be proper, convex, and lower semicontinuous such that $D(\phi) \cap \mathcal{V} \neq \emptyset$. Then, one could possibly consider the following approximations

$$\phi_n(u) := \phi(u) + \frac{1}{n}\|u\|_{\mathcal{V}}^2 \ \text{ if } \ u \in \mathcal{V} \ \text{ and } \ \phi_n(u) := +\infty \ \text{ otherwise,}$$

which of course are still proper, convex, and lower semicontinuous. The aim of this choice is indeed to build an approximation with a possibly smaller domain. Since ϕ_n are decreasing to ϕ we readily recall [1, Thm 3.20.(ii), p. 298] and deduce that actually ϕ_n M-converges to ϕ which is the lower semicontinuous envelope of $\inf_{n\in\mathbb{N}} \phi_n$. Hence, our abstract machinery applies.

We shall now leave the abstract framework and reformulate the argument in a Sobolev space setting which is suitable of being exploited in some PDE application. In particular, let $\Omega \in \mathbb{R}^m$ $(m \in \mathbb{N})$ denote a regular and bounded open set and define $\mathcal{H} := L^2(\Omega)$. We will also make use of the spaces $H_0^1(\Omega)$ and $H^2(\Omega)$, the reader is referred to [17] for definitions and details. Moreover, let us fix $\psi := \mathbb{R} \to (-\infty, +\infty]$ proper, convex, and lower semicontinuous, possibly being $\psi = I_K$ where $K \subset \mathbb{R}$ is a non-empty closed interval. One introduces the functional $\phi : \mathcal{H} \to (-\infty, +\infty]$ as

$$\phi(u) := \begin{cases} \displaystyle\int_\Omega \psi(u(x))\, dx & \text{if } \psi(u) \in L^1(\Omega), \\ +\infty & \text{if } \psi(u) \notin L^1(\Omega), \end{cases}$$

which of course turns out to be proper, convex, and lower semicontinuous as well [2, Prop. 2.8, p. 61]. We shall consider the possibility of replacing ϕ by

$$\phi_n(u) := \phi(u) + \frac{1}{n}\|u\|^2_{H^1(\Omega)} \quad \text{for } u \in H^1(\Omega) \quad \text{and} \quad \phi_n(u) := +\infty \quad \text{otherwise.}$$

We shall refer to the paper [15] for some related results and motivation.

Indeed, the latter choice fits into the lines of our analysis. Namely, we are in the position of proving the M-convergence of ϕ_n to ϕ. For this aim, let us fix $u \in \mathcal{H}$ and denote by $u_n \in H^2(\Omega) \cap H_0^1(\Omega)$ the solution of the singular perturbation problem

$$u_n - \frac{1}{n}\Delta u_n = u \quad \text{a.e. in } \Omega. \tag{4.5}$$

It is easy to check that the latter problem admits indeed a unique solution such that [16]

$$u_n \to u \text{ strongly in } \mathcal{H} \text{ and } \frac{1}{n}\|u_n\|^2_{H^1(\Omega)} \to 0.$$

Next, by testing equation (4.5) by v_n where $[u_n, v_n] \in \partial\phi$, taking the integral in space, and exploiting the well known result [6, Lemma 2], we readily obtain that

$$\phi(u_n) - \phi(u) \le (v_n, u_n - u) = \frac{1}{n}(\Delta u_n, v_n) \le 0.$$

Thus, we have proved that

$$\limsup_{n\to+\infty} \phi_n(u_n) \le \limsup_{n\to+\infty} \phi(u_n) + \lim_{n\to+\infty} \frac{1}{n}\|u_n\|^2_{H^1(\Omega)} \le \phi(u).$$

On the other hand, owing to the lower semicontinuity of ϕ, one has that, for any $u_n, u \in \mathcal{H}$,

$$u_n \to u \text{ weakly in } \mathcal{H} \quad \Rightarrow \quad \phi(u) \;\le\; \liminf_{n\to+\infty} \phi(u_n) \;\le\; \liminf_{n\to+\infty} \phi_n(u_n).$$

The latter relations entail in particular the validity of (2.3)-(2.4). Let us refer again the reader to [15] for some related result.

5 Convergence results for Prandtl-Ishlinskiĭ operators

We shall now extend the results for the stop operator to the Prandtl-Ishlinskiĭ case. To this aim, we start by introducing the latter operator in the present context (slightly different from the classical one [13, 24]). In particular, we basically follow SHOWALTER & STEFANELLI [23]. Let us denote by $(Y, \mathcal{P}, \mu)$ a suitable measure space, where μ is a finite Borel measure. Moreover, we introduce a parameterized family of non-empty, convex, and closed sets $K_y \subset \mathcal{H}$ and initial data $u_y^0 \in K_y$ for $y \in Y$. Let us recall that a function $[y, u] \mapsto \phi_y(u) \in [0, +\infty]$ such that $\phi_y(\cdot)$ is convex, lower semicontinuous and $\phi_y(0) = 0$ for all $y \in Y$ is called a *normal integrand* [21, Prop. 14.39, p. 666] if there exists a countable collection $M = \{m : y \mapsto m_y\}$ of measurable functions from Y to $\mathcal{H}$ (recall that $\mathcal{H}$ is separable) such that

$$\begin{aligned} y \mapsto \phi_y(m_y) \quad & \text{is measurable for all } \ m \in M \ \text{ and} \\ \{m_y \,:\, m \in M\} \cap D(\phi_y) \quad & \text{is dense in } \ D(\phi_y) \quad \text{for all } \ y \in Y. \end{aligned}$$

When the interior of $D(\phi_y)$ is non-empty for every $y \in Y$, ϕ is a normal integrand iff the function $y \mapsto \phi_y(u)$ is measurable for each $u \in \mathcal{H}$ fixed. Also note that ϕ is a normal integrand if it is independent of $y \in Y$. This holds more generally in the *discrete* situation where the measure μ assigns the mass $\mu_y > 0$ to each point $y \in Y$.

The latter definition is motivated by the fact that, whenever ϕ is a normal integrand, the function $y \mapsto \phi_y(u_y)$ is measurable for every given measurable function $u : Y \to \mathcal{H}$. Hence, we can define $\widehat{\phi} : L^2(Y; \mathcal{H}) \to [0, +\infty]$ as

$$\widehat{\phi}(u) := \begin{cases} \displaystyle\int_Y \phi_y(u_y)\, d\mu & \text{if } \ y \mapsto \phi_y(u_y) \in L^1(Y) \\ +\infty & \text{if } \ y \mapsto \phi_y(u_y) \notin L^1(Y) \end{cases} \tag{5.1}$$

which is again convex and lower semicontinuous. See ROCKAFELLAR [22, 19, 20] for these and additional issues of measurability. We shall make use of the following lemma, whose elementary proof can be found, for instance, in [23].

Lemma 5.1 (Subdifferential characterization). *Let ϕ be a normal integrand and $\widehat{\phi}$ be defined as above. Then, $[u, v] \in \partial\widehat{\phi}$ iff $[u_y, v_y] \in \partial\phi_y$ for μ-almost every $y \in Y$.*

In order to introduce the Prandtl-Ishlinskiĭ operator P, we shall assume in particular that

$$\begin{aligned} [y, u] \mapsto I_{K_y}(u) \quad & \text{is a normal integrand,} \\ u^0 : y \mapsto u_y^0 \quad & \text{is measurable.} \end{aligned}$$

Hence, we state the following.

Definition 5.2 (Prandtl-Ishlinskiĭ operator). *We say that $[v, u] \in P \subset (H^1(0, T; \mathcal{H}))^2$ iff $u = \int_Y u_y \, d\mu$ where*

$$u_y' + \partial I_{K_y}(u_y) \ni v' \quad \textit{a.e. in} \quad (0, T) \quad \textit{and} \quad u_y(0) = u_y^0, \quad \textit{for } \mu\textit{-a.e. } y \in Y. \tag{5.2}$$

Owing to Lemma 5.1 we readily check that the latter definition makes sense. Indeed, for all $\widehat{v}' \in L^2(0, T; L^2(Y; \mathcal{H}))$ and $u^0 \in D(\widehat{\phi})$, we find that $\widehat{u} \in H^1(0, T; L^2(Y; \mathcal{H}))$ given by $\widehat{u}(t)(y) = u_y(t)$ for μ-almost every $y \in Y$ is the unique solution to

$$\widehat{u}' + \partial\widehat{\phi}(\widehat{u}) \ni \widehat{v}' \quad \text{a.e. in} \quad (0, T) \quad \text{and} \quad \widehat{u}(0) = u^0.$$

This entails in particular that $u = \int_Y u_y \, d\mu$ is well defined [25, Cor. 2, p. 134]. Moreover, it is straightforward to check that indeed $y \mapsto u_y$ belongs to $L^\infty(Y; H^1(0,T;\mathcal{H}))$.

As before, we firstly focus our attention on the situation of converging inputs. We have the following results.

Lemma 5.3 (Strongly converging inputs). *Let $[v_n, u_n] \in P$, $v_n' \to v'$ strongly in $L^2(0,T;\mathcal{H})$, and $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$. Then $[v,u] \in P$ and $u_n \to u$ strongly in $H^1(0,T;\mathcal{H})$.*

Proof. Let us recall that $u_n = \int_Y u_{n,y} d\mu$ for some $u_{n,y}$ solving (5.2) with v_n instead of v. Taking into account the above introduced notation and the discussion of the previous section, we readily check that $u_{n,y}$ is μ-essentially bounded in $H^1(0,T;\mathcal{H})$. Hence, we easily find u_y such that $u_{n,y} \to u_y$ strongly in $H^1(0,T;\mathcal{H})$ and (5.2) is fulfilled (note that the whole sequence converges). In order to check that u_n actually converges strongly to $\int_Y u_y \, d\mu$ in $H^1(0,T;\mathcal{H})$ let us consider the functions $\psi_n : Y \to [0,+\infty)$ defined μ-almost everywhere by $\psi_n(y) := \|u_y - u_{n,y}\|_{H^1(0,T;\mathcal{H})}$. It is straightforward to check that $\psi_n \to 0$ μ-almost everywhere being μ-essentially bounded. Finally $\|u_n - \int_Y u_y \, d\mu\|_{H^1(0,T;\mathcal{H})} \leq \int_Y \psi_n \, d\mu \to 0$ and the assertion is proved. □

Lemma 5.4 (Weakly converging inputs). *Let $[v_n, u_n] \in P$, and $u_n = \int_Y u_{n,y} d\mu$ as in Definition 5.2. Moreover, let $v_n' \to v'$ weakly in $L^2(0,T;\mathcal{H})$, $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$, and*

$$\limsup_{n\to+\infty} \int_0^T (v_n', u_{n,y}) \leq \int_0^T (v', u_y) \quad \textit{for a.e.} \ \ y \in Y, \tag{5.3}$$

for u_y solving (5.2). Then $[v,u] \in P$. Moreover, if also

$$\limsup_{n\to+\infty} \int_0^T (v_n', u_{n,y}') \leq \int_0^T (v', u_y') \quad \textit{for a.e.} \ \ y \in Y, \tag{5.4}$$

then $u_n \to u$ strongly in $H^1(0,T;\mathcal{H})$.

Proof. Following the proof of Lemma 5.3, for μ-almost every $y \in Y$, we exploit Lemma 4.3 and find that $u_{n,y} \to u_y$ weakly in $H^1(0,T;\mathcal{H})$, where u_y fulfills (5.2). In order to conclude for $[v,u] \in P$ we now need to check that indeed $u = \int_Y u_y \, d\mu$. To this aim, let us fix $w \in L^2(0,T;\mathcal{H})$ and define μ-almost everywhere the functions $\psi_n : Y \to \mathbb{R}$ as $\psi_n(y) := \int_0^T (u_y - u_{n,y}, w)$. Again, we readily check that $\psi_n \to 0$ μ-almost everywhere and is μ-essentially bounded. Namely, $\int_0^T (u_n - \int_Y u_y \, d\mu, w) = \int_Y \psi_n \, d\mu \to 0$. Finally, $[v,u] \in P$ since weak limits in $L^2(0,T;\mathcal{H})$ are unique. Moreover, if condition (5.4) holds, the μ-almost everywhere convergence $u_{n,y} \to u_y$ is strong and we can follow the lines of the proof of Lemma 5.3. □

Differently from the situation of the stop operator, here condition (5.3) is quite restrictive and is not necessary for weak continuity. Indeed, let us recall the above introduced example $\mathcal{H} := L^2(0,2\pi)$, $T = 2$, $v_n(t) := t \sin nx$ for $[x,t] \in (0,2\pi) \times (0,2)$, and let $Y = \mathbb{R}$, $\mu = (\delta_{-1} + \delta_1)/2$ (Dirac masses), $K_1 = K_{-1} := \{w \in L^2(0,2\pi) : w \in [-1,1] \ \text{a.e. in} \ (0,2\pi)\}$, and

$u^0_{\pm 1} := \pm 1$. Then, $[v_n, u_{\pm 1,n}] \in S$ yields

$$u_{1,n}(x,t) = -t(\sin nx)^- + 1, \quad u_{-1,n}(x,t) = t(\sin nx)^+ - 1,$$
$$\text{and} \quad u_n(x,t) = \frac{1}{2}(u_{-1,n}(x,t) + u_{1,n}(x,t)) = v_n(x,t) \quad \text{for} \quad [x,t] \in (0,2\pi) \times (0,2).$$

In particular, u_n, $v_n \to 0$ weakly in $H^1(0,T;\mathcal{H})$ and $[0,0] \in P$. On the other hand we have already checked that $\int_0^T (v'_n, u_{1,n}) = \pi$ and we readily prove that $\int_0^T (v'_n, u_{1,n}) = \pi$ as well so that (5.3) does not hold. Note also that, since of course $\int_0^T (v'_n, u_n) = \pi$, even the weaker condition $\limsup_{n\to+\infty} \int_0^T (v'_n, u_n) \leq 0$ is not satisfied in the present situation.

Let us now turn to some approximation of the Prandtl-Ishlinskiĭ operator P by means of a sequence of subsets $P_n \subset (H^1(0,T;\mathcal{H}))^2$. In particular, let ϕ_n be normal integrands and $u^0_n \in D(\widehat{\phi}_n)$. As for the parameter space approximation, we restrict ourselves to the case $Y = \mathbb{R}$, $d\mu_n = g_n d\mathcal{L}_1$, $g_n \in L^1(Y)$. Hence, we say that $[v,u] \in P_n \subset (H^1(0,T;\mathcal{H}))^2$ iff $u = \int_Y u_y \, d\mu_n$ where

$$u'_y + \partial\phi_{n,y}(u_y) \ni v' \quad \text{a.e. in} \quad (0,T), \quad u_y(0) = u^0_{n,y}, \quad \text{for a.e.} \quad y \in Y. \tag{5.5}$$

We have the following.

Lemma 5.5 (Approximation + strong convergence). *Let $\phi_{n,y}$ M-converge to I_{K_y}, $g_n \to g$ strongly in $L^1(Y)$, $u^0_{n,y} \to u^0_y$ strongly in $\mathcal{H}$, and $\phi_{n,y}(u^0_{n,y}) \to I_{K_y}(u^0_y) = 0$ for μ-almost every $y \in Y$. Moreover, let $[v_n, u_n] \in P_n$, $v'_n \to v'$ strongly in $L^2(0,T;\mathcal{H})$, and $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$. Then $[v,u] \in P$ and $u_n \to u$ strongly in $H^1(0,T;\mathcal{H})$.*

Proof. By exploiting Lemma 4.4, for μ-almost every $y \in Y$, we find that $u_{n,y} \to u_y$ strongly in $H^1(0,T;\mathcal{H})$, where u_y fulfills (5.2). Let again $\psi_n : Y \to [0,+\infty)$ be defined μ-almost everywhere as $\psi_n(y) := \|u_y - u_{n,y}\|_{H^1(0,T,\mathcal{H})}$. Since $\psi_n \to 0$ μ-almost everywhere and is μ-essentially bounded we readily deduce that

$$\left\| u_n - \int_Y u_y \, g \right\|_{H^1(0,T;\mathcal{H})} \leq \int_Y \psi_n \, g + \int_Y \|u_{n,y}\|_{H^1(0,T;\mathcal{H})} |g_n - g| \to 0,$$

and the assertion follows. ☐

Lemma 5.6 (Approximation + weak convergence). *Let $\phi_{n,y}$ M-converge to I_{K_y}, $g_n \to g$ strongly in $L^1(Y)$, $u^0_{n,y} \to u^0_y$ strongly in $\mathcal{H}$, and $\phi_{n,y}(u^0_{n,y}) \to I_{K_y}(u^0_y) = 0$ for μ-almost every $y \in Y$. Moreover, let $[v_n, u_n] \in P_n$, namely $u_n = \int_Y u_{n,y} d\mu_n$ where $u_{n,y}$ fulfill (5.5). Finally let $v'_n \to v'$ weakly in $L^2(0,T;\mathcal{H})$, and $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$, and*

$$\limsup_{n\to+\infty} \int_0^T (v'_n, u_{n,y}) \leq \int_0^T (v', u_y) \quad \text{for a.e.} \quad y \in Y, \tag{5.6}$$

where u_y solves (5.2). Then $[v,u] \in P$. Moreover, if also

$$\limsup_{n\to+\infty} \int_0^T (v'_n, u'_{n,y}) \leq \int_0^T (v', u'_y) \quad \text{for a.e.} \quad y \in Y, \tag{5.7}$$

then $u_n \to u$ strongly in $H^1(0,T;\mathcal{H})$.

Proof. For μ-almost every $y \in Y$, we are in the position of exploiting Lemma 4.5 and check that $u_{n,y} \to u_y$ weakly in $H^1(0,T;\mathcal{H})$, where u_y fulfill (5.2). Thus, it suffices to prove that $u = \int_Y u_y g$. To this end, let us choose any $w \in L^2(0,T;\mathcal{H})$ and define the functions $\psi_n : Y \to \mathbb{R}$ as $\psi_n(y) := \int_0^T (u_y - u_{n,y}, w)$. Again, we readily check that $\psi_n \to 0$ μ-almost everywhere and is μ-essentially bounded. Hence,

$$\int_0^T \left(u_n - \int_Y u_y g, w \right) \leq \int_Y \psi_n \, d\mu + \int_Y \left| \int_0^T (u_{n,y}, w) \right| |g_n - g| \to 0.$$

Finally, $[v,u] \in P$ since weak limits in $L^2(0,T;\mathcal{H})$ are unique. Moreover, if condition (5.7) holds, the convergence $u_{n,y} \to u_y$ is strong and we can follow the lines of the proof of Lemma 5.5. □

6 Convergence for semilinear parabolic problems

We shall now exploit the above introduced abstract material in order to possibly pass to the limit within some semilinear PDE problem of parabolic type. In particular, let $\Omega \in \mathbb{R}^m$ $(m \in \mathbb{N})$ denote a regular and bounded open set and define $\mathcal{H} := L^2(\Omega)$. We are concerned with the problem of finding $v \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ and $u \in H^1(0,T;\mathcal{H})$ such that

$$\partial_t v - \Delta v + u = f \quad \text{a.e. in } \Omega \times (0,T), \tag{6.1}$$

$$[v,u] \in H, \tag{6.2}$$

$$v(0) = v^0 \quad \text{a.e. in } \Omega, \tag{6.3}$$

with obvious notations. In the latter, $f \in L^2(0,T;\mathcal{H})$ is a given datum, and Dirichlet homogeneous boundary conditions are chosen just for the sake of simplicity. The symbol $H \subset (H^1(0,T;\mathcal{H}))^2$ in (6.2) represents a general hysteresis relation and v^0 is some initial datum. Of course, relation H (and its approximations H_n, see below) has to fulfill some basic requirements. In particular, let us reduce ourseves from the very beginning to the situation of a *causal* operator H, namely, for all v_1, $v_2 \in H^1(0,T;\mathcal{H})$ and $t \in [0,T]$, $v_1 = v_2$ on $[0,t]$ yields the equality of sets $H(v_1) = H(v_2)$ on $[0,t]$. Moreover H is said to be *linearly bounded in* $E \times E$ iff there exists a positive constant c such that

$$\|u\|_E \leq c\big(\|v\|_E + 1\big) \quad \forall [v,u] \in H.$$

Finally, we recall that H is said to be *piecewise monotone* [24, p. 62] iff, for any $[v,u] \in H$, one has that $\partial_t v \partial_t u \geq 0$ almost everywhere in $\Omega \times (0,T)$. Let us stress that both the stop and Prandtl-Ishlinskiĭ operators of stop-type are causal, piecewise monotone, and linearly bounded in $(H^1(0,T;\mathcal{H}))^2$.

Our aim is to exploit the possibility of approximating the solution v and u of problem (6.1)-(6.3) by means of solutions to some auxiliary problem. Indeed, let $v_n \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ and $u_n \in H^1(0,T;\mathcal{H})$ be such that

$$\partial_t v_n - \Delta v_n + u_n = f_n \quad \text{a.e. in } \Omega \times (0,T), \tag{6.4}$$

$$[v_n,u_n] \in H_n, \tag{6.5}$$

$$v_n(0) = v_n^0 \quad \text{a.e. in } \Omega, \tag{6.6}$$

where H_n, f_n, and v_n^0 suitably converge to H, f, and v^0, respectively. Of course we focus ourselves here just on the proof of the convergence of $[v_n, u_n]$ to $[v, u]$, omitting completely the discussion of the possible well-posedness of the approximating and limit problems. The reader is referred to the monograph [24, Ch. X, p. 295] for an extensive discussion on this subject. The spirit of our results is that of providing some minimal requirements on data in order to establish a priori estimates that allow us to identify the limits.

Let us firstly turn to the study of the situation where the strong convergence in $H^1(0,T;\mathcal{H})$ of the functions v_n can be inferred by compactness. Then, we shall use the following notion of convergence for the sets H_n.

Definition 6.1 (Convergence in the strong sense). *Let $H, H_n \subset (H^1(0,T;\mathcal{H}))^2$. We say that H_n* converges in the strong sense *to H iff, for all sequences $n_k \to +\infty$ and $[v_k, u_k] \in H_{n_k}$ such that $v_k \to v$ strongly in $H^1(0,T;\mathcal{H})$ and $u_k \to u$ weakly in $H^1(0,T;\mathcal{H})$, one has that $[v, u] \in H$.*

The above definition is motivated by Lemmas 4.4 and 5.5. In particular, whenever one considers sequences of approximating operators S_n or P_n fulfilling the requirements of Lemmas 4.4 and 5.5, respectively, one finds that they turn out to be convergent in the above defined strong sense. Let us however stress that the aforementioned convergence is weaker than the Kuratowski convergence $H_n \to H$ with respect to the strong $\times$ weak topology in $(H^1(0,T;\mathcal{H}))^2$ [1, Cor. 1.35, p. 95].

The above definition turns out to be useful in order to pass to the limit in (6.4)-(6.6). Indeed, one can prove the following.

Theorem 6.2 (Semilinear problem 1). *Let $H_n \in (H^1(0,T;\mathcal{H}))^2$ be uniformly linearly bounded in $(H^1(0,T;\mathcal{H}))^2$, and converge in the strong sense to $H \subset (H^1(0,T;\mathcal{H}))^2$. Moreover, let $f_n \to f$ weakly in $H^1(0,T;\mathcal{H})$, $v_n^0 \to v^0$ weakly in $H^2(\Omega) \cap H_0^1(\Omega)$, $f_n(0) - H_n(v_n)(0) + \Delta v_n^0$ are bounded in $H^1(\Omega)$, and $v_n \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ and $u_n \in H^1(0,T;\mathcal{H})$ solve* (6.4)-(6.6). *Then, there exist not relabeled subsequences such that $v_n \to v$ strongly in $H^1(0,T;\mathcal{H})$, $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$, and v, u solve* (6.1)-(6.3).

Sketch of the proof. The idea of the proof is to obtain a uniform a priori estimate on v_n in $H^2(0,T;\mathcal{H}) \cap H^1(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ by taking a time derivative of (6.4), multiply it by $\partial_{tt} v_n$, integrate in space time, and use some comparison. Of course by now this procedure is just formal. However, let us stress that it could be rigorously justified at some further approximation level. Hence, by standard compactness results and possibly extracting some not relabeled subsequences, we may infer the convergences

$$
\begin{aligned}
v_n &\to v \text{ strongly in } H^1(0,T;\mathcal{H}) \\
&\qquad \text{and weakly in } H^2(0,T;\mathcal{H}) \cap H^1(0,T;H^2(\Omega) \cap H_0^1(\Omega)), \\
u_n &\to u \text{ weakly in } H^1(0,T;\mathcal{H}).
\end{aligned}
$$

Now, owing in particular to the convergence in the strong sense of the H_n, it is a standard matter to pass to the limit in (6.4)-(6.6) and prove the theorem. □

Let us now turn to the study of the possibility of identifying limits related to weak convergent sequences v_n in $H^1(0,T;\mathcal{H})$. In particular, we focus on the case of less regular data, where the above compactness argument is not expected to work. Exactly as in the maximal monotone case (namely, where H, H_n are realizations of local in time maximal monotone operators in $\mathcal{H}$), we shall provide some supplementary condition in order to identify weak limits. Namely, one needs to make precise the following notion of convergence for the graphs H_n.

Definition 6.3 (Convergence in the weak sense). *Let $H, H_n \subset (H^1(0,T;\mathcal{H}))^2$. We say that H_n* converges in the weak sense *to H iff, for all increasing sequences $n_k \to +\infty$ and $[v_k, u_k] \in H_{n_k}$ such that $v_k \to v$ weakly in $H^1(0,T;\mathcal{H})$, $u_k \to u$ weakly in $H^1(0,T;\mathcal{H})$, and $\limsup_{k\to+\infty} \int_0^T (v_k', u_k) \leq \int_0^T (v', u)$, one has that $[v,u] \in H$.*

Of course, the above notion of convergence is motivated by Lemmas 4.5 and 5.6. We shall stress that the choice of the terms *strong* and *weak* in Definitions 6.1 and 6.3 just refers to the topology of $H^1(0,T;\mathcal{H})$ under which the sequence v_n is assumed to be converging. Indeed, one easily checks that if H_n converges to H in the weak sense then it also converges in the strong sense. This is to say that the convergence in the weak sense is stronger than convergence in the strong sense (this is however natural since it provides the identification of limits in a weaker setting). Finally, one has that the convergence in the weak sense is weaker than the Kuratowski convergence $H_n \to H$ with respect to the weak $\times$ weak topology in $(H^1(0,T;\mathcal{H}))^2$.

Theorem 6.4 (Semilinear problem 2). *Let $H_n \in (H^1(0,T;\mathcal{H}))^2$ be piecewise monotone, uniformly bounded on bounded sets, and converge in the weak sense to $H \subset (H^1(0,T;\mathcal{H}))^2$. Moreover, let $f_n \to f$ weakly in $H^1(0,T;\mathcal{H})$ and strongly in $L^2(0,T;\mathcal{H})$, $v_n^0 \to v^0$ weakly in $H^2(\Omega)\cap H_0^1(\Omega)$, and $v_n \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and $u_n \in H^1(0,T;\mathcal{H})$ solve* (6.4)-(6.6)*. Then there exist not relabeled subsequences such that $v_n \to v$ weakly in $H^1(0,T;\mathcal{H})$ and $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$, and v,u solve* (6.1)-(6.3).

Sketch of the proof. We shall derive some suitable a priori estimates on the sequence of approximate solutions. To this aim let us take the derivative in time of equation (6.1), multiply it by $\partial_t v_n$, and exploit the piecewise monotonicity of H_n. Of course, by now this procedure is just formal and it should be justified at some approximation level. By integrating on space and time and exploiting some boundedness and comparison we are indeed in the position of claiming that v_n is uniformly bounded in $W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$ and u_n is bounded in $H^1(0,T;\mathcal{H})$. Owing to well known compactness results we readily find two functions v and u such, at least for some not relabeled subsequences,

$$\begin{aligned} v_n &\to v \text{ weakly star in } W^{1,\infty}(0,T;\mathcal{H}) \cap H^1(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega)) \\ u_n &\to u \text{ weakly in } H^1(0,T;\mathcal{H}). \end{aligned}$$

In particular, we can pass to the limit in (6.4) and (6.6) and establish (6.1) and (6.3). Now, let us multiply (6.4) by $\partial_t v_n$ and integrate in space and time. One readily gets that

$$\int_0^T (\partial_t v_n, u_n) = -\int_0^T \|\partial_t v_n\|^2 - \frac{1}{2}\int_\Omega |\nabla v_n(t)|^2 + \frac{1}{2}\int_\Omega |\nabla v_n^0|^2 + \int_0^T (f_n, \partial_t v_n).$$

Hence, passing to the $\limsup$ in the latter relation for $n \to +\infty$ and performing an integration by parts in the last term, we obtain

$$\limsup_{n\to+\infty} \int_0^T (\partial_t v_n, u_n) \leq \int_0^T (\partial_t v, u),$$

and (6.2) follows from the convergence in the weak sense of the sets H_n. □

Finally, we provide a second convergence result where no piecewise monotonicity for the sets H_n is required.

Theorem 6.5 (Semilinear problem 3). *Let $H_n \in (H^1(0,T;\mathcal{H}))^2$ be uniformly bounded on bounded sets of $H^1(0,T;\mathcal{H})$, uniformly linearly bounded in $(L^2(0,T;\mathcal{H}))^2$, and converge in the weak sense to $H \subset (H^1(0,T;\mathcal{H}))^2$. Moreover, let $f_n \to f$ strongly in $L^2(0,T;\mathcal{H})$, $v_n^0 \to v^0$ strongly in $H^1(\Omega)$, and $v_n \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and $u_n \in H^1(0,T;\mathcal{H})$ solve (6.4)-(6.6). Then there exist not relabeled subsequences such that $v_n \to v$ weakly in $H^1(0,T;\mathcal{H})$ and $u_n \to u$ weakly in $H^1(0,T;\mathcal{H})$, and v,u solve (6.1)-(6.3).*

Sketch of the proof. We shall check that v_n and u_n are uniformly bounded in $H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ and $H^1(0,T;\mathcal{H})$, respectively, by testing (6.4) by $\partial_t v_n$ and exploiting boundedness and comparisons. Hence, we possibly find two functions v and u and some not relabeled subsequences such that

$$\begin{aligned} v_n &\to v \text{ weakly star in } H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \\ u_n &\to u \text{ weakly in } H^1(0,T;\mathcal{H}). \end{aligned}$$

In particular, we can pass to the limit and establish (6.1) and (6.3). Again, arguing exactly as in the proof of Theorem 6.4, we readily check that

$$\limsup_{n\to+\infty} \int_0^T (\partial_t v_n, u_n) \le \int_0^T (\partial_t v, u),$$

and (6.2) follows from the convergence in the weak sense of the sets H_n. □

7 Convergence for quasilinear parabolic problems

With the same notations as above, we shall now focus on the problem of finding $v \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and $u \in H^1(0,T;\mathcal{H})$ such that

$$\partial_t(v+u) - \Delta v = f \quad \text{a.e. in } \Omega\times(0,T), \tag{7.1}$$

$$[v,u] \in H, \tag{7.2}$$

$$v(0) = v^0 \quad \text{a.e. in } \Omega. \tag{7.3}$$

Again, $f \in L^2(0,T;\mathcal{H})$ and v^0 are given data, and Dirichlet homogeneous boundary conditions are chosen just for the sake of simplicity. In the latter $H \subset (H^1(0,T;\mathcal{H}))^2$ is a relation, possibly of hysteresis type. We shall not address here well-posedness issues for the latter problem and refer the reader at once to the monograph [24, Ch. IX, p. 257] for a some related material. Let us now approximate the solution v, u by means of solutions to some auxiliary problem. In particular, let $v_n \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and $u_n \in H^1(0,T;\mathcal{H})$ be such that

$$\partial_t(v_n+u_n) - \Delta v_n = f_n \quad \text{a.e. in } \Omega\times(0,T), \tag{7.4}$$

$$[v_n,u_n] \in H_n, \tag{7.5}$$

$$v_n(0) = v_n^0 \quad \text{a.e. in } \Omega, \tag{7.6}$$

where H_n, f_n, and u_n^0 suitably converge to H, f, and u^0, respectively. We will exploit the above introduced convergence in the weak sense in order to pass to the limit into (7.4)-(7.6). Namely, we have the following.

Theorem 7.1 (Quasilinear problem). *Let* $H_n \subset (H^1(0,T;\mathcal{H}))^2$ *be piecewise monotone, uniformly linearly bounded in* $(H^1(0,T;\mathcal{H}))^2$*, and converge in the weak sense to* $H \subset (H^1(0,T;\mathcal{H}))^2$*. Moreover, let* $f_n \to f$ *weakly in* $L^2(0,T;\mathcal{H})$*,* $v_n^0 \to v^0$ *weakly in* $H^1(\Omega)$*, and* $v_n \in H^1(0,T;\mathcal{H}) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ *and* $u_n \in H^1(0,T;\mathcal{H})$ *solve* (7.4)-(7.6)*. Then there exist not relabeled subsequences such that* $v_n \to v$ *weakly in* $H^1(0,T;\mathcal{H})$ *and* $u_n \to u$ *weakly in* $H^1(0,T;\mathcal{H})$*, and* v, u *solve* (7.1)-(7.3)*.*

Sketch of the proof. We shall establish some a priori estimates on v_n in $H^1(0,T;\mathcal{H}) \cap$ $\cap L^\infty(0,T;H^1(\Omega)) \cap$ $\cap L^2(0,T;H^2(\Omega))$ and u_n in $H^1(0,T;\mathcal{H})$ by simply multiplying (7.4) by ∂v_n, take the integral on space and time, and use some boundedness assumptions and comparison. Hence, possibly passing to not relabeled subsequences and owing to well known compactness results, we easily deduce that

$$\begin{aligned} v_n &\to v \text{ weakly star in } H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\ &\qquad \text{and strongly in } C([0,T];\mathcal{H}) \cap L^2(0,T;H_0^1(\Omega)), \qquad (7.7) \\ u_n &\to u \text{ weakly in } H^1(0,T;\mathcal{H}), \qquad (7.8) \end{aligned}$$

which are sufficient in order to establish (7.1) and (7.3). We are now in the position of multiplying (7.4) by v_n, integrate on space and time, exploit piecewise monotonicity, and perform an integration by parts in order to get that

$$\int_0^T (\partial_t v_n, u_n) = \frac{1}{2}\|v_n(T)\|^2 - \frac{1}{2}\|v_n^0\|^2$$
$$+ (u_n(T), v_n(T)) - (u_n(0), v_n^0) + \int_0^T \int_\Omega |\nabla v_n|^2 - \int_0^T (f_n, v_n).$$

Hence, passing to the $\limsup$ as $n \to +\infty$, and using (7.1) and (7.7)-(7.8), we get that

$$\int_0^T (\partial_t v_n, u_n) \to \int_0^T (\partial_t v, u).$$

Namely, owing to the convergence in the weak sense of the operators H_n, the above convergences suffices to pass to the limit in (7.5) and conclude the proof of the lemma. □

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