Global Existence of a Strong Solution to the One-Dimensional Full Model for Irreversible Phase Transitions

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Abstract

This note deals with the analysis of a model proposed by M. Frémond in order to describe some irreversible phase transition phenomena resulting as macroscopic effects of the microscopic movements of molecules. This model consists in a nonlinear system of partial differential equations of parabolic type and several simplifications have been studied recently. Nevertheless, up to now the question of the existence of a solution to the full problem was still open. This paper answers affirmatively to this question in the one dimensional setting by exploiting a regularization – a priori estimates – passage to the limit procedure.

Key words: microscopic movements, phase transition, irreversibility.

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1 Introduction

This analysis is concerned with a system of partial differential equations governing the evolution of the two unknown fields ϑ and χ. In particular, letting $T > 0$ be some reference time and $Q := (0,1) \times (0,T)$ we investigate the existence of solutions to

$$\vartheta_t + \vartheta\chi_t - \vartheta_{xx} = \chi_t^2 \qquad \text{a.e. in } Q, \tag{1.1}$$

$$\chi_t + \alpha(\chi_t) - \chi_{xx} + \beta(\chi) \ni \vartheta - \vartheta_c \qquad \text{a.e. in } Q. \tag{1.2}$$

Here, ϑ_c is a positive constant and α, $\beta \in \mathbb{R} \times \mathbb{R}$ are maximal monotone graphs. More precisely $\alpha = \partial I_{[0,+\infty)}$, where $I_{[0,+\infty)}$ is the indicator function of the set $[0,+\infty)$, given by $I_{[0,+\infty)}(x) = 0$ if $x \in [0,+\infty)$ and $I_{[0,+\infty)}(x) = +\infty$ elsewhere. Thus, from the definition of subdifferential, it turns out that

$$y \in \alpha(x) \quad \text{if and only if} \quad x \geq 0 \ \text{ and } \ y(x-z) \geq 0 \ \ \forall z \geq 0,$$

which may be equivalently written as

$$\alpha(x) = \begin{cases} \emptyset & \text{if } x < 0 \\ (-\infty, 0] & \text{if } x = 0 \\ \{0\} & \text{if } x > 0. \end{cases} \tag{1.3}$$

On the other hand the graph β is chosen completely arbitrarily.

The system (1.1)-(1.2) arises in the study of an irreversible phase transition model recently proposed by M. Frémond [3]. In particular, we assume that the domain Ω is filled with a two-phase substance whose evolution is described by means of its absolute temperature ϑ and a *phase field* χ. The latter parameter is to be interpreted as a local proportion between the two phases. In this respect, we remark that the physical choice for the graph β is $\beta = \partial I_{[0,1]}$ where we recall that

$$y \in \partial I_{[0,1]}(x) \quad \text{if and only if} \quad x \in [0,1] \ \text{ and } \ y(x-z) \geq 0 \ \ \forall z \in [0,1],$$

or, equivalently, $\partial I_{[0,1]}(x) = \emptyset$ if $x < 0$ or $x > 1$, $\partial I_{[0,1]}(0) = (-\infty, 0]$, $\partial I_{[0,1]}(x) = \{0\}$ if $x \in (0,1)$, and $\partial I_{[0,1]}(1) = [0,+\infty)$. In particular, let us stress that the choice $\beta = \partial I_{[0,1]}$ forces χ to attain solely values in $[0,1]$.

Hence, the model consists of the coupling of an energy balance equation with a doubly nonlinear inclusion governing the phase dynamics. Indeed, we aim to analyze the following system

$$c_s\vartheta_t + L\chi_t - k\vartheta_{xx} = -\frac{L}{\vartheta_c}(\vartheta - \vartheta_c)\chi_t + \mu(\chi_t)^2 + \xi\chi_t, \tag{1.4}$$

$$\mu\chi_t + \xi - \nu\chi_{xx} + \eta = \frac{L}{\vartheta_c}(\vartheta - \vartheta_c), \tag{1.5}$$

where

$$\eta \in \beta(\chi) \quad \text{and} \quad \xi \in \alpha(\chi_t), \tag{1.6}$$

and the quantities c_s, L, k, ϑ_c, μ, and ν are positive physical parameters. The reader is referred to [3] for a full discussion on the derivation of the model as well as for the physical meaning of the latter parameters. On the contrary, here we prefer not to go into the details of the modeling and we only restrict ourselves to some considerations.

First of all, let us observe that the second inclusion in (1.6) implies that χ_t has to be nonnegative. Hence, this model is suitable of describing some irreversible phase transition dynamics such as, for instance, glue hardening or glass formation.

Secondly, we remark the nonlinear features of the right hand side of the energy equation (1.4). This nonlinear character is closely related to the assumption that the microscopic movements of the molecules (accounted for in the quantity χ_t) are responsible of macroscopic effects as well.

Finally, let us notice that the term $\xi\chi_t$ in (1.4) vanishes almost everywhere due to the second inclusion of (1.6). Thus, normalizing most of the physical constants to 1, the system (1.4)-(1.6) reduces, in particular, to a system of the type of (1.1)-(1.2).

In this respect we also observe that the rather classical energy balance relation for phase change problems

$$c_s\vartheta_t + L\chi_t - k\theta_{xx} = 0, \tag{1.7}$$

consists essentially in a linearization of (1.4) in a neighborhood of the critical temperature ($\vartheta \approx \vartheta_c$) combined with the smallness assumption $\mu\,(\chi_t)^2 \approx 0$.

Of course, for the sake of a mathematical study, the system (1.1)-(1.2) has to be complemented with suitable boundary and initial conditions. Actually, we prescribe

$$\vartheta_x(0,\cdot) = \vartheta_x(1,\cdot) = 0 \quad \text{a.e. in } (0,T), \tag{1.8}$$
$$\chi_x(0,\cdot) = \chi_x(1,\cdot) = 0 \quad \text{a.e. in } (0,T), \tag{1.9}$$
$$\vartheta(\cdot,0) = \vartheta^0 \quad \text{a.e. in } \Omega, \tag{1.10}$$
$$\chi(\cdot,0) = \chi^0 \quad \text{a.e. in } \Omega, \tag{1.11}$$

for a pair of functions $\vartheta^0, \chi^0 : (0,1) \longrightarrow \mathbb{R}$. The reader should notice that our choice of Neumann boundary condition (1.9) has an evident physical justification at least as χ is concerned. From the mathematical point of view, we stress that different choices of boundary conditions can be accounted for in our analysis and that we restrict ourselves to (1.8) just for the sake of simplicity.

We shall now briefly present some literature on the system (1.4)-(1.6). A first result in the direction of the existence of solutions to some reduced model has been obtained in [3] where, nevertheless, relation (1.7) is considered in place of (1.4). Then, the paper [8] deals with the existence of a solution to (1.4)-(1.6) by assuming that $\mu = 0$ in the right hand side of (1.4) (but not in (1.5)). Finally, the full problem is solved in [6] in the non-diffusive case ($\nu = 0$) and in [9] where the original graph α is replaced by $\partial I_{[0,\lambda]}$ for some $\lambda \in (0,+\infty)$. The latter graph forces the quantity χ_t to attain only bounded values and may account for a limit velocity in the phase transition. We conclude this review with the paper [10] where the authors prove the existence of a strong solution to (1.4)-(1.6) in the one-dimensional case when $\alpha \equiv 0$.

Up to now, no existence result for the full model (1.1)-(1.2) has been proved, even in the one-dimensional case. This fact is probably related to the specific mathematical difficulties of the model, in particular to its high-order nonlinearities in (1.1). The main novelty of this paper is the use of a rather particular approximation of the system (1.1)-(1.2), combined with an a priori estimate – passage to the limit argument, that allows us to answer affirmatively to the question of global existence in the one-dimensional setting.

The plan of the paper is the following: the main result is stated in Section 2 while Section 3 addresses the approximating problem. Then, the a priori estimates are derived in Section 4 and the limit procedure is detailed in Section 5.

2 Main result

We start by fixing some notations. Let $\Omega = (0,1)$ and denote $Q_t := \Omega \times (0,t)$, for all $t \in (0,T]$, and $Q := Q_T$. Moreover, for the sake of simplicity we set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{u \in H^2(\Omega) \text{ such that } u_x(0) = u_x(1) = 0\},$$

endowed with the usual scalar products and we denote by $\|\cdot\|$ the norm in H, and by $\|\cdot\|_E$ the norm of the generic normed space E. Finally, let V^* be the dual of V, $\langle\cdot,\cdot\rangle$ be the duality pairing between V^* and V, and $J : V \longrightarrow V^*$ be the corresponding Riesz isomorphism.

Next, we introduce our assumptions on data by requiring that

$$\vartheta_c > 0 \quad \text{is a prescribed constant}, \tag{2.1}$$

$$\alpha = \partial I_{[0,+\infty)}, \tag{2.2}$$

$$\hat{\beta} : \mathbb{R} \longrightarrow (-\infty, +\infty] \quad \text{is a proper, convex, and lower semicontinuous function,}$$
$$\min \hat{\beta} = \hat{\beta}(0) = 0, \quad \text{and} \quad \beta = \partial \hat{\beta}, \tag{2.3}$$

$$\vartheta^0 \in V, \quad \vartheta^0 > 0 \quad \text{in} \quad \overline{\Omega}, \tag{2.4}$$

$$\chi^0 \in W^{2,\infty}(\Omega),\ \hat{\beta}(\chi^0) \in L^1(\Omega),$$
$$\text{and there exists} \quad \eta^0 \in L^\infty(\Omega) \quad \text{such that} \quad \eta^0 \in \beta(\chi^0) \quad \text{a.e. in} \quad \Omega. \tag{2.5}$$

We are now in the position of stating our main result.

Theorem 2.1. *Under the assumptions (2.1)-(2.5), there exists a quadruple* $(\vartheta, \chi, \xi, \eta)$ *such that*

$$\vartheta \in H^1(0,T;H) \cap C([0,T];V) \cap L^2(0,T;W), \tag{2.6}$$

$$\chi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{2.7}$$

$$\xi \in L^\infty(0,T;H), \tag{2.8}$$

$$\eta \in L^\infty(0,T;H), \tag{2.9}$$

and the following relations hold

$$\vartheta_t + \vartheta\chi_t - \vartheta_{xx} = \chi_t^2 \qquad a.e.\ in\ \ Q, \tag{2.10}$$

$$\chi_t + \xi - \chi_{xx} + \eta = \vartheta - \vartheta_c \qquad a.e.\ in\ \ Q, \tag{2.11}$$

$$\xi \in \alpha(\chi_t) \qquad a.e.\ in\ \ Q, \tag{2.12}$$

$$\eta \in \beta(\chi) \qquad a.e.\ in\ \ Q, \tag{2.13}$$

$$\vartheta_x(0,\cdot) = \vartheta_x(1,\cdot) = 0 \quad a.e.\ in\ \ (0,T), \tag{2.14}$$

$$\chi_x(0,\cdot) = \chi_x(1,\cdot) = 0 \quad a.e.\ in\ \ (0,T), \tag{2.15}$$

$$\vartheta(\cdot,0) = \vartheta^0 \quad a.e.\ in\ \ \Omega, \tag{2.16}$$

$$\chi(\cdot,0) = \chi^0 \quad a.e.\ in\ \ \Omega. \tag{2.17}$$

Moreover, we also have that there exists a constant $\vartheta^* > 0$ *such that*

$$\vartheta \geq \vartheta^* > 0 \quad a.e.\ in\ \ Q. \tag{2.18}$$

The proof of this result will be performed throughout the rest of the paper.

3 An approximated problem

In order to achieve the proof of Theorem 2.1 we proceed as follows. First of all we establish a global existence result for an approximating problem. Then, we perform some a priori estimates which enable us to pass to the limit.

In this respect, the present section is devoted to the proof of the existence of a solution (ϑ, χ) to the problem (2.10)-(2.13) when α is replaced by $\partial I_{[0,\lambda]}$ for some $\lambda > 0$ and β is Lipschitz continuous. As the first assumption is concerned, let us stress that it entails (see

(2.12)) that χ_t is essentially bounded. On the other hand, we will regularize β by means of its Yosida approximation β_λ defined by

$$\beta_\lambda := \lambda\bigl(\mathrm{id} - (\mathrm{id} + \lambda^{-1}\beta)^{-1}\bigr),$$

where id denotes the identity in $\mathbb{R}$. Of course other approximations are also possible and we limit ourselves to the present one for the sake of simplicity. In the remainder of the section we will prove the following

Theorem 3.1. *Let the assumptions (2.1), (2.3)-(2.5) hold, assign $\lambda > 0$ and take*

$$\alpha_\lambda := \partial I_{[0,\lambda]}, \quad \beta_\lambda := \lambda\bigl(\mathrm{id} - (\mathrm{id} + \lambda^{-1}\beta)^{-1}\bigr). \tag{3.1}$$

Then, there exists a quadruple $(\vartheta_\lambda, \chi_\lambda, \xi_\lambda, \eta_\lambda)$ such that

$$\vartheta_\lambda \in H^1(0,T;H) \cap C([0,T];V) \cap L^2(0,T;W), \tag{3.2}$$
$$\chi_\lambda \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{3.3}$$
$$\xi_\lambda \in L^\infty(0,T;H), \tag{3.4}$$
$$\eta_\lambda \in L^\infty(0,T;H), \tag{3.5}$$

and the following relations hold

$$\vartheta_{\lambda t} + \vartheta_\lambda \chi_{\lambda t} - \vartheta_{\lambda xx} = \chi_{\lambda t}^2 \qquad a.e.\ in\ Q, \tag{3.6}$$
$$\chi_{\lambda t} + \xi_\lambda - \chi_{\lambda xx} + \eta_\lambda = \vartheta_\lambda - \vartheta_c \qquad a.e.\ in\ Q, \tag{3.7}$$
$$\xi_\lambda \in \alpha_\lambda(\chi_{\lambda t}) \qquad a.e.\ in\ Q, \tag{3.8}$$
$$\eta_\lambda \in \beta_\lambda(\chi_\lambda) \qquad a.e.\ in\ Q, \tag{3.9}$$
$$\vartheta_{\lambda x}(0,\cdot) = \vartheta_{\lambda x}(1,\cdot) = 0 \quad a.e.\ in\ (0,T), \tag{3.10}$$
$$\chi_{\lambda x}(0,\cdot) = \chi_{\lambda x}(1,\cdot) = 0 \quad a.e.\ in\ (0,T), \tag{3.11}$$
$$\vartheta_\lambda(\cdot,0) = \vartheta^0 \quad a.e.\ in\ \Omega, \tag{3.12}$$
$$\chi_\lambda(\cdot,0) = \chi^0 \quad a.e.\ in\ \Omega. \tag{3.13}$$

Moreover, we may also find a constant $\vartheta_\lambda^* > 0$ such that

$$\vartheta_\lambda \ge \vartheta_\lambda^* > 0 \quad a.e.\ in\ Q. \tag{3.14}$$

Proof. Since $\lambda > 0$ is now fixed, we omit the subscript λ from functions for the remainder of this section.

The forthcoming proof is really close to that of [9, Thm 2.1]. We thus only sketch the main steps of the proof and refer to [9] for details.

As a first step, we establish the existence of a solution to a time-discrete problem. To this aim, let $N \in \mathbb{N}$, $\tau := T/N$ and look for a solution $\{(\vartheta_i, \chi_i)\}_{i=0}^N \in (H \times H)^{N+1}$ to the scheme

$$\frac{\vartheta_i - \vartheta_{i-1}}{\tau} + \vartheta_{i-1}\frac{\chi_i - \chi_{i-1}}{\tau} + A\vartheta_i = \left(\frac{\chi_i - \chi_{i-1}}{\tau}\right)^2 \quad \text{for } i = 1, \dots, N, \tag{3.15}$$
$$\frac{\chi_i - \chi_{i-1}}{\tau} + \xi_i + A\chi_i + \eta_i = \vartheta_{i-1} - \vartheta_c \quad \text{for } i = 1, \dots, N, \tag{3.16}$$
$$\xi_i \in \alpha_\lambda\left(\frac{\chi_i - \chi_{i-1}}{\tau}\right), \quad \eta_i = \beta_\lambda(\chi_i) \quad \text{for } i = 1, \dots, N, \tag{3.17}$$
$$\vartheta_0 = \vartheta^0, \ \chi_0 = \chi^0, \tag{3.18}$$

where $A : H \longrightarrow H$ is the operator defined by

$$D(A) := W, \quad Au = -u_{xx} \qquad \forall u \in D(A).$$

It is straightforward to prove that the latter *semi-implicit* scheme has a unique solution (see [9, Lemma 3.1]). Let us introduce some useful notation. Let $\{u_i\}_{i=0}^{N} \in H^{N+1}$ be a vector. Then, we denote by u_τ and $\overline{u}_\tau$ two functions of the time interval $[0,T]$ which interpolate the values of the vector piecewise linearly and backward constantly, respectively. That is,

$$\begin{aligned} &u_\tau(0) := u_0, \quad u_\tau(t) := a_i(t)u_i + \big(1 - a_i(t)\big)u_{i-1}, \\ &\overline{u}_\tau(0) := u_0, \quad \overline{u}_\tau(t) := u_i, \qquad \text{for } t \in ((i-1)\tau, i\tau], \ i = 1, \dots, N, \end{aligned} \tag{3.19}$$

where $a_i(t) := (t - (i-1)\tau)/\tau$ for $t \in ((i-1)\tau, i\tau]$, $i = 1, \dots, N$. Moreover, let us introduce the translation operator $\mathcal{T}_\tau$ related to the the time step τ by setting, for all $u : [0,T] \longrightarrow \mathbb{R}$,

$$(\mathcal{T}_\tau u)(t) := u(0) \ \ \forall t \in [0,\tau) \quad \text{and} \quad (\mathcal{T}_\tau u)(t) := u(t-\tau) \qquad \forall t \in [\tau, T].$$

According to the latter positions, it is a standard matter to rewrite (3.15)-(3.17) as

$$\partial_t\vartheta_\tau + \mathcal{T}_\tau\overline{\vartheta}_\tau\partial_t\chi_\tau + A\overline{\vartheta}_\tau = (\partial_t\chi_\tau)^2 \quad \text{a.e. in } Q, \tag{3.20}$$

$$\partial_t\chi_\tau + \overline{\xi}_\tau + A\overline{\chi}_\tau + \overline{\eta}_\tau = \mathcal{T}_\tau\overline{\vartheta}_\tau - \vartheta_c \quad \text{a.e. in } Q, \tag{3.21}$$

$$\overline{\xi}_\tau \in \alpha_\lambda(\partial_t\chi_\tau), \quad \overline{\eta}_\tau = \beta_\lambda(\overline{\chi}_\tau) \quad \text{a.e. in } Q. \tag{3.22}$$

Our next aim is that of proving some a priori estimates for $(\vartheta_\tau, \chi_\tau, \overline{\xi}_\tau, \overline{\eta}_\tau)$ independently of τ. Henceforth, C_λ will denote a positive constant, possibly depending on the data and on λ but independent of the time step τ. Of course C_λ may vary from line to line.

First of all, let us observe that the inclusion in (3.22) entails

$$0 \le \partial_t\chi_\tau \le \lambda \quad \text{a.e. in } Q. \tag{3.23}$$

Hence, owing to well-known results on time-implicit schemes for parabolic equations, it is a standard matter to obtain the bound

$$\|\vartheta_\tau\|_{H^1(0,T;H)\cap C([0,T];V)\cap L^2(0,T;W)} \le C_\lambda. \tag{3.24}$$

As regards χ_τ, we argue as in [9] and multiply equation (3.21) by $\tau(A\delta\chi_i + \delta\eta_i)$ where $\delta\chi_i := (\chi_i - \chi_{i-1})/\tau$, and $\delta\eta_i := (\eta_i - \eta_{i-1})/\tau$. Moreover, we take the integral on Ω, and obtain

$$\begin{aligned} &\tau\|\delta\chi_{i,x}\|^2 + \tau(\delta\chi_i, \delta\eta_i) + \tau(A\chi_i + \eta_i, A\delta\chi_i + \delta\eta_i) \\ &\quad + \tau(\xi_i, \delta A\chi_i) + \tau(\xi_i, \delta\eta_i) \le \tau(\vartheta_{i-1} - \vartheta_c, A\delta\chi_i + \delta\eta_i). \end{aligned} \tag{3.25}$$

At first, we readily get that

$$\begin{aligned} &\tau(A\chi_i + \eta_i, A\delta\chi_i + \delta\eta_i) \\ &= \frac{1}{2}\big(\|A\chi_i + \eta_i\|^2 + \|(A\chi_i + \eta_i) - (A\chi_{i-1} + \eta_{i-1})\|^2 - \|A\chi_{i-1} + \eta_{i-1}\|^2\big). \end{aligned}$$

Moreover, the relation

$$\tau(\delta\chi^i, \delta\eta^i) \ge 0,$$

is a consequence of the monotonicity of β_λ while

$$\tau(\xi^i, A\delta\chi^i) \ge 0,$$

follows from the monotonicity of α_λ and [5, Lemma 2].

Our next aim is to control the last term on the left hand side of (3.25). We readily get that

$$\tau(\xi_i, \delta\eta_i) = \tau \int_\omega \xi_i\, \delta\eta_i\, \frac{\delta\chi_i}{\delta\chi_i} \geq 0$$

where $\omega = \{\delta\chi_i \neq 0\}$ and we exploited the monotonicity of α_λ and β_λ, together with the fact that $0 \in \alpha_\lambda(0)$.

Now, let us take the sum in (3.25) for $i = 1, \ldots, m$ ($m \leq N$). One has

$$\begin{aligned}\sum_{i=1}^{m} \tau\|\delta\chi_{i,x}\|^2 &+ \frac{1}{2}\|A\chi_m + \eta_m\|^2 + \frac{1}{2}\sum_{i=1}^{m}\|\tau(A\delta\chi_i + \delta\eta_i)\|^2 \\ &\leq \frac{1}{2}\|A\chi^0 + \eta^0\|^2 + \sum_{i=1}^{m}\tau(\vartheta_{i-1} - \vartheta_c, A\delta\chi_i + \delta\eta_i). \end{aligned} \tag{3.26}$$

The latter right hand side is to be controlled by means of a discrete integration by parts procedure. Namely, we easily infer that

$$\begin{aligned}&\sum_{i=1}^{m}\tau(\vartheta_{i-1} - \vartheta_c, A\delta\chi_i + \delta\eta_i) \\ &= -\sum_{i=1}^{m-1}\tau(\delta\vartheta_i, A\chi_i + \eta_i) + (\vartheta_{m-1} - \vartheta_c, A\chi_m + \eta_m) - (\vartheta^0 - \vartheta_c, A\chi^0 + \eta^0).\end{aligned}$$

Whence, looking back to (2.4)-(2.5), (3.24), and (3.26), one readily gets that

$$\sum_{i=1}^{m}\tau\|\delta\chi_{i,x}\|^2 + \frac{1}{4}\|A\chi_m + \eta_m\|^2 \leq C_\lambda\left(1 + \sum_{i=1}^{m-1}\tau\|A\chi_i + \eta_i\|^2\right).$$

Finally, an application of the discrete Gronwall lemma (see, e.g., the version reported in [7, Prop. 2.2.1]), the monotonicity of β_λ, and standard elliptic estimates ensure that

$$\|\chi_\tau\|_{H^1(0,T;V)\cap C([0,T];W)} + \|\overline{\eta}_\tau\|_{L^\infty(0,T;H)} \leq C_\lambda. \tag{3.27}$$

Thus, a comparison in (3.21) entails also

$$\|\overline{\xi}_\tau\|_{L^\infty(0,T;H)} \leq C_\lambda. \tag{3.28}$$

Now, the passage to the limit procedure as $\tau \longrightarrow 0$ is performed owing to well-known compactness and monotonicity arguments. Since it follows exactly [9, Sec. 5] and it is very similar to the argument developed in Section 5 of the present paper we omit it here.

As regards the positivity property (3.14), we argue as follows. First of all, we rewrite equation (3.6) as

$$\vartheta_{\lambda t} - \vartheta_{\lambda xx} = -\chi_{\lambda t}\vartheta_\lambda + \chi_{\lambda t}^2 =: a\vartheta_\lambda + b. \tag{3.29}$$

Now, owing to (3.3), we readily deduce that

$$a \in L^1(0,T;L^\infty(\Omega)) \quad \text{and} \quad b \geq 0 \ \text{ a.e. in } Q,$$

so that

$$\vartheta_{\lambda t} - \vartheta_{\lambda xx} \geq -\|a\|_{L^\infty(\Omega)}\vartheta_\lambda.$$

Putting $\underline{\vartheta}^0 := \min_{x\in[0,1]} \vartheta^0(x)$ and noticing that

$$\Theta \ : \ t \ \mapsto \ \underline{\vartheta}^0 \exp\left(- \int_0^t \|a(s)\|_{L^\infty(\Omega)} ds \right)$$

satisfies $\Theta(0) \leq \vartheta^0$ and $\Theta' + \|a\|_{L^\infty(\Omega)}\Theta = 0$, the comparison principle entails that

$$\vartheta_\lambda(\cdot, t) \geq \Theta(t) \quad \text{a.e. in } \ \Omega, \ \forall t \in [0,T].$$

Hence, relation (3.14) is proved with $\vartheta_\lambda^* := \underline{\vartheta}^0 \exp\big(- \|a\|_{L^1(0,T;L^\infty(\Omega))}\big)$.

□

4 A priori estimates

In this section we shall establish a set of a priori estimates, independent of the approximation parameter λ, in order to pass to the limit in the approximated problem (3.6)-(2.17). Henceforth, C denotes any constant, possibly depending on $\|\vartheta^0\|_V$, $\|\chi^0\|_{W^{2,\infty}(\Omega)}$, $\|\hat{\beta}(\chi^0)\|_{L^1(\Omega)}$, and $\|\eta^0\|_{L^\infty(\Omega)}$ but independent of λ. Of course, C may vary from line to line.

4.1 First estimate

Let us multiply equation (3.6) by 1 and equation (3.7) by $\chi_{\lambda t}$, respectively. Then, we add the two resulting equations, notice some cancellations and integrate on Q_t, for $t \in (0,T)$. Owing to the monotonicity of α_λ and the non-negativity of $\vartheta_c \chi_{\lambda t}$, one has that

$$\int_\Omega \Big(\vartheta_\lambda(t) + \frac{1}{2}\chi_{\lambda x}^2(t) + \hat{\beta}_\lambda(\chi_\lambda(t))\Big) \leq \int_\Omega \Big(\vartheta^0 + \frac{1}{2}(\chi_x^0)^2 + \hat{\beta}(\chi^0)\Big).$$

Hence, taking into account (2.4)-(2.5) and recalling (3.14) we readily get that

$$\|\vartheta_\lambda\|_{L^\infty(0,T;L^1(\Omega))} + \|\chi_{\lambda x}\|_{L^\infty(0,T;H)} \leq C. \tag{4.1}$$

4.2 Second estimate

Since $\vartheta_\lambda^{-1} \in L^\infty(Q)$ by (3.14) we may multiply equation (3.6) by the function $-\vartheta_\lambda^{-1}$ and integrate on Q_t, for $t \in (0,T)$. Exploiting (2.4) and (4.1) we obtain

$$\begin{aligned} - \int_\Omega \ln\big(\vartheta_\lambda(t)\big) + \iint_{Q_t} \left(\frac{\vartheta_{\lambda x}^2}{\vartheta_\lambda^2} + \frac{\chi_{\lambda t}^2}{\vartheta_\lambda}\right) &\leq - \int_\Omega \ln(\vartheta^0) + \iint_{Q_t} \chi_{\lambda t} \\ \leq - \int_\Omega \ln(\vartheta^0) + \frac{1}{2}\iint_{Q_t} \left(\frac{\chi_{\lambda t}^2}{\vartheta_\lambda} + \vartheta_\lambda\right) &\leq C + \frac{1}{2}\iint_{Q_t} \frac{\chi_{\lambda t}^2}{\vartheta_\lambda}. \end{aligned}$$

Using again (4.1) together with the concavity of $r \mapsto \ln r$, we further obtain

$$\iint_Q \left(\frac{\vartheta_{\lambda x}}{\vartheta_\lambda}\right)^2 + \frac{\chi_{\lambda t}^2}{\vartheta_\lambda} \leq C.$$

Then, the latter relation, (4.1), and the continuity of the inclusion $W^{1,1}(\Omega) \subset L^\infty(\Omega)$ entail in particular that

$$
\begin{aligned}
\int_0^T \|\vartheta_\lambda\|_{L^\infty(\Omega)} = \int_0^T \|{\vartheta_\lambda}^{1/2}\|^2_{L^\infty(\Omega)} &\le C \int_0^T \Big(\|({\vartheta_\lambda}^{1/2})_x\|^2_{L^1(\Omega)} + \|\vartheta_\lambda\|^2_{L^1(\Omega)} \Big) \\
\le C \left(1 + \int_0^T \Big(\int_\Omega \frac{\vartheta_{\lambda x}}{{\vartheta_\lambda}^{1/2}} \Big)^2 \right) &\le C \left(1 + \int_0^T \Big(\|\vartheta_{\lambda x}/\vartheta_\lambda\| \, \|{\vartheta_\lambda}^{1/2}\| \Big)^2 \right) \\
&\le C \left(1 + \int_0^T \|\vartheta_{\lambda x}/\vartheta_\lambda\|^2 \right) \le C.
\end{aligned}
$$

Hence, in particular we have that

$$
\|\vartheta_\lambda\|_{L^1(0,T;L^\infty(\Omega))} \le C, \tag{4.2}
$$

and finally, by interpolation with (4.1),

$$
\|\vartheta_\lambda\|_{L^2(0,T;H)} \le C. \tag{4.3}
$$

4.3 Third estimate

Let us multiply (3.7) by $\chi_{\lambda t}$ and integrate on Q_t, for $t \in (0,T)$. Owing to the monotonicity of α_λ one has that

$$
\begin{aligned}
&\iint_{Q_t} \chi_{\lambda t}^2 + \frac{1}{2}\|\chi_{\lambda x}(t)\|^2 + \int_\Omega \hat\beta_\lambda(\chi_\lambda(t)) \\
&\le \frac{1}{2}\|\chi^0_x(t)\|^2 + \int_\Omega \hat\beta(\chi^0) + \iint_{Q_t} (\vartheta_\lambda - \vartheta_c)\, \chi_{\lambda t}.
\end{aligned}
$$

Hence, owing to the estimate (4.3), it is straightforward to check that

$$
\|\chi_{\lambda t}\|_{L^2(0,T;H)} \le C. \tag{4.4}
$$

4.4 Fourth estimate

First of all let us take the time derivative of (3.7), multiply it by $\chi_{\lambda t}$ and integrate on Q_t, for $t \in (0,T)$. Let us stress that, at this stage, the differentiation is just formal. However, the procedure can be justified at some approximation level and we prefer to skip these details for clarity. Taking into account the special form of α_λ one easily deduces that

$$
\iint_{Q_t} \xi_{\lambda t}\, \chi_{\lambda t} = \int_\Omega \hat\alpha^*_\lambda(\xi_\lambda(t)) - \hat\alpha^*_\lambda(\xi_\lambda(0)), \tag{4.5}
$$

where $\hat\alpha^*_\lambda$ is the *convex conjugate* of $\hat\alpha_\lambda$. Now, it is a standard matter to prove that

$$
\hat\alpha^*_\lambda(r) \;:=\; \sup_{s\in\mathbb{R}} \Big(rs - I_{[0,\lambda]}(s) \Big) \;=\; \sup_{s\in[0,\lambda]} (rs) \;=\; \lambda \max\{0, r\}. \tag{4.6}
$$

Moreover $\xi_\lambda(0)$ fulfills

$$
\xi_\lambda(0) \in \alpha_\lambda(\chi_{\lambda t}(0)) \quad \text{a.e. in } Q,
$$

where $\chi_{\lambda t}(0)$ is defined by

$$\chi_{\lambda t}(0) := (\mathrm{id} + \alpha_\lambda)^{-1}\big(- A\chi^0 - \eta^0 + \vartheta^0 - \vartheta_c\big).$$

Owing to (2.5) and (3.1), it is straightforward to check that $\chi_{\lambda t}(0)$ is bounded in $L^\infty(\Omega)$, uniformly with respect to λ. Hence, upon choosing λ large enough, we have that $\xi_\lambda(0) \leq 0$ almost everywhere in Ω, so that $\hat{\alpha}^*_\lambda(\xi_\lambda(0)) = 0$ by (4.6), and relation (4.5) turns out to ensure that

$$\iint_{Q_t} \xi_{\lambda t}\, \chi_{\lambda t} \geq 0.$$

The latter considerations and the monotonicity of β_λ yield

$$\frac{1}{2}\|\chi_{\lambda t}(t)\|^2 + \iint_{Q_t} \chi_{\lambda x t}^2 \leq \frac{1}{2}\|\chi_{\lambda t}(0)\|^2 + \iint_{Q_t} \vartheta_{\lambda t}\, \chi_{\lambda t},$$

and assumptions (2.4)-(2.5) entail the bound

$$\begin{aligned} \frac{1}{2}\|\chi_{\lambda t}(t)\|^2 + \iint_{Q_t} \chi_{\lambda x t}^2 &\leq C + \int_0^t \|\vartheta_{\lambda t}\|_{V^*} \|\chi_{\lambda t}\|_V \\ &\leq C + \frac{1}{2}\int_0^t \big(\|\vartheta_{\lambda t}\|^2_{V^*} + \|\chi_{\lambda t}\|^2 + \|\chi_{\lambda x t}\|^2\big)\,. \end{aligned} \tag{4.7}$$

Next, we look at (3.6) as a relation in V^*, add to both sides ϑ_λ (so that $\vartheta_\lambda - \vartheta_{\lambda xx} = J\vartheta_\lambda$), test it with $J^{-1}\vartheta_{\lambda t}$, and integrate the result over $(0,t)$, for $t \in (0,T)$. We get

$$\begin{aligned} &\|\vartheta_{\lambda t}\|^2_{L^2(0,t;V^*)} + \frac{1}{2}\|\vartheta_\lambda(t)\|^2 \\ &\leq \frac{1}{2}\|\vartheta^0\|^2 + \int_0^t \langle \chi_{\lambda t}^2, J^{-1}\vartheta_{\lambda t}\rangle + \int_0^t \langle \vartheta_\lambda \chi_{\lambda t}, J^{-1}\vartheta_{\lambda t}\rangle + \int_0^t \langle \vartheta_\lambda, J^{-1}\vartheta_{\lambda t}\rangle. \end{aligned} \tag{4.8}$$

Our next aim is that of controlling the above right hand side. By th continuous embedding $V \subset L^\infty(\Omega)$ we readily get that

$$\begin{aligned} \int_0^t \langle \chi_{\lambda t}^2, J^{-1}\vartheta_{\lambda t}\rangle &\leq \frac{1}{8}\|\vartheta_{\lambda t}\|^2_{L^2(0,t;V^*)} + C\|\chi_{\lambda t}^2\|^2_{L^2(0,t;L^1(\Omega))} \\ &\leq \frac{1}{8}\|\vartheta_{\lambda t}\|^2_{L^2(0,t;V^*)} + C\int_0^t \|\chi_{\lambda t}\|^2\|\chi_{\lambda t}\|^2, \\ \int_0^t \langle \vartheta_\lambda \chi_{\lambda t}, J^{-1}\vartheta_{\lambda t}\rangle &\leq \frac{1}{8}\|\vartheta_{\lambda t}\|^2_{L^2(0,t;V^*)} + C\|\vartheta_\lambda \chi_{\lambda t}\|^2_{L^2(0,t;L^1(\Omega))} \\ &\leq \frac{1}{8}\|\vartheta_{\lambda t}\|^2_{L^2(0,t;V^*)} + C\int_0^t \|\vartheta_\lambda\|^2\|\chi_{\lambda t}\|^2, \\ \int_0^t \langle \vartheta_\lambda, J^{-1}\vartheta_{\lambda t}\rangle &\leq \frac{1}{2}\|\vartheta_\lambda(t)\|^2_{V^*} + \frac{1}{2}\|\vartheta^0\|^2 \leq C, \end{aligned}$$

where we have used (4.1), (2.4), and the embedding $L^1(\Omega) \subset V^*$ to obtain the last bound. Hence, relation (4.8) turns out to become

$$\frac{3}{4}\|\vartheta_{\lambda t}\|^2_{L^2(0,t;V^*)} + \frac{1}{2}\|\vartheta_\lambda(t)\|^2 \leq C + \int_0^t \big(\|\vartheta_\lambda\|^2 + \|\chi_{\lambda t}\|^2\big)\|\chi_{\lambda t}\|^2. \tag{4.9}$$

Finally, we simply sum (4.7) and (4.9) so that, using (4.4), we get

$$\frac{1}{2}\|\chi_{\lambda t}(t)\|^2 + \frac{1}{2}\iint_{Q_t} \chi_{\lambda xt}^2 + \frac{1}{4}\|\vartheta_{\lambda t}\|^2_{L^2(0,t;V^*)} + \frac{1}{2}\|\vartheta_\lambda(t)\|^2$$
$$\leq C + \int_0^t \left(\|\vartheta_\lambda\|^2 + \|\chi_{\lambda t}\|^2\right)\|\chi_{\lambda t}\|^2,$$

and the bound (4.4) and Gronwall's lemma entail that

$$\begin{aligned} \|\vartheta_\lambda\|_{L^\infty(0,T;H)} + \|\vartheta_{\lambda t}\|_{L^2(0,T;V^*)} &\leq C, \qquad (4.10)\\ \|\chi_{\lambda t}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} &\leq C. \qquad (4.11)\end{aligned}$$

4.5 Fifth estimate

By (3.6) we have

$$\vartheta_{\lambda t} - \vartheta_{\lambda xx} = \chi_{\lambda t}^2 - \chi_{\lambda t}\vartheta_\lambda$$

and (4.10)-(4.11) and the continuous embedding $V \subset L^\infty(\Omega)$ entail that both $\chi_{\lambda t}^2$ and $\chi_{\lambda t}\vartheta_\lambda$ are bounded in $L^2(Q)$ uniformly with respect to λ. Recalling (2.4), standard parabolic estimates yield

$$\|\vartheta_\lambda\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq C. \qquad (4.12)$$

4.6 Sixth estimate

Let us multiply relation (3.7) by $\partial_t(-\chi_{\lambda xx}+\eta_\lambda)$, and integrate on Q_t, for $t \in (0,T)$. Of course this procedure is formal since $-\chi_{\lambda xx}$ need not be differentiable with respect to time. However, let us stress once again that the latter argument can be justified at the approximation level of Section 3. We readily get that

$$\int_0^t \|\chi_{\lambda xt}\|^2 + \frac{1}{2}\|(-\chi_{\lambda xx}+\eta_\lambda)(t)\|^2 + \iint_{Q_t} \xi_\lambda \partial_t(-\chi_{\lambda xx}+\eta_\lambda)$$
$$= \frac{1}{2}\|-\chi_{xx}^0+\eta^0\|^2 + \iint_{Q_t} (\vartheta_\lambda - \vartheta_c)\partial_t(-\chi_{\lambda xx}+\eta_\lambda).$$

To control the term above involving ξ_λ we take into account the monotonicity of β_λ and [5, Lemma 2] and get that

$$\iint_{Q_t} \xi_\lambda \partial_t(-\chi_{\lambda xx}+\eta_\lambda) \geq 0.$$

Thus, assumptions (2.4)-(2.5), the estimate (4.12), and an integration by parts entail that

$$\frac{1}{2}\|(-\chi_{\lambda xx}+\eta_\lambda)(t)\|^2 \leq \frac{1}{2}\|-\chi_{xx}^0+\eta^0\|^2 + \int_\Omega (\vartheta_\lambda - \vartheta_c)(t)(-\chi_{\lambda xx}+\eta_\lambda)(t)$$
$$- \int_\Omega (\vartheta^0 - \vartheta_c)(-\chi_{xx}^0+\eta^0) - \iint_{Q_t} \vartheta_{\lambda t}(-\chi_{\lambda xx}+\eta_\lambda)$$
$$\leq \frac{1}{4}\|(-\chi_{\lambda xx}+\eta_\lambda)(t)\|^2 + C\left(1 + \int_0^t \|-\chi_{\lambda xx}+\eta_\lambda\|^2\right).$$

Finally, an application of the Gronwall lemma ensures the bound

$$\|-\chi_{\lambda xx}+\eta_\lambda\|_{L^\infty(0,T;H)} \leq C.$$

Hence, by the monotonicity on β_λ and standard elliptic estimates, we deduce that

$$\|\chi_\lambda\|_{L^\infty(0,T;W)} \leq C, \tag{4.13}$$

$$\|\eta_\lambda\|_{L^\infty(0,T;H)} \leq C. \tag{4.14}$$

We conclude by a comparison in (3.7) that

$$\|\xi_\lambda\|_{L^\infty(0,T;H)} \leq C. \tag{4.15}$$

5 Passage to the limit

This section will conclude the proof of the existence of a global strong solution to (2.10)-(2.13) by passing to the limit in (3.6)-(3.9) as $\lambda \longrightarrow +\infty$. First of all, let us observe that the two operator convergences

$$\alpha_\lambda \longrightarrow \alpha, \quad \beta_\lambda \longrightarrow \beta \tag{5.1}$$

hold in the sense of the G-convergence of graphs in $\mathbb{R} \times \mathbb{R}$. Namely, for all $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $y \in \alpha(x)$ ($\beta(x)$, respectively) there exists a sequence $(x_\lambda, y_\lambda) \in \mathbb{R} \times \mathbb{R}$ such that $y_\lambda \in \alpha_\lambda(x_\lambda)$ ($\beta_\lambda(x_\lambda)$, respectively) and (x_λ, y_λ) converges to (x, y) as $\lambda \longrightarrow +\infty$. The latter convergences will turn out to be crucial in the limit procedure.

Owing to (4.11)-(4.12), (4.13)-(4.15) and well-known compactness results, we may find a quadruple $(\vartheta, \chi, \eta, \xi)$ such that, possibly taking subsequences (not relabeled), one has that

$$\begin{aligned}
\vartheta_\lambda \longrightarrow \vartheta \qquad & \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \\
& \text{and strongly in } C([0,T];H) \cap L^2(0,T;V), && (5.2)\\
\chi_\lambda \longrightarrow \chi \qquad & \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \\
& \text{and strongly in } C([0,T];V), && (5.3)\\
\eta_\lambda \longrightarrow \eta \qquad & \text{weakly star in } L^\infty(0,T;H), && (5.4)\\
\xi_\lambda \longrightarrow \xi \qquad & \text{weakly star in } L^\infty(0,T;H). && (5.5)
\end{aligned}$$

Hence, we may pass to the limit in (3.7) obtaining (2.11) along with the properties (2.6)-(2.9).

We now focus on the interpretation of the limits η and ξ. First of all, the convergences (5.3) and (5.4) entail

$$\lim_{\lambda \to +\infty} \iint_Q \eta_\lambda \chi_\lambda = \iint_Q \eta \chi.$$

Thus, relation (2.13) is an easy consequence of the operator convergence (5.1) and classical results on monotone operators (see, e.g., [2, Prop. 1.1.iv, p. 42]).

Let us now multiply equation (3.7) by $\chi_{\lambda t}$ and integrate over Q obtaining

$$\begin{aligned}
\iint_Q \xi_\lambda \chi_{\lambda t} = & - \iint_Q \chi_{\lambda t}^2 - \iint_Q \chi_{\lambda x} \chi_{\lambda x t} \\
& - \iint_Q \eta_\lambda \chi_{\lambda t} + \iint_Q (\vartheta_\lambda - \vartheta_c) \chi_{\lambda t}.
\end{aligned} \tag{5.6}$$

Our next aim is that of passing to the limsup as $\lambda \longrightarrow +\infty$ in the above equation. To this end, we exploit (5.3) and let $\hat{\beta}_\lambda$ be convex functions such that $\partial \hat{\beta}_\lambda = \beta_\lambda$, $\min \hat{\beta}_\lambda = \hat{\beta}_\lambda(0) = 0$. It is well known the the functionals induced by $\hat{\beta}_\lambda$ on H, namely

$$B_\lambda(v) := \int_\Omega \hat{\beta}_\lambda(v(x))\, dx \qquad \text{for } v \in H, \tag{5.7}$$

turn out to converge in the sense of Mosco [1, Prop. 3.56, p. 354] in H to the functional

$$B(v) := \begin{cases} \displaystyle\int_\Omega \hat\beta(v(x))\,dx & \text{if } v \in H \text{ and } \hat\beta(v) \in L^1(\Omega) \\ +\infty & \text{if } v \in H \text{ and } \hat\beta(v) \notin L^1(\Omega). \end{cases} \tag{5.8}$$

In particular, owing to (5.3), one has that

$$\int_\Omega \hat\beta(\chi(T)) \le \liminf_{\lambda\to+\infty} \int_\Omega \hat\beta_\lambda(\chi_\lambda(T)).$$

Hence, since we have that $\hat\beta_\lambda(r) \le \hat\beta(r)$ for any $r \in \mathbb{R}$ (see, e.g., [4]), one readily deduces that

$$\begin{aligned} \limsup_{\lambda\to+\infty} \left(-\iint_Q \eta_\lambda \chi_{\lambda t} \right) &= -\liminf_{\lambda\to+\infty} \int_\Omega \left(\hat\beta_\lambda(\chi_\lambda(T)) - \hat\beta_\lambda(\chi^0) \right) \\ &\le \int_\Omega \left(\hat\beta(\chi(T)) - \hat\beta(\chi^0) \right) = \iint_Q \eta \chi_t. \end{aligned} \tag{5.9}$$

Then, thanks also to (5.2)-(5.3), equation (5.6) implies that

$$\begin{aligned} \limsup_{\lambda\to+\infty} \iint_Q \xi_\lambda \chi_{\lambda t} &\le -\iint_Q \chi_t^2 - \iint_Q \chi_x \chi_{xt} \\ &- \iint_Q \eta \chi_t + \iint_Q (\vartheta - \vartheta_c)\chi_t = \iint_Q \xi \chi_t. \end{aligned} \tag{5.10}$$

Owing to (5.10), the inclusion (2.12) is again a standard consequence of the convergences (5.1), (5.5), and classical results on monotone operators. Moreover, the limsup in (5.10) is actually a limit and the equality sign holds. Hence, we exploit once more (5.6), make use of (5.2)-(5.3) and (5.9)-(5.10), in order to deduce that

$$\limsup_{\lambda\to+\infty} \iint_Q \chi_{\lambda t}^2 \le \iint_Q \chi_t^2$$

which, together with (5.3), implies that

$$\chi_{\lambda t} \longrightarrow \chi_t \quad \text{strongly in } L^2(0,T;H).$$

The latter strong convergence suffices in order to pass to the limit in equation (3.6).

Finally, owing to (2.7), the claim (2.18) follows at once from (2.4) and (3.6) by the comparison principle, the proof being similar to that of (3.14).

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