One-dimensional thermo-visco-plastic processes with hysteresis and phase transitions

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Abstract

We consider a strongly coupled system of partial differential equations as a model for the dynamics of a thermo-visco-elasto-plastic solid under phase transitions. It consists of the momentum balance equation for the displacement, the energy balance equation for the absolute temperature, and an order parameter equation describing the dynamics of the phase transition. Both the phase transition and the strain-stress law involve hysteresis dependence represented by hysteresis operators. We show the thermodynamic consistency of the model, and prove its well-posedness.

Keywords: Phase-field systems, phase transitions, hysteresis operators, thermo-visco-plasticity

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1 Introduction

The paper is devoted to the problem of well-posedness of the system of equations

$$\begin{aligned} \rho\, u_{tt} + \gamma\, u_{xxxx} - \mu\, u_{xxt} - \sigma_x &= f\,, && (1.1)\\ (C_V\,\theta + F_1[u_x, w])_t - \kappa\,\theta_{xx} &= \mu\, u_{xt}^2 + \sigma\, u_{xt} + g(x,t,\theta)\,, && (1.2)\\ H_1[u_x, w] + \theta H_2[u_x, w] &= \sigma\,, && (1.3)\\ \nu\, w_t + H_3[u_x, w] + \theta H_4[u_x, w] &= 0\,, && (1.4) \end{aligned}$$

as a continuation of the analysis started in [6], where the case $\gamma = 0$ was considered under different boundary conditions. It constitutes a model for the one-dimensional thermomechanical motion of a thin thermo-visco-elasto-plastic wire in which a solid-solid phase transition takes place. In this connection, the unknowns u, θ, σ, w denote displacement, absolute (Kelvin) temperature, thermo-elasto-plastic stress component, and phase variable (usually called *generalized freezing index*, cf. [5]), respectively. The positive physical constants $\rho, \gamma, \mu, C_V, \kappa, \nu$ denote mass density, couple stress coefficient, viscosity, specific heat, heat conductivity, and a relaxation coefficient, in that order. The analysis in independent of the actual value of the constants. For the sake of notational convenience, we therefore set $\rho = \gamma = \mu = C_V = \kappa = \nu = 1$, and the length of the wire is normalized to π.

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The expressions H_i, $1 \le i \le 4$, and F_1, are mappings which account for the material memory. The assumptions stated below in Hypothesis 2.1 below are typically satisfied in the case of *hysteresis operators*, see [6].

Eqs. (1.1), (1.2), (1.4), represent the equation of motion, the balance of internal energy, and the phase evolution equation, in that order; Eq. (1.3) is the constitutive law relating temperature, strain and phase variable to the thermo-elasto-plastic stress component.

The motivation to study systems of the above type is twofold. On the one hand, it is well-known that for many materials the macroscopic strain-stress (ε-σ, where $\varepsilon = u_x$ is the linearized strain and u is the displacement) relations measured in uniaxial load-deformation experiments strongly depend on the absolute (Kelvin) temperature θ and, at the same time, exhibit a strong elastoplasticity witnessed by the occurrence of *hysteresis loops* that are *rate-independent*, i. e. independent of the speed with which there are traversed. Due to the hysteresis, which reflects the presence of a *rate-independent memory* in the material, the stress-strain relation can no longer be expressed in terms of a simple single-valued function. Among the materials showing very strong temperature-dependent hysteretic effects are the so-called *shape memory alloys* (see Chapter 5 in [1]); but even quite ordinary steels are well-known to exhibit this kind of behaviour, although to a smaller extent.

For a more detailed discussion about the model and a more complete list of related publications, we refer the reader to [6]. The situation here is simpler due to the presence of the fourth order term in Eq. (1.1), analogously as in the case without phase transitions and with temperature-dependent hysteresis in [4]. The solution is constructed by an easy two-step approximation method. First, a cut-off system is solved via Galerkin-type approximations and the compactness argument, and then additional estimates are used for removing the cut-off constraint.

The paper is organized as follows: In Section 2, we give a detailed statement of the mathematical problem and of Theorem 2.2 as the main result. We also show that the model is compatible with the Second Principle of Thermodynamics in the form of Clausius-Duhem inequality. In Section 3 we define the approximation scheme. Section 4 is devoted to estimates independent of the Galerkin approximations. Section 5 brings the proof of existence for the cut-off system and additional estimates, and in the concluding Section 6 we finish the proof of existence, uniqueness and continuous dependence for the original problem.

2 Statement of the problem

In the rectangle $Q =]0,\pi[\,\times\,]0,T[$, where $T > 0$ is a given final time, we consider

Problem (*P*). Find $u, \theta, w : Q \to \mathbb{R}$ satisfying the equations

$$
\begin{aligned}
u_{tt} + u_{xxxx} - u_{xxt} - \sigma_x &= f\,, && (2.1)\\
(\theta + F_1[u_x, w])_t - \theta_{xx} &= u_{xt}^2 + \sigma u_{xt} + g(x,t,\theta)\,, && (2.2)\\
H_1[u_x, w] + \theta H_2[u_x, w] &= \sigma\,, && (2.3)\\
w_t + H_3[u_x, w] + \theta H_4[u_x, w] &= 0\,, && (2.4)
\end{aligned}
$$

coupled with initial and boundary conditions

$$\begin{aligned} u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) &= 0\,, && (2.5)\\ \theta_x(0,t) = \theta_x(\pi,t) &= 0\,, && (2.6)\\ u(x,0) &= u^0(x)\,, && (2.7)\\ u_t(x,0) &= u^1(x)\,, && (2.8)\\ \theta(x,0) &= \theta^0(x)\,, && (2.9)\\ w(x,0) &= w^0(x)\,. && (2.10) \end{aligned}$$

Before giving the list of hypotheses, let us fix the notation. We consider the spaces

$C[0,T]$ of continuous functions $u:[0,T]\to\mathbb{R}$, endowed with the family of seminorms

$$|u|_{[0,t]} = \max\{|u(s)|\,;\, 0\le s\le t\} \qquad \text{for } \ t\in[0,T]\,,$$

$L^p(0,\pi)$ endowed with the norms $|\cdot|_p$ for $1\le p\le\infty$,

$W^{k,2}(0,\pi)$ of functions $u\in L^2(0,\pi)$ such that the j-th derivative $u^{(j)}$ belongs to $L^2(0,\pi)$ for $1\le j\le k$, $k\in\mathbb{N}$,

$L^p(Q)$ endowed with the norms $\|\cdot\|_p$ for $1\le p\le\infty$,

$L^\infty(0,T;L^2(0,\pi))$ endowed with the norm $\|\cdot\|_{\infty,2}$,

$C(\bar{Q})$ of continuous functions $\bar{Q}\to\mathbb{R}$ endowed with the norm $\|\cdot\|_\infty$.

In Eqs. (2.2) – (2.4), H_i and F_1, $i=1,2,3,4$, are given *causal operators* $C(\bar{Q})\times C(\bar{Q})\to C(\bar{Q})$. Recall that an operator $G: C([0,T];X)\to C([0,T];Y)$, where X,Y are Banach spaces, is said to be causal, if for arbitrary functions $v_1,v_2\in C([0,T];X)$, the implication

$$v_1(s)=v_2(s)\ \ \forall s\in[0,t] \quad\Rightarrow\quad G[v_1](t)=G[v_2](t) \tag{2.11}$$

holds for every $t\in[0,T]$. If moreover G is (Lipschitz) continuous, then we obtain in particular as a consequence of (2.11) that there exists a (Lipschitz) continuous function $\psi_G: X\to Y$ such that for every $v\in C([0,T];X)$ we have

$$G[v](0)=\psi_G(v(0))\,. \tag{2.12}$$

Another operator F_2 will be introduced in the sequel in order to make the model compatible with the Second Principle of Thermodynamics. Details will be given at the end of this section.

Hypothesis 2.1

(i) H_i, $i=1,2,3,4$, *and* F_j, $j=1,2$, *are causal operators* $C(\bar{Q})\times C(\bar{Q})\to C(\bar{Q})$ *generated by operators* $H_i^*, F_j^*: C[0,T]\times C[0,T]\to C[0,T]$ *according to the formula*

$$H_i[\varepsilon,w](x,t)=H_i^*[\varepsilon(x,\cdot),w(x,\cdot)](t)\,,\quad F_j[\varepsilon,w](x,t)=F_j^*[\varepsilon(x,\cdot),w(x,\cdot)](t)\,, \tag{2.13}$$

under the assumption that there exists a constant $K_0 > 0$ *such that for every* ε, ε_1, ε_2, w, w_1, $w_2 \in C[0,T]$ *and every* $t \in [0,T]$ *we have*

$$|H_i^*[\varepsilon_1, w_1](t) - H_i^*[\varepsilon_2, w_2](t)| \leq K_0 \left(|\varepsilon_1 - \varepsilon_2|_{[0,t]} + |w_1 - w_2|_{[0,t]}\right), \quad i = 1,2,3,4, \tag{2.14}$$

$$|F_1^*[\varepsilon_1, w_1](t) - F_1^*[\varepsilon_2, w_2](t)| \leq K_0(|\varepsilon_1|_{[0,t]} + |w_1|_{[0,t]} + |\varepsilon_2|_{[0,t]} + |w_2|_{[0,t]}) \times \left(|\varepsilon_1 - \varepsilon_2|_{[0,t]} + |w_1 - w_2|_{[0,t]}\right), \tag{2.15}$$

$$|H_i^*[\varepsilon, w]|_{[0,T]} \leq K_0 \quad \textit{for} \quad i = 2,3,4, \tag{2.16}$$

$$F_1^*[\varepsilon, w](t) \geq 0, \tag{2.17}$$

and if moreover ε, w *are absolutely continuous, then*

$$|(H_i^*[\varepsilon, w])_t(t)| \leq K_0(|\varepsilon_t(t)| + |w_t(t)|) \quad \textit{a. e.} \quad \textit{for} \quad i = 1,2,3,4, \tag{2.18}$$

$$|(F_1^*[\varepsilon, w])_t(t)| \leq K_0(|\varepsilon|_{[0,t]} + |w|_{[0,t]})(|\varepsilon_t(t)| + |w_t(t)|) \quad \textit{a. e.}, \tag{2.19}$$

$$(F_1^*[\varepsilon, w])_t(t) \leq H_1^*[\varepsilon, w]\, \varepsilon_t + H_3^*[\varepsilon, w]\, w_t \quad \textit{a. e.}, \tag{2.20}$$

$$(F_2^*[\varepsilon, w])_t(t) \leq H_2^*[\varepsilon, w]\, \varepsilon_t + H_4^*[\varepsilon, w]\, w_t \quad \textit{a. e.}, \tag{2.21}$$

(ii) $f, f_t \in L^2(Q)$,

(iii) *The function* $g(\cdot,\cdot,\theta)$ *is measurable in* Q *for every* $\theta \in \mathbb{R}$, *and there exist a function* $g^0 \in L^2(Q)$ *and a constant* $K_g > 0$ *such that* $g(x,t,0) = g^0(x,t) \geq 0$ *a. e., and*

$$|g(x,t,\theta_1) - g(x,t,\theta_2)| \leq K_g\, |\theta_1 - \theta_2| \quad \textit{a. e.} \quad \forall \theta_1, \theta_2 \in \mathbb{R}, \tag{2.22}$$

(iv) $w^0 \in W^{1,2}(0,\pi)$,

(v) $\theta^0 \in W^{1,2}(0,\pi)$, $\theta^0(x) \geq \theta^* > 0$ *for all* $x \in [0,\pi]$,

(vi) $u^0 \in W^{4,2}(0,\pi)$, $u^1 \in W^{2,2}(0,\pi)$, $u^0(0) = u^0(\pi) = u^0_{xx}(0) = u^0_{xx}(\pi) = 0$, $u^1(0) = u^1(\pi) = 0$.

According to the discussion preceding Eq. (2.12), it follows from the above hypotheses that there exist functions $\psi_i, \Psi_1 : \mathbb{R}^2 \to \mathbb{R}$, $i = 1,2,3,4$, such that

$$H_i^*[\varepsilon, w](0) = \psi_i(\varepsilon(0), w(0)), \quad i = 1,2,3,4, \tag{2.23}$$

$$F_1^*[\varepsilon, w](0) = \Psi_1(\varepsilon(0), w(0)), \tag{2.24}$$

for all $\varepsilon, w \in C[0,T]$, and we have

$$\max\left\{\left|\frac{\partial \psi_i}{\partial \varepsilon}(\varepsilon, w)\right|, \left|\frac{\partial \psi_i}{\partial w}(\varepsilon, w)\right|\right\} \leq K_0, \tag{2.25}$$

$$\max\left\{\left|\frac{\partial \Psi_1}{\partial \varepsilon}(\varepsilon, w)\right|, \left|\frac{\partial \Psi_1}{\partial w}(\varepsilon, w)\right|\right\} \leq 2\, K_0(|\varepsilon| + |w|) \tag{2.26}$$

for a. e. $\varepsilon, w \in \mathbb{R}$.

We are now ready to state our main result. The rest of the paper is devoted to its proof.

Theorem 2.2 *Let Hypothesis 2.1 hold. Then Problem* $(\mathbf{P})$ *admits a unique solution such that* $u, \theta, w, \sigma, u_t, u_x, u_{xt}, u_{xx}, w_t \in C(\bar{Q})$, $u_{tt}, u_{xxt}, u_{xxxx}, \sigma_x \in L^\infty(0,T; L^2(0,\pi))$, $\theta_t, \theta_{xx} \in L^2(Q)$, *Eqs. (2.1), (2.2), (2.6) hold almost everywhere, and Eqs. (2.3) – (2.5), (2.7) – (2.10) hold for all* $(x,t) \in Q$. *Moreover, there exists a constant* $C^* > 0$ *such that* $\theta(x,t) \geq \theta^* e^{-C^* t}$ *for every* $(x,t) \in \bar{Q}$.

Let us note that (2.20), (2.21) are the *hysteresis energy inequalities* corresponding to *clockwise admissible hysteresis potentials*, cf. [1, 3]. They will turn out to be substantial for the positivity of temperature which in turn yields the crucial energy estimates. Moreover, they ensure that the model itself is thermodynamically consistent.

Indeed, we define the *free energy* Ψ in the form

$$\Psi \; = \; \theta(1 - \log\theta) + F_1[u_x, w] + \theta\, F_2[u_x, w] + \frac{1}{2} u_{xx}^2 \,. \tag{2.27}$$

The *entropy* S is given by the relation

$$S \; = \; -\frac{\partial \Psi}{\partial \theta} \; = \; \log\theta - F_2[u_x, w]\,, \tag{2.28}$$

and the *internal energy* has the form

$$U \; = \; \Psi + \theta\, S \; = \; \theta + F_1[u_x, w] + \frac{1}{2} u_{xx}^2 \,. \tag{2.29}$$

The energy conservation law

$$U_t + q_x = u_{xt}\,(u_{xt} + \sigma) + u_{xx} u_{xxt} + g\,, \tag{2.30}$$

where q is the heat flux which we consider in the Fourier form $q = -\theta_x$, and g is the heat source density, is nothing but Eq. (2.2). As a criterion of thermodynamical consistency, we require that

$$\theta(x,t) > 0 \tag{2.31}$$

for every $(x,t) \in Q$, and that the Clausius-Duhem inequality

$$S_t + \left(\frac{q}{\theta}\right)_x \; \geq \; \frac{g}{\theta} \tag{2.32}$$

holds almost everywhere in Q for every solution of our system. Condition (2.31) is ensured by Theorem 2.2. To check that (2.32) holds, we first notice that it is equivalent to

$$U_t \; \leq \; \theta\, S_t + u_{xt}\,(u_{xt} + \sigma) + u_{xx} u_{xxt} - \frac{q\, \theta_x}{\theta} \tag{2.33}$$

as a consequence of (2.30). We have by (2.20), (2.21) that

$$U_t - \theta\, S_t \; = \; F_1[u_x, w]_t + \theta\, F_2[u_x, w]_t + u_{xx} u_{xxt} \; \leq \; \sigma\, u_{xt} - w_t^2 + u_{xx} u_{xxt}\,, \tag{2.34}$$

hence (2.33) is satisfied due to the choice of q.

3 Approximation

The solution to Problem $(\boldsymbol{P})$ will be approximated in two steps. We first fix a cut-off parameter $r > 0$ and consider the following truncated problem.

Problem $(\boldsymbol{P}(r))$. Find $u, \theta, w : Q \to \mathbb{R}$ satisfying the equations

$$
\begin{aligned}
u_{tt} + u_{xxxx} - u_{xxt} - \sigma_x &= f\,, && (3.1)\\
(\theta + F_1[u_x, w])_t - \theta_{xx} &= u_{xt}^2 + \sigma u_{xt} + g(x, t, \varrho_r(\theta))\,, && (3.2)\\
H_1[u_x, w] + \varrho_r(\theta)\, H_2[u_x, w] &= \sigma\,, && (3.3)\\
w_t + H_3[u_x, w] + \varrho_r(\theta)\, H_4[u_x, w] &= 0\,, && (3.4)
\end{aligned}
$$

coupled with initial and boundary conditions (2.5) – (2.10), where $\varrho_r(\theta)$ is given by

$$\varrho_r(\theta) = \max\{0, \min\{r, \theta\}\}\,. \tag{3.5}$$

We approximate Problem $(\boldsymbol{P}(r))$ by a finite-dimensional Galerkin system with basis functions

$$\mathbf{a}_k(x) = \sqrt{\frac{2}{\pi}} \sin kx \qquad \text{for } \ k \in \mathbb{N}\,, \tag{3.6}$$

$$\mathbf{b}_k(x) = \begin{cases} \sqrt{\frac{2}{\pi}} \cos kx & \text{for } \ k \in \mathbb{N}\,, \\ \sqrt{\frac{1}{\pi}} & \text{for } \ k = 0\,. \end{cases} \tag{3.7}$$

For a fixed $n \in \mathbb{N}$, the problem reads as follows.

Problem $(\boldsymbol{P}^{(n)}(r))$. Find $w^{(n)} : \bar{Q} \to \mathbb{R}$, $u_1, \dots, u_n, \theta_0, \dots, \theta_n : [0, T] \to \mathbb{R}$, satisfying for $k = (0), 1, 2, \dots, n$ the system

$$
\begin{aligned}
\ddot{u}_k + k^2 \dot{u}_k + k^4 u_k + \int_0^\pi \sigma^{(n)}\, k \mathbf{b}_k\, dx &= \int_0^\pi f\, \mathbf{a}_k\, dx\,, && (3.8)\\
\dot{\theta}_k + \int_0^\pi \left(F_1[u_x^{(n)}, w^{(n)}]\right)_t \mathbf{b}_k\, dx + k^2 \theta_k &= \int_0^\pi \left((u_{xt}^{(n)})^2 + \sigma^{(n)} u_{xt}^{(n)} + g^{(n)}\right) \mathbf{b}_k\, dx\,, && (3.9)\\
H_1[u_x^{(n)}, w^{(n)}] + \varrho_r(\theta^{(n)}) H_2[u_x^{(n)}, w^{(n)}] &= \sigma^{(n)}\,, && (3.10)\\
w_t^{(n)} + H_3[u_x^{(n)}, w^{(n)}] + \varrho_r(\theta^{(n)})\, H_4[u_x^{(n)}, w^{(n)}] &= 0\,, && (3.11)
\end{aligned}
$$

where

$$
\begin{aligned}
g^{(n)}(x, t) &= g(x, t, \varrho_r(\theta^{(n)}(x, t)))\,, && (3.12)\\
u^{(n)}(x, t) &= \sum_{k=1}^n u_k(t)\, \mathbf{a}_k(x)\,, && (3.13)\\
\theta^{(n)}(x, t) &= \sum_{k=0}^n \theta_k(t)\, \mathbf{b}_k(x)\,, && (3.14)
\end{aligned}
$$

coupled with initial conditions

$$u_k(0) = \int_0^\pi u^0(x)\, \mathbf{a}_k(x)\, dx\,, \tag{3.15}$$

$$\dot u_k(0) = \int_0^\pi u^1(x)\, \mathbf{a}_k(x)\, dx\,, \tag{3.16}$$

$$\theta_k(0) = \int_0^\pi \theta^0(x)\, \mathbf{b}_k(x)\, dx\,, \tag{3.17}$$

$$w^{(n)}(x,0) = w^0(x)\,. \tag{3.18}$$

We consider (3.8) – (3.18) as a first order system for the unknowns $U = u_1, \dots, u_n$, $V = \dot u_1, \dots, \dot u_n$, $Z = z_0, \dots, z_n$, $w^{(n)}$, where $z_k = \theta_k + \int_0^\pi F_1[u_x^{(n)}, w^{(n)}]\, \mathbf{b}_k\, dx$ for $k = 0, \dots, n$, and with a right-hand given by causal locally Lipschitz continuous operators. In a neighbourhood of the initial condition we construct a local solution $(U, V, Z, w^{(n)}) \in C([0,\tau]; \mathbb{R}^n) \times C([0,\tau]; \mathbb{R}^n) \times C([0,\tau]; \mathbb{R}^{n+1}) \times C([0,\tau] \times [0,\pi])$ for $\tau > 0$ sufficiently small in a standard way by the Banach Contraction Principle. In the next section, we derive estimates which, on the one hand, imply that this local solution can be extended to a global one and, on the other hand, will enable us to pass to the limit as $n \to \infty$ and $r \to \infty$.

4 Estimates

Throughout the section, we denote by $C_1(r), C_2(r), \dots$ any positive constant independent of n, and possibly dependent on r, and by $K_1, K_2, \dots$ any positive constant depending only on the quantity

$$R = \|f\|_2 + \|(f)_t\|_2 + \|g^0\|_2 + |w^0|_2 + |(w^0)_x|_2 + |\theta^0|_2 + |(\theta^0)_x|_2 + |(u^0)_{xxxx}|_2 + |(u^1)_{xx}|_2\,, \tag{4.1}$$

and, in particular, independent of both n and r. The dependence on T is not taken into account here, as T is assumed to be fixed.

We first recall an easy embedding and interpolation result.

Lemma 4.1 *Let V be the space of functions $v \in L^2(Q)$ such that $v_x \in L^\infty(0,T; L^2(0,\pi))$, $v_t \in L^2(Q)$, endowed with the norm $\|v\|_2 + \|v_t\|_2 + \|v_x\|_{\infty,2}$. Then there exists a constant K_1 such that for every $v \in V$ we have*

$$\|v\|_\infty^2 \le K_1 \left(\|v\|_2^2 + \|v\|_2 (\|v_t\|_2 + \|v_x\|_{\infty,2}) + \|v\|_2^{1/2} \|v_t\|_2^{1/2} \|v_x\|_{\infty,2} \right), \tag{4.2}$$

and the inequality

$$|v(x,t) - v(y,s)| \le K_1 \left(\|v_t\|_2 + \|v_x\|_{\infty,2} \right) \left(|t-s|^{1/4} + |x-y|^{1/2} \right) \tag{4.3}$$

holds for every $t, s \in [0,T]$, $x, y \in [0,\pi]$. In particular, V is compactly embedded into $C(\bar Q)$.

Proof. For every $s, t \in [0,T]$ we have

$$\int_0^\pi v^2(x,t)\, dx \le \int_0^\pi v^2(x,s)\, dx + 2 \int_0^T \int_0^\pi |v||v_t|\, dx\, dt\,.$$

Integrating the above inequality with respect to s and passing to the maximum with respect to t we obtain

$$\|v\|^2_{\infty,2} \;\le\; \frac{1}{T}\,\|v\|^2_2 + 2\,\|v\|_2\|v_t\|_2\,. \tag{4.4}$$

Similarly, for $x, y \in [0,\pi]$ and $t \in [0,T]$ we have

$$v^2(x,t) \;\le\; v^2(y,t) + 2\int_0^\pi |v(\xi,t)\, v_x(\xi,t)|\, d\xi\,, \tag{4.5}$$

and integrating with respect to y we obtain

$$\|v\|^2_\infty \;\le\; \frac{1}{\pi}\|v\|^2_{\infty,2} + 2\|v\|_{\infty,2}\|v_x\|_{\infty,2}\,, \tag{4.6}$$

and inequality (4.2) follows easily from (4.4), (4.6). The Hölder-type estimate (4.3) is a special case of Theorem V.2.4 of [3], and the compact embedding follows from the Arzelà-Ascoli Theorem. ■

4.1 Estimate I

Consider a maximal solution of (3.8) – (3.18) defined in a time interval $]0,T_n[$. Testing Eq. (3.8) by $\dot u_k$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\left|u_t^{(n)}(t)\right|_2^2 + \left|u_{xx}^{(n)}(t)\right|_2^2\right) \;+\; \left|u_{xt}^{(n)}(t)\right|_2^2 \;&\le\; \left|\sigma^{(n)}(t)\right|_2\left|u_{xt}^{(n)}(t)\right|_2 + |f(t)|_2\left|u_t^{(n)}(t)\right|_2 \\ &\le\; \frac{1}{2}\left(\left|\sigma^{(n)}(t)\right|_2^2 + \left|u_{xt}^{(n)}(t)\right|_2^2 + |f(t)|_2^2 + \left|u_t^{(n)}(t)\right|_2^2\right)\end{aligned} \tag{4.7}$$

for every $t \in]0,T_n[$. We have by Hypothesis 2.1 that

$$|w_t^{(n)}(x,t)| \;\le\; K_0(1+\varrho_r(\theta^{(n)}(x,t)))\,, \tag{4.8}$$

hence

$$|w^{(n)}(x,t)| \;\le\; |w^0|_\infty + K_0\left(t + \int_0^t \varrho_r(\theta^{(n)}(x,s))\, ds\right)\,. \tag{4.9}$$

This enables us to estimate

$$\begin{aligned}\left|\sigma^{(n)}(x,t)\right| \;&\le\; |H_1[0,0](x,t)| + \left|H_1[u_x^{(n)},w^{(n)}](x,t) - H_1[0,0](x,t)\right| \\ &\quad + \varrho_r(\theta^{(n)}(x,t))\left|H_2[u_x^{(n)},w^{(n)}](x,t)\right| \\ &\le\; K_2\left(1+\varrho_r(\theta^{(n)}(x,t)) + \left|u_x^{(n)}(x,\cdot)\right|_{[0,t]} + \left|w^{(n)}(x,\cdot)\right|_{[0,t]}\right) \\ &\le\; K_3\left(1+\varrho_r(\theta^{(n)}(x,t)) + \int_0^t\left(\left|u_{xt}^{(n)}(x,s)\right| + \varrho_r(\theta^{(n)}(x,s))\right) ds\right)\end{aligned} \tag{4.10}$$

for all $(x,t) \in Q$, where we also used the fact that

$$|u_x^{(n)}(x,0)| \le \sqrt{\pi}|u_{xx}^0|_2\,. \tag{4.11}$$

Hence,

$$\left|\sigma^{(n)}(t)\right|_2^2 \;\le\; K_4 \left(1 + \left|\varrho_r(\theta^{(n)}(t))\right|_2^2 + \int_0^t \left|\varrho_r(\theta^{(n)})(s)\right|_2^2 ds + \int_0^t \left|u_{xt}^{(n)}(s)\right|_2^2 ds\right) . \tag{4.12}$$

Using the inequalities $|u_t^{(n)}(0)|_2 \le |u^1|_2$, $|u_{xx}^{(n)}(0)|_2 \le |u_{xx}^0|_2$, and applying the Gronwall argument to (4.7) we finally obtain for every $t \in]0, T_n[$ the estimate

$$\left|u_t^{(n)}(t)\right|_2^2 + \left|u_{xx}^{(n)}(t)\right|_2^2 + \int_0^t \left|u_{xt}^{(n)}(s)\right|_2^2 ds \;\le\; K_5 \left(1 + \int_0^t \left|\varrho_r(\theta^{(n)})(s)\right|_2^2 ds\right) . \tag{4.13}$$

4.2 Estimate II

We now test Eq. (3.9) by $\dot\theta_k$. From Hypothesis 2.1 it follows that

$$\begin{aligned} \left|\theta_t^{(n)}(t)\right|_2^2 \;+\; \frac{1}{2}\frac{d}{dt}\left|\theta_x^{(n)}(t)\right|_2^2 \;\le\; & \left|\theta_t^{(n)}(t)\right|_2 \Bigg(\left|(F_1[u_x^{(n)}, w^{(n)}])_t(t)\right|_2 \\ & + \left|u_{xt}^{(n)}(t)\right|_4^2 + \left|\left(\sigma^{(n)}\, u_{xt}^{(n)}\right)(t)\right|_2 + \left|g^0(t)\right|_2 + K_g \left|\varrho_r(\theta^{(n)})(t)\right|_2 \Bigg), \end{aligned} \tag{4.14}$$

where we have by (2.19) that

$$|(F_1[u_x^{(n)}, w^{(n)}])_t(x,t)| \;\le\; K_0 \left(|u_x^{(n)}(x,\cdot)|_{[0,t]}^2 + |w^{(n)}(x,\cdot)|_{[0,t]}^2 + |u_{xt}^{(n)}(x,t)|^2 + |w_t^{(n)}(x,t)|^2\right) . \tag{4.15}$$

The estimates (4.8) – (4.10), (4.12) – (4.15) yield an upper bound for the maximal solution of Problem $(\boldsymbol{P}^{(n)}(r))$ which may possibly depend on n and r, but does not depend on T_n. The solution is therefore defined in the whole interval $[0,T]$, that is, $T_n = T$. Inequality (4.13) can then be written in the form

$$\left\|u_t^{(n)}\right\|_{\infty,2}^2 + \left\|u_{xx}^{(n)}\right\|_{\infty,2}^2 + \left\|u_{xt}^{(n)}\right\|_2^2 \;\le\; K_5 \left(1 + \left\|\varrho_r(\theta^{(n)})\right\|_2^2\right) . \tag{4.16}$$

Using (4.15), (4.8), (4.9) we obtain

$$\left\|(F_1[u_x^{(n)}, w^{(n)}])_t\right\|_2^2 \;\le\; K_6 \left(1 + \left\|u_{xt}^{(n)}\right\|_4^4 + \left\|\varrho_r(\theta^{(n)})\right\|_4^4\right) , \tag{4.17}$$

and combining (4.14) with (4.12), (4.17), and (4.16), we end up with the inequality

$$\begin{aligned} \left\|\theta_t^{(n)}\right\|_2^2 + \left\|\theta_x^{(n)}\right\|_{\infty,2}^2 \;\le\; & \left|\theta_x^{(n)}(0)\right|_\infty^2 + \left\|(F_1[u_x^{(n)}, w^{(n)}])_t\right\|_2^2 \\ & + \left\|u_{xt}^{(n)}\right\|_\infty^2 \left(\left\|u_{xt}^{(n)}\right\|_2^2 + \left\|\sigma^{(n)}\right\|_2^2\right) + \left\|g^0\right\|_2^2 + K_g^2 \left\|\varrho_r(\theta^{(n)})\right\|_2^2 \\ \le\; & K_7 \left(1 + \left\|u_{xt}^{(n)}\right\|_\infty^2 + \left\|\varrho_r(\theta^{(n)})\right\|_\infty^2\right)\left(1 + \left\|\varrho_r(\theta^{(n)})\right\|_2^2\right) . \end{aligned} \tag{4.18}$$

4.3 Estimate III

Differentiating Eq. (3.8) and testing by $\ddot{u}_k$ yields

$$\frac{1}{2}\frac{d}{dt}\left(\left|u_{tt}^{(n)}(t)\right|_2^2 + \left|u_{xxt}^{(n)}(t)\right|_2^2\right) + \left|u_{xtt}^{(n)}(t)\right|_2^2 \leq \left|\sigma_t^{(n)}(t)\right|_2 \left|u_{xtt}^{(n)}(t)\right|_2 + |f_t(t)|_2 \left|u_{tt}^{(n)}(t)\right|_2 \quad (4.19)$$
$$\leq \frac{1}{2}\left(\left|\sigma_t^{(n)}(t)\right|_2^2 + \left|u_{xtt}^{(n)}(t)\right|_2^2 + 2T\,|f_t(t)|_2^2 + \frac{1}{2T}\left|u_{tt}^{(n)}(t)\right|_2^2\right),$$

hence

$$\frac{1}{2}\left\|u_{tt}^{(n)}\right\|_{\infty,2}^2 + \left\|u_{xxt}^{(n)}\right\|_{\infty,2}^2 + \left\|u_{xtt}^{(n)}\right\|_2^2 \leq \left\|\sigma_t^{(n)}\right\|_2^2 + 2T\,\|f_t\|_2^2 + \left|u_{tt}^{(n)}(0)\right|_2^2 + \left|u_{xxt}^{(n)}(0)\right|_2^2, \quad (4.20)$$

where

$$\left|u_{xxt}^{(n)}(0)\right|_2^2 \leq |u_{xx}^1|_2^2, \quad (4.21)$$

$$\left|u_{tt}^{(n)}(0)\right|_2 \leq |f(0)|_2 + |u_{xxxx}^0|_2 + |u_{xx}^1|_2 + |\sigma_x^{(n)}(0)|_2. \quad (4.22)$$

By (2.23), (2.25) we have

$$|\sigma_x^{(n)}(x,0)| \leq K_0(1 + \varrho_r(\theta^{(n)}(x,0))(|u_{xx}^{(n)}(x,0))| + |w_x^0(x)|) \quad (4.23)$$
$$+ |\theta_x^{(n)}(x,0)|(|\psi_2(0,0)| + K_0(|u_x^{(n)}(x,0))| + |w^0(x)|).$$

Similarly as in (4.5) with $\theta^{(n)}(x,0)$ instead of $v(x,t)$ we have

$$|\theta^{(n)}(0)|_\infty^2 \leq \frac{1}{\pi}|\theta^0|_2^2 + 2|\theta^0|_2|\theta_x^0|_2, \quad (4.24)$$

and using (4.11) we obtain

$$|\sigma_x^{(n)}(0)|_2 \leq K_0(1 + |\theta^{(n)}(0)|_\infty)(|u_{xx}^0|_2 + |w_x^0|_2) + K_8\,|\theta_x^0|_2\,(1 + |u_{xx}^0|_2 + +|w^0|_\infty), \quad (4.25)$$

hence

$$\left|u_{tt}^{(n)}(0)\right|_2 \leq K_9. \quad (4.26)$$

It remains to estimate the term $\left\|\sigma_t^{(n)}\right\|_2^2$ in (4.20). We have

$$\left|\sigma_t^{(n)}(x,t)\right| \leq K_0\left((1 + \varrho_r(\theta^{(n)}(x,t)))(|u_{xt}^{(n)}(x,t)| + |w_t^{(n)}(x,t)|) + |\theta_t^{(n)}(x,t)|\right), \quad (4.27)$$

and from (4.18) it follows that

$$\left\|\sigma_t^{(n)}\right\|_2^2 \leq K_{10}\left(1 + \left\|\varrho_r(\theta^{(n)})\right\|_2^2\right)\left(1 + \left\|u_{xt}^{(n)}\right\|_\infty^2 + \left\|\varrho_r(\theta^{(n)})\right\|_\infty^2\right), \quad (4.28)$$

hence also

$$\left\|u_{tt}^{(n)}\right\|_{\infty,2}^2 + \left\|u_{xxt}^{(n)}\right\|_{\infty,2}^2 + \left\|u_{xtt}^{(n)}\right\|_2^2 \leq K_{11}\left(1 + \left\|\varrho_r(\theta^{(n)})\right\|_2^2\right)\left(1 + \left\|u_{xt}^{(n)}\right\|_\infty^2 + \left\|\varrho_r(\theta^{(n)})\right\|_\infty^2\right). \quad (4.29)$$

Using Lemma 4.1 we can find for every $\delta > 0$ some $K_\delta > 0$ such that

$$\left\|u_{xt}^{(n)}\right\|_\infty^2 \ \leq \ K_\delta \left\|u_{xt}^{(n)}\right\|_2^2 + \delta\left(\left\|u_{xxt}^{(n)}\right\|_{\infty,2}^2 + \left\|u_{xtt}^{(n)}\right\|_2^2\right). \tag{4.30}$$

Since $\left\|\varrho_r(\theta^{(n)})\right\|_2^2 \leq \pi \left\|\varrho_r(\theta^{(n)})\right\|_\infty^2 \leq \pi r^2$, we obtain for $\delta = \delta(r)$ sufficiently small that

$$\left\|u_{tt}^{(n)}\right\|_{\infty,2}^2 + \left\|u_{xxt}^{(n)}\right\|_{\infty,2}^2 + \left\|u_{xtt}^{(n)}\right\|_2^2 \ \leq \ C_1(r) \tag{4.31}$$

as a consequence of (4.29) and (4.16), hence

$$\left\|\theta_t^{(n)}\right\|_2 + \left\|\theta_x^{(n)}\right\|_{\infty,2} \ \leq \ C_2(r)\,, \tag{4.32}$$

$$\left\|u_{xt}^{(n)}\right\|_\infty \ \leq \ C_3(r)\,, \tag{4.33}$$

$$\left\|\sigma_t^{(n)}\right\|_2 \ \leq \ C_4(r) \tag{4.34}$$

as a consequence of (4.18), (4.31), and (4.28). Furthermore, differentiating Eq. (3.11) with respect to t, we obtain from Hypothesis 2.1 and from (4.8), (4.32), (4.33) that

$$\left\|w_{tt}^{(n)}\right\|_2 \ \leq \ C_5(r)\,. \tag{4.35}$$

4.4 Estimate IV

By Hypothesis 2.1 we have for $x, y \in [0,\pi]$ and $t \in [0,T]$ the inequalities

$$\begin{aligned}\left|w_t^{(n)}(x,t) - w_t^{(n)}(y,t)\right| \ &\leq \ K_0\Bigg(\left|\theta^{(n)}(x,t) - \theta^{(n)}(y,t)\right| \\ &+ \left(1 + \varrho_r(\theta^{(n)}(x,t))\right)\left(\left|u_x^{(n)}(x,\cdot) - u_x^{(n)}(y,\cdot)\right|_{[0,t]} + \left|w^{(n)}(x,\cdot) - w^{(n)}(y,\cdot)\right|_{[0,t]}\right)\Bigg) \\ &\leq \ K_0(1+r)\Bigg(\left|\theta^{(n)}(x,t) - \theta^{(n)}(y,t)\right| + \left|u_x^{(n)}(x,0) - u_x^{(n)}(y,0)\right| + \left|w^0(x) - w^0(y)\right| \\ &+ \int_0^t \left(\left|u_{xt}^{(n)}(x,s) - u_{xt}^{(n)}(y,s)\right| + \left|w_t^{(n)}(x,s) - w_t^{(n)}(y,s)\right|\right) ds\Bigg)\end{aligned} \tag{4.36}$$

Using Gronwall's lemma and letting y tend to x, we see that the following two inequalities hold a. e. in Q:

$$\left|w_{xt}^{(n)}(x,t)\right| \ \leq \ C_6(r)\left(\left|\theta_x^{(n)}(x,t)\right| + \left|u_{xx}^{(n)}(x,0)\right| + |w_x^0(x)| + \int_0^t \left(\left|u_{xxt}^{(n)}(x,s)\right| + \left|\theta_x^{(n)}(x,s)\right|\right) ds\right), \tag{4.37}$$

$$\left|w_x^{(n)}(x,t)\right| \ \leq \ C_7(r)\left(\left|u_{xx}^{(n)}(x,0)\right| + |w_x^0(x)| + \int_0^t \left(\left|u_{xxt}^{(n)}(x,s)\right| + \left|\theta_x^{(n)}(x,s)\right|\right) ds\right). \tag{4.38}$$

The argument of (4.36) – (4.38) yields analogously that

$$\left|\sigma_x^{(n)}(x,t)\right| \ \leq \ C_8(r)\left(\left|u_{xx}^{(n)}(x,0)\right| + |w_x^0(x)| + \int_0^t \left(\left|u_{xxt}^{(n)}(x,s)\right| + \left|\theta_x^{(n)}(x,s)\right|\right) ds\right). \tag{4.39}$$

We have in particular the following estimates.

$$\left\|w_t^{(n)}\right\|_\infty + \left\|w^{(n)}\right\|_\infty + \left\|w_{xt}^{(n)}\right\|_{\infty,2} + \left\|w_x^{(n)}\right\|_{\infty,2} \leq C_9(r)\,, \tag{4.40}$$

$$\left\|\sigma_x^{(n)}\right\|_{\infty,2} \leq C_{10}(r)\,. \tag{4.41}$$

Directly from Eq. (3.8) and from estimates (4.41), (4.31), (4.17), (4.34), we obtain that

$$\left\|u_{xxxx}^{(n)}\right\|_{\infty,2} \leq C_{11}(r)\,, \tag{4.42}$$

$$\left\|\theta_{xx}^{(n)}\right\|_2 \leq C_{12}(r)\,. \tag{4.43}$$

5 Solution of Problem (P(r))

The estimated established in the previous section and the compact embedding in Lemma 4.1 enable us to pass to the limit as $n \to \infty$, keeping r fixed for the moment. Selecting a subsequence, if necessary, we find functions $u, \theta, w, \sigma \in C(\bar{Q})$ such that

$$u_t, u_x, u_{xt}, u_{xx}, w_t \in C(\bar{Q})\,, \tag{5.1}$$

$$u_{tt}, u_{xxt}, u_{xxxx}, w_{xt}, w_x, \theta_x, \sigma_x \in L^\infty(0,T; L^2(0,\pi))\,, \tag{5.2}$$

$$u_{xtt}, \theta_t, \theta_{xx}, \sigma_t, w_{tt} \in L^2(Q)\,, \tag{5.3}$$

and

- $u^{(n)} \to u$, $u_t^{(n)} \to u_t$, $u_x^{(n)} \to u_x$, $u_{xt}^{(n)} \to u_{xt}$, $u_{xx}^{(n)} \to u_{xx}$, $\theta^{(n)} \to \theta$, $w^{(n)} \to w$, $w_t^{(n)} \to w_t$, $\sigma^{(n)} \to \sigma$, all uniformly in $C(\bar{Q})$,
- $u_{tt}^{(n)} \to u_{tt}$, $u_{xxt}^{(n)} \to u_{xxt}$, $u_{xxxx}^{(n)} \to u_{xxxx}$, $w_{xt}^{(n)} \to w_{xt}$, $w_x^{(n)} \to w_x$, $\theta_x^{(n)} \to \theta_x$, $\sigma_x^{(n)} \to \sigma_x$, all weakly* in $L^\infty(0,T; L^2(0,\pi))$,
- $u_{xtt}^{(n)} \to u_{xtt}$, $\theta_t^{(n)} \to \theta_t$, $\theta_{xx}^{(n)} \to \theta_{xx}$, $\sigma_t^{(n)} \to \sigma_t$, $w_{tt}^{(n)} \to w_{tt}$, all weakly in $L^2(Q)$.

The above convergences imply immediately that u, θ, w, σ satisfy Problem $(\boldsymbol{P}(r))$ in the sense of Theorem 2.2. Moreover, passing to the limit in (4.16), (4.18), (4.29) we obtain the estimates

$$\|u_t\|_{\infty,2}^2 + \|u_{xx}\|_{\infty,2}^2 + \|u_{xt}\|_2^2 \leq K_5\left(1 + \|\theta\|_2^2\right)\,, \tag{5.4}$$

$$\|\theta_t\|_2^2 + \|\theta_x\|_{\infty,2}^2 \leq K_7\left(1 + \|\theta\|_2^2\right)\left(1 + \|u_{xt}\|_\infty^2 + \|\theta\|_\infty^2\right)\,, \tag{5.5}$$

$$\|u_{tt}\|_{\infty,2}^2 + \|u_{xxt}\|_{\infty,2}^2 + \|u_{xtt}\|_2^2 \leq K_{11}\left(1 + \|\theta\|_2^2\right)\left(1 + \|u_{xt}\|_\infty^2 + \|\theta\|_\infty^2\right)\,. \tag{5.6}$$

Note that the above estimates are *independent of* r. Additional estimates will be needed for removing the cut-off. We first show that $\theta(x,t)$ remains bounded away from zero in $\bar{Q}$.

5.1 Positivity of temperature

We rewrite Eqs. (3.2), (3.3) in the form

$$\theta_t - \theta_{xx} = -F_1[u_x, w]_t + u_{xt}^2 + (H_1[u_x, w] + \varrho_r(\theta)\, H_2[u_x, w])\, u_{xt} + g(x, t, \varrho_r(\theta)), \tag{5.7}$$

and using the relations (2.20), (3.5) we obtain a. e. in Q the inequality

$$\begin{aligned} \theta_t - \theta_{xx} &\geq -H_3[u_x, w]\, w_t + \varrho_r(\theta)\, H_2[u_x, w]\, u_{xt} - K_g\, \varrho_r(\theta) \\ &\geq \varrho_r(\theta)\ (H_2[u_x, w]\, u_{xt} + H_3[u_x, w]\, H_4[u_x, w] - K_g) \\ &= h(x, t)\, \theta, \end{aligned} \tag{5.8}$$

where $h := \frac{\varrho_r(\theta)}{\theta}(H_2[u_x, w]\, u_{xt} + H_3[u_x, w]\, H_4[u_x, w] - K_g)$ is a function from $L^\infty(Q)$,

$$\|h\|_\infty \leq K_0(C_3(r) + K_0 + K_g) =: C^*(r). \tag{5.9}$$

Let us consider an auxiliary function

$$z(x, t) := e^{C^*(r)t}\theta(x, t) - \theta^* \tag{5.10}$$

with θ^* from Hypothesis 2.1. Then (5.8) reads

$$z_t - z_{xx} \geq (C^*(r) + h(x, t))\, (z + \theta^*) \geq (C^*(r) + h(x, t))\, z \tag{5.11}$$

a. e. in Q. Let us test Eq. (5.11) by the negative part z^- of z. Then

$$\frac{1}{2}|z^-(t)|_2^2 + \int_0^t |z_x^-(s)|_2^2\, ds \leq \frac{1}{2}|z^-(0)|_2^2 + 2C^*(r) \int_0^t |z^-(s)|_2^2\, ds. \tag{5.12}$$

We have by hypothesis $z^-(x, 0) = 0$ for every $x \in [0, \pi]$, hence

$$|z^-(t)|_2^2 \leq 4C^*(r) \int_0^t |z^-(s)|_2^2\, ds, \tag{5.13}$$

and Gronwall's argument yields $z^-(x, t) \equiv 0$, that is,

$$\theta(x, t) \geq e^{-C^*(r)t}\theta^* > 0 \tag{5.14}$$

for every $(x, t) \in \bar{Q}$.

5.2 Estimate V

Test Eq. (3.1) by u_t, (3.2) by 1, and sum up. The positivity of θ and F_1 then yield

$$\frac{d}{dt}\left(\frac{1}{2}|u_t(t)|_2^2 + \frac{1}{2}|u_{xx}(t)|_2^2 + |\theta(t)|_1 + |F_1[u_x, w]|_1\right) \leq |f(t)|_2|u_t(t)|_2 + |g^0(t)|_1 + K_g\, |\theta(t)|_1, \tag{5.15}$$

hence, by Gronwall's lemma,

$$|u_t(t)|_2^2 + |u_{xx}(t)|_2^2 + |\theta(t)|_1 + |F_1[u_x, w](t)|_1 \leq K_{12} \quad \forall t \in [0, T]. \tag{5.16}$$

5.3 Estimate VI

Testing Eq. (3.2) by $1/\theta$ yields

$$\begin{aligned}\int_0^t\int_0^\pi \left(\frac{u_{xt}^2}{\theta}+\frac{\theta_x^2}{\theta^2}\right) dx\,ds \;&\le\; \int_0^\pi \log\theta(x,t)\,dx - \int_0^\pi \log\theta^0(x)\,dx \\ &\quad + \int_0^t\int_0^\pi \frac{F_1[u_x,w]_t - \sigma\,u_{xt} - g(x,s,\varrho_r(\theta))}{\theta} dx\,ds\,,\end{aligned} \tag{5.17}$$

where

$$\int_0^\pi \log\theta(x,t)\,dx \;\le\; \int_0^\pi \theta(x,t)\,dx \;\le\; K_{12}\,, \tag{5.18}$$

$$-\int_0^\pi \log\theta^0(x)\,dx \;\le\; -\pi\,\log\theta^*\,, \tag{5.19}$$

$$\begin{aligned}\frac{1}{\theta}(F_1[u_x,w]_t-\sigma\,u_{xt}) \;&\le\; \frac{1}{\theta}(H_3[u_x,w]\,w_t - \varrho_r(\theta)\,H_2[u_x,w]\,u_{xt}) \\ &\le\; -\frac{\varrho_r(\theta)}{\theta}(H_3[u_x,w]\,H_4[u_x,w]+H_2[u_x,w]\,u_{xt}) \\ &\le\; K_{13}\,(1+|u_{xt}|)\,,\end{aligned} \tag{5.20}$$

$$\frac{-g(x,s,\varrho_r(\theta))}{\theta} \;=\; \frac{-g^0(x,s)+g(x,s,0)-g(x,s,\varrho_r(\theta))}{\theta} \;\le\; K_g\,. \tag{5.21}$$

Hence,

$$\begin{aligned}\int_0^t\int_0^\pi \left(\frac{u_{xt}^2}{\theta}+\frac{\theta_x^2}{\theta^2}\right) dx\,ds \;&\le\; K_{14}\left(1+\int_0^t\int_0^\pi |u_{xt}|\,dx\,ds\right) \\ &\le\; K_{14}\left(1+\int_0^t\int_0^\pi \frac{|u_{xt}|}{\sqrt{\theta}}\sqrt{\theta}\,dx\,ds\right) \\ &\le\; K_{14}\left(1+\left(\int_0^t\int_0^\pi \frac{u_{xt}^2}{\theta}\,dx\,ds\right)^{1/2}\left(\int_0^t\int_0^\pi \theta(x,s)\,dx\,ds\right)^{1/2}\right) \\ &\le\; K_{15}\left(1+\left(\int_0^t\int_0^\pi \frac{u_{xt}^2}{\theta}\,dx\,ds\right)^{1/2}\right)\end{aligned} \tag{5.22}$$

and this yields the estimate

$$\int_0^t\int_0^\pi \left(\frac{u_{xt}^2}{\theta}+\frac{\theta_x^2}{\theta^2}\right) dx\,ds \;\le\; K_{16}\,. \tag{5.23}$$

Furthermore,

$$\sqrt{\theta(x,t)} \;\le\; \sqrt{\theta(y,t)} + \int_0^\pi \frac{|\theta_x(\xi,t)|}{2\sqrt{\theta(\xi,t)}}\,d\xi \tag{5.24}$$

holds for every $x,y\in[0,\pi]$ and $t\in[0,T]$, hence, by integration over y and by the Cauchy-Schwarz inequality,

$$\sqrt{\theta(x,t)} \;\le\; \frac{1}{\sqrt{\pi}}\left(\int_0^\pi \theta(y,t)\,dy\right)^{1/2} + \frac{1}{2}\left(\int_0^\pi \frac{\theta_x^2}{\theta^2}(\xi,t)\,d\xi\right)^{1/2}\left(\int_0^\pi \theta(\xi,t)\,d\xi\right)^{1/2} \tag{5.25}$$

Combining (5.25) with (5.23) and (5.16) we obtain

$$\int_0^T |\theta(t)|_\infty \, dt \quad \le \quad K_{17}\,, \tag{5.26}$$

$$\|\theta\|_2^2 = \int_0^T \int_0^\pi \theta^2(x,t)\, dx\, dt \quad \le \quad \int_0^T |\theta(t)|_\infty \int_0^\pi \theta(x,t)\, dx\, dt \le K_{12}\, K_{17}\,. \tag{5.27}$$

Coming back to Eqs. (5.4) – (5.6), we see that Eq. (5.27) entails

$$\|u_t\|_{\infty,2} + \|u_{xx}\|_{\infty,2} + \|u_{xt}\|_2 \quad \le \quad K_{18}\,, \tag{5.28}$$

$$\|\theta_t\|_2^2 + \|\theta_x\|_{\infty,2}^2 \quad \le \quad K_{19}\left(1 + \|u_{xt}\|_\infty^2 + \|\theta\|_\infty^2\right)\,, \tag{5.29}$$

$$\|u_{tt}\|_{\infty,2}^2 + \|u_{xxt}\|_{\infty,2}^2 + \|u_{xtt}\|_2^2 \quad \le \quad K_{20}\left(1 + \|u_{xt}\|_\infty^2 + \|\theta\|_\infty^2\right)\,. \tag{5.30}$$

We now apply Lemma 4.1 consecutively with $v = \theta$ and $v = u_{xt}$. From Eqs. (5.27) – (5.30) we obtain in particular that

$$\|\theta_t\|_2 + \|\theta_x\|_{\infty,2} + \|\theta\|_\infty \;\le\; K_{21}\,. \tag{5.31}$$

6 Solution of Problem (P)

Choosing now $r := K_{21}$, we have $\varrho_r(\theta(x,t)) = \theta(x,t)$ for every $(x,t) \in \bar{Q}$, hence u, θ, w, σ satisfies the existence part of the assertion of Theorem 2.2. The lower bound for $\theta(x,t)$ follows from (5.14) with $C^* = C^*(r)$.

The uniqueness is a consequence of the following stronger result on locally Lipschitz continuous dependence. All constants K_i introduced in the previous sections will be now considered as functions $K_i = K_i(R)$ of the quantity R defined by (4.1) according to the discussion at the beginning of Section 4.

Theorem 6.1 *Let Hypothesis 2.1(i) hold, and let $R > 0$ be given. Then there exists a constant L_R such that for every data f_i, g_i, w_i^0, θ_i^0, u_i^0, u_i^1, $i = 1,2$, satisfying Hypothesis 2.1 (ii) – (vi) and such that*

$$\|f_i\|_2 + \|(f_i)_t\|_2 + \|g_i^0\|_2 + |w_i^0|_2 + |(w_i^0)_x|_2 + |\theta_i^0|_2 + |(\theta_i^0)_x|_2 + |(u_i^0)_{xxxx}|_2 + |(u_i^1)_{xx}|_2 \le R$$

for $i = 1,2$, the differences $\bar{u} = u_1 - u_2$, $\bar{w} = w_1 - w_2$, $\bar{\theta} = \theta_1 - \theta_2$ of the corresponding solutions u_i, θ_i, w_i satisfy the inequality

$$\|\bar{u}_t\|_{\infty,2} + \|\bar{u}_{xx}\|_{\infty,2} + \|\bar{u}_{xt}\|_2 + \|\bar{\theta}\|_{\infty,2} + \|\bar{w}_t\|_{\infty,2} \;\le\; L_R\,\Delta\,, \tag{6.1}$$

where we set

$$\Delta \;:=\; \|\bar{f}\|_2 + \|\bar{g}\|_2 + |\bar{w}^0|_2 + |\bar{u}^0_{xx}|_2 + |\bar{u}^1|_2 + |\bar{\theta}^0|_2\,,$$

with $\bar{f} = f_1 - f_2$, $\bar{g}(x,t) = \max\{|g_1(x,t,\theta) - g_2(x,t,\theta)|\,;\, 0 \le \theta \le K_{21}(R)\}$, $\bar{w}^0 = w_1^0 - w_2^0$, $\bar{u}^0 = u_1^0 - u_2^0$, $\bar{u}^1 = u_1^1 - u_2^1$, $\bar{\theta}^0 = \theta_1^0 - \theta_2^0$.

Proof. By Hypothesis 2.1 and using (5.31), we obtain from (2.4) that

$$|\bar{w}_t(t)|_2^2 \;\le\; K_{22}(R)\left(|\bar{\theta}(t)|_2^2 + |\bar{u}_x^0|_2^2 + |\bar{w}^0|_2^2 + \int_0^t \left(|\bar{u}_{xt}(s)|_2^2 + |\bar{w}_t(s)|_2^2\right) ds\right), \tag{6.2}$$

and from Gronwall's lemma it follows that

$$|\bar{w}_t(t)|_2^2 \;\le\; K_{23}(R)\left(\Delta^2 + |\bar{\theta}(t)|_2^2 + \int_0^t \left(|\bar{u}_{xt}(s)|_2^2 + |\bar{\theta}(s)|_2^2\right) ds\right). \tag{6.3}$$

We analogously have for $\bar{\sigma} = \sigma_1 - \sigma_2$ that

$$|\bar{\sigma}(t)|_2^2 \;\le\; K_{24}(R)\left(\Delta^2 + \int_0^t \left(|\bar{u}_{xt}(s)|_2^2 + |\bar{\theta}(s)|_2^2\right) ds\right). \tag{6.4}$$

Testing the equation

$$\bar{u}_{tt} + \bar{u}_{xxxx} - \bar{u}_{xxt} \;=\; \bar{\sigma}_x + \bar{f} \tag{6.5}$$

by $\bar{u}_t$ we therefore obtain

$$\begin{aligned} &\frac{d}{dt}\left(|\bar{u}_t(t)|_2^2 + |\bar{u}_{xx}(t)|_2^2\right) + |\bar{u}_{xt}(t)|_2^2 \\ &\quad\le\; K_{25}(R)\left(\Delta^2 + |\bar{f}(t)|_2^2 + |\bar{u}_t(t)|_2^2 + \int_0^t \left(|\bar{u}_{xt}(s)|_2^2 + |\bar{\theta}(s)|_2^2\right) ds\right), \end{aligned} \tag{6.6}$$

and Gronwall's lemma yields again for $t \in [0,T]$ that

$$|\bar{u}_t(t)|_2^2 + |\bar{u}_{xx}(t)|_2^2 + \int_0^t |\bar{u}_{xt}(s)|_2^2\, ds \;\le\; K_{26}(R)\left(\Delta^2 + \int_0^t |\bar{\theta}(s)|_2^2\, ds\right). \tag{6.7}$$

We now integrate the difference of Eqs. (2.2) for θ_1 and θ_2 with respect to t and test with $\bar{\theta}$. This and the above estimates yield that

$$\begin{aligned} \left|\bar{\theta}(t)\right|_2^2 + \frac{d}{dt}\left|\int_0^t \bar{\theta}_x(s)\, ds\right|_2^2 \;&\le\; K_{27}(R)\Bigg(|\bar{\theta}^0|_2^2 + |\bar{u}_x^0|_2^2 + |\bar{w}^0|_2^2 \\ &\quad + \int_0^t \left(|\bar{u}_{xt}(s)|_2^2 + |\bar{w}_t(s)|_2^2 + |\bar{\sigma}(s)|_2^2 + |\bar{g}(s)|_2^2 + K_g\, |\bar{\theta}(s)|_2^2\right) ds\Bigg) \\ &\le\; K_{28}(R)\left(\Delta^2 + \int_0^t |\bar{\theta}(s)|_2^2\, ds\right). \end{aligned} \tag{6.8}$$

and the assertion follows from a repeated use of Gronwall's lemma. ■

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