Study of a System for the Isothermal Separation of Components in a Binary Alloy with Change of Phase

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Abstract

An isothermal model describing the separation of the components of a binary metallic alloy is considered. A process of phase transition is also assumed to occur in the solder; hence, the state of the material is described by two order parameters, i.e., the concentration c of the first component and the *phase field* φ. A physical derivation is provided starting from energy balance considerations and the thermodynamic consistency of the model is shown. The resulting system of PDE's consists of a rather regular second order parabolic equation for φ, coupled with a fourth order relation of Cahn-Hilliard type for c with constraint and solution-dependent mobility. Global existence of solutions is proved and several regularity properties are discussed under more restrictive assumptions on the physical parameters. Continuous dependence on data is shown in a special case. An asymptotic analysis of the model is also performed, yielding at the limit step a coupling of the original phase field equation with a Hele-Shaw like system for c.

Key words: binary alloy, phase transition, fourth order parabolic system, constraint, variational formulation, maximum principle, Faedo-Galerkin scheme.

AMS (MOS) subject classification: 35K35, 35R35, 47H05, 74N25.

1 Introduction

In this paper, we aim to study a model describing the diffusive separation of components in a binary metallic alloy possibly presenting a phase transition phenomenon. As a basic simplification, the whole process is assumed to be isothermal.

We finally arrive at the system

$$\partial_t \varphi - \Delta \varphi = F_1(\varphi) + c F_2(\varphi), \tag{1.1}$$

$$\partial_t c - \operatorname{div}(\mu(c,\varphi)\nabla w) = 0, \tag{1.2}$$

$$w \in -\Delta c + \beta(c) + \gamma(c) + g(\varphi), \tag{1.3}$$

that, of course, has to be coupled with the no-flux conditions and with the Cauchy conditions for φ and c. Here, we have set $\beta := \partial I_{[0,1]}$, and...

Here is the outline of the paper. In the next section we provide some analytical preliminaries that are required for stating the precise mathematical abstract formulation of the problem. This is presented in Section 3 together with our main related results. In Sec. 4 we approximate the system by regularizing the subdifferential term, by use of the Yosida approximation, and then by exploiting a Faedo-Galerkin technique and prove local existence by an a priori estimates – passage to the limit argument. Section 5 is devoted to the analysis of further properties of the solution, as additional regularity, continuous dependence on data, uniqueness, and physical interpretation. Finally, in Sec. 6, the singular limit problem is considered and a related convergence result is proved.

2 Mathematical preliminaries

Let Ω be a smooth, bounded, and connected domain in $\mathbb{R}^d$, $1 \le d \le 3$, and let $T > 0$ be the final time of the process. Set $\Gamma := \partial\Omega$, $\Sigma := \Gamma \times (0,T)$, $Q_t := \Omega \times (0,t)$ for $t \in (0,T]$, and $Q := Q_T$. Set also $H := L^2(\Omega)$ and $V := H^1(\Omega)$ and endow the latter space with the usual scalar product

$$((v,w)) := \int_\Omega v\,w\,dx + \int_\Omega \nabla v \cdot \nabla w\,dx. \tag{2.1}$$

Identify H and its dual, in order that the compact inclusion $H \subset V'$ holds and (V,H,V') form a *Hilbert triplet* [25, p. 202]. Denote by $(\cdot,\cdot)$ the scalar product of both H and H^d and by $|\cdot|$ the associated norms. Finally, indicate by $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V and by $((\cdot,\cdot))_*$ the associated scalar product on V'.

Let us introduce some notations for functions and functionals with zero mean value. Namely, for any $\zeta \in V'$, let us set

$$\zeta_\Omega := \frac{1}{|\Omega|}\langle \zeta, 1\rangle, \tag{2.2}$$

$$V_0' := \{\zeta \in V' : \zeta_\Omega = 0\}, \qquad V_0 := V \cap V_0'. \tag{2.3}$$

The above notation V_0' is suggested just by the sake of convenience; indeed, we mainly see V_0, V_0' as (closed) subspaces of V, V', inheriting their norms, rather than as a couple of spaces in duality.

Let now $0 < \alpha \le \mu_0$ be assigned constants and let

$$\mu \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^2), \quad \text{with } \alpha \le \mu \le \mu_0 \quad \text{a.e. in } \mathbb{R}^2. \tag{2.4}$$

Let also $v, z : \Omega \to \mathbb{R}$ be measurable functions. Then, we can naturally associate to the couple (v, z) the elliptic operator

$$B_{(v,z)} : V \to V', \qquad \langle B_{(v,z)} u, y \rangle := \int_\Omega \mu(v, z) \nabla u \cdot \nabla y \, dx \quad \text{for } \ u, y \in V. \tag{2.5}$$

Note indeed that

$$\mu(v, z) \in L^\infty(\Omega) \qquad \text{with } \ \alpha \le \mu(v, z) \le \mu_0 \quad \text{a.e. in } \Omega. \tag{2.6}$$

Analogously, we introduce the realization of the Laplace operator with homogeneous Neumann boundary conditions as

$$B : V \to V', \qquad \langle B u, y \rangle := \int_\Omega \nabla u \cdot \nabla y \, dx \quad \text{for } \ u, y \in V. \tag{2.7}$$

Clearly, $B, B_{(v,z)}$ map V onto V_0' and their restrictions to V_0 turn out to be isomorphisms of V_0 onto V_0'. Then, we can denote by $\mathcal{N} : V_0' \to V_0$ the inverse of B and by $\mathcal{N}_{(v,z)} : V_0' \to V_0$ the inverse of $B_{(v,z)}$. Just by applying the definition (2.5), one can readily check that for any $u \in V$ and $\zeta \in V_0'$, there holds

$$\langle B_{(v,z)} u, \mathcal{N}_{(v,z)} \zeta \rangle = \langle B_{(v,z)} \mathcal{N}_{(v,z)} \zeta, u \rangle = \langle \zeta, u \rangle. \tag{2.8}$$

We also recall the Poincaré-Wirtinger inequality in the form

$$\|v\|_H^2 \le C_\Omega \|\nabla v\|_H^2 \qquad \text{for any } \ v \in V_0 \tag{2.9}$$

and for some $C_\Omega > 0$ only depending on the domain Ω. We have the following preliminary result, for whose proof we refer, e.g., to [7, Lemma 2.2],

Lemma 2.1. *For all $\zeta \in V_0'$ and for all measurable functions $v, z : \Omega \to \mathbb{R}$, we have that*

$$\|\mathcal{N}_{(v,z)} \zeta\|_V \le C_{\alpha,\Omega} \|\zeta\|_{V'} \tag{2.10}$$

where $C_{\alpha,\Omega} > 0$ only depends on α, Ω – and one can choose $C_{\alpha,\Omega} = (1 + C_\Omega)/\alpha$.

A natural coercivity property [7, Lemma 2.3] is also satisfied by $\mathcal{N}_{(v,z)}$,

Lemma 2.2. *There exists $C_{\alpha,\Omega,\mu} > 0$ such that, for any v, z, ζ as above, it is*

$$\langle \zeta, \mathcal{N}_{(v,z)} \zeta \rangle \ge C_{\alpha,\Omega,\mu} \|\zeta\|_{V'}^2 \tag{2.11}$$

and a valid choice is $C_{\alpha,\Omega,\mu} = \alpha/\mu_0^2(1 + C_\Omega)$.

Let us now extend the operators $B_{(v,z)}$ and $\mathcal{N}_{(v,z)}$ to a time dependent setting. So, let now $v, z : Q \to \mathbb{R}$ be measurable and, for a.e. $t \in (0,T)$, set $B_{(v,z)}(t) := B_{(v(t),z(t))}$ and $\mathcal{N}_{(v,z)}(t) := \mathcal{N}_{(v(t),z(t))}$. Clearly, since the analogue of (2.6) clearly holds in Q, we can extend the above maps to time dependent functions as follows. Let $p \in [1,\infty]$ and let u, ζ be measurable functions of time with values in V and V_0', respectively. Then, for a.e. $t \in (0,T)$, we can put $(B_{(v,z)}u)(t) := (B_{(v,z)}(t))(u(t))$ and $(\mathcal{N}_{(v,z)}\zeta)(t) := (\mathcal{N}_{(v,z)}(t))(\zeta(t))$. Denoting by $\mathcal{L}(X,Y)$ the space of the linear and continuous operators between the Banach spaces X and Y, a natural property turns out to hold (the proof follows, with minor modifications, that of [7, Lemma 3.1]).

Lemma 2.3. *For v, z as above and for any $p \in [1,+\infty]$, we have that the operators*

$$B_{(v,z)} : L^p(0,T;V) \to L^p(0,T;V_0') \quad \textit{and} \quad \mathcal{N}_{(v,z)} : L^p(0,T;V_0') \to L^p(0,T;V_0),$$

introduced as noted above, are well defined. In addition, the $B_{(v,z)}$ are (surjective) isomorphisms of $L^p(0,T;V_0)$ onto $L^p(0,T;V_0')$. Finally, we have that

$$\|B_{(v,z)}\|_{\mathcal{L}(L^p(0,T;V),L^p(0,T;V_0'))} \le \mu_0, \qquad \|\mathcal{N}_{(v,z)}\|_{\mathcal{L}(L^p(0,T;V_0'),L^p(0,T;V_0))} \le C_{\alpha,\Omega}, \tag{2.12}$$

where the constant $C_{\alpha,\Omega}$ is the same as in (2.10) and, in particular, does not depend on the functions v, z. ∎

Let us now present some lemmas, extending to the less regular case of monotone multivalued maps some properties that are obvious in a more regular setting. Thus, for the sequel, let $J : H \to [0,+\infty]$ be a convex, lower semicontinuous, and proper function and $\mathcal{A}$ be its subdifferential, regarded as a multivalued operator of H into itself. Firstly, we have (see [9, Lemme 3.3, p. 73]):

Lemma 2.4. *Let $T > 0$, $u \in H^1(0,T;H)$, $\xi \in L^2(0,T;H)$, $\xi(t) \in \mathcal{A}(u(t))$ for a.e. $t \in (0,T)$. Then, the function $t \mapsto J(u(t))$ is absolutely continuous in $(0,T)$ and*

$$(\xi(t), \partial_t u(t)) = \frac{d}{dt} J(u)(t) \qquad \textit{for a.e. } t \in (0,T). \tag{2.13}$$

In the following, anyway, a more precise statement will be needed,

Lemma 2.5. *Let $T > 0$, $u \in H^1(0,T;V') \cap L^2(0,T;V)$, $\eta \in L^2(0,T;V)$. Let also $\eta(t) \in \mathcal{A}u(t)$ for a.e. $t \in (0,T)$. Moreover, let us suppose that there exist $\kappa_1, \kappa_2 > 0$ such that*

$$J(v) \ge \kappa_1 |v|^2 - \kappa_2 \qquad \textit{for all } v \in H. \tag{2.14}$$

Then, the function $t \mapsto J(u(t))$ is absolutely continuous in $(0,T)$. In particular,

$$\int_s^t \langle \partial_t u(r), \eta(r) \rangle \, dr = J(u(t)) - J(u(s)) \qquad \forall\, s, t \in (0,T). \tag{2.15}$$

PROOF. Let us define a new convex functional $J_{\text{ext}} : V' \to \mathbb{R}$ as

$$J_{\text{ext}}(v) := J(v), \quad \text{if } v \in H, \qquad J_{\text{ext}}(v) := +\infty, \quad \text{otherwise}. \tag{2.16}$$

Owing to (2.14), it is not difficult to prove that J_{ext} is lower semicontinuous on V'. Furthermore, by definition of subdifferential and using that $\eta \in V$, for any $\zeta \in H$ we have

$$\langle \zeta - u, \eta \rangle = (\eta, \zeta - u) \leq J(\zeta) - J(u) = J_{\text{ext}}(\zeta) - J_{\text{ext}}(u). \tag{2.17}$$

Note that this relation still holds if $\zeta \in V'$. Actually, due to (2.16), if $\zeta \in V' \setminus H$, then $J_{\text{ext}}(\zeta) = +\infty$, and (2.17) is trivial. Then, denoting by $\mathcal{R} : V \to V'$ the Riesz isomorphism, it is not difficult to infer that

$$\mathcal{R}\eta \in \partial_{V'} J_{\text{ext}}(u),$$

where $\partial_{V'} J_{\text{ext}}$ of course denotes the subdifferential of J_{ext} with respect to the Hilbert structure of V'. Then, Lemma 2.4 can be applied in the space V'. Furthermore, for all $s, t \in [0, T]$, a simple integration yields

$$\begin{aligned}\int_s^t \langle \partial_t u(r), \eta(r) \rangle \, dr &= \int_s^t ((\partial_t u(r), \mathcal{R}\eta(r)))_* \, dr \\ &= J_{\text{ext}}(u(t)) - J_{\text{ext}}(u(s)) = J(u(t)) - J(u(s)),\end{aligned}$$

since it is known that $u(\cdot) \in H$ a.e. in $(0, T)$. ∎

Finally, for the proof of the next property see, e.g., [11, Lemma 2].

Lemma 2.6. *Let $z, \xi \in H$, and let J be the realization in H of a convex, l.s.c., and proper function $j : \mathbb{R} \to [0, +\infty]$, e.g.,*

$$J(v) := \int_\Omega j(v(x)) \, dx \qquad \text{for } v \in H,$$

where it is intended that the value of the integral might be $+\infty$ for some v. Then, let $\xi \in \partial J(z)$. Let also μ as in (2.4), $u, v, B_{(u,v)}$ as in (2.5). Finally, let $B_{(u,v)} z \in H$. Then, $(B_{(u,v)} z, \xi) \geq 0$.

3 Main results

Let us give the main assumptions on the data of the problem. Let $K > 0$ and let

$$F_1, F_2, \gamma \in W^{1,\infty}(\mathbb{R}), \tag{3.1}$$

$$|F_1|, |F_2|, |\gamma|, |F_1'|, |F_2'|, |\gamma'| \leq K \quad \text{a.e. in } \mathbb{R}, \tag{3.2}$$

$$g \in \operatorname{Lip}_{\text{loc}}(\mathbb{R}), \tag{3.3}$$

$$\varphi_0 \in H, \qquad c_0 \in V, \tag{3.4}$$

$$\beta \subset \mathbb{R} \times \mathbb{R} \qquad \textit{maximal monotone graph s.t. } 0 \in \beta(0). \tag{3.5}$$

Let us fix a convex and lower semicontinuous function $\psi : \mathbb{R} \to [0, +\infty]$ such that $\beta = \partial\psi$ and $\psi(0) = 0$. Moreover, we recall that the *domain* of the graph β is defined as

$$D(\beta) := \{r \in \mathbb{R} : \quad \beta(r) \neq \emptyset\}.$$

Then, we also require

$$\psi(c_0) \in L^1(\Omega), \tag{3.6}$$

$$c_\Omega \in \operatorname{int} D(\beta), \quad \text{where} \quad c_\Omega := (c_0)_\Omega. \tag{3.7}$$

We are now able to state our main existence result,

Theorem 3.1. *Let assumptions* (2.4), (3.1–3.7) *hold. Let us additionally suppose that*

$$g \in W^{1,\infty}(\mathbb{R}), \qquad |g|, |g'| \le K \quad \text{a.e. in } \mathbb{R}, \tag{3.8}$$

where K is as in (3.2)*. Then, there exists a quadruple (φ, c, w, ξ) such that*

$$\varphi \in H^1(0,T;V') \cap L^2(0,T;V), \tag{3.9}$$

$$c \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \tag{3.10}$$

$$w \in L^2(0,T;V), \tag{3.11}$$

$$\xi \in L^2(0,T;H). \tag{3.12}$$

The quadruple (φ, c, w, ξ) satisfies

$$\partial_t\varphi + B\varphi = F_1(\varphi) + cF_2(\varphi) \qquad \text{in } V', \quad \text{a.e. in } (0,T), \tag{3.13}$$

$$\partial_t c + B_{(\varphi,c)} w = 0 \qquad \text{in } V', \quad \text{a.e. in } (0,T), \tag{3.14}$$

$$w = Bc + \xi + \gamma(c) + g(\varphi) \qquad \text{in } V', \quad \text{a.e. in } (0,T), \tag{3.15}$$

$$\xi \in \beta(c) \qquad \text{a.e. in } Q, \tag{3.16}$$

$$\varphi(\cdot,0) = \varphi_0(\cdot), \quad c(\cdot,0) = c_0(\cdot) \qquad \text{a.e. in } \Omega. \tag{3.17}$$

Moreover, c is a conserved order parameter, *i.e.*,

$$c(t)_\Omega = c_\Omega \qquad \text{for all } t \in [0,T]. \tag{3.18}$$

Remark 3.2. Using (3.12) and the last of (3.10), one can see that (3.15) turns out to hold a.e. in Q. However, we prefer to state it in V', since this is the natural output space for the operator B.

Let us come to some regularity results.

Theorem 3.3. *In addition to* (2.4) *and* (3.1–3.7), *let*

$$\varphi_0 \in V. \tag{3.19}$$

Then, the function φ whose existence is ensured by Theorem 3.1 *satisfies*

$$\varphi \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;H^2(\Omega)). \tag{3.20}$$

Theorem 3.4. *In addition to* (2.4) *and* (3.1–3.7), *let*

$$\varphi_0 \in H^2(\Omega), \qquad \partial_{\mathbf{n}}\varphi_0 = 0 \quad \text{a.e. on } \Gamma, \tag{3.21}$$

where $\boldsymbol{n}$ denotes the outer unit normal vector to Ω. Then, the function φ given by Theorem 3.1 *satisfies*

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^3(\Omega)). \tag{3.22}$$

The following continuous dependence results holds in the regularity setting of Theorem 3.1, but only in case the function μ in (2.4) is replaced by a constant.

Theorem 3.5. *Let us assume* (3.1–3.3) *and* (3.5), *and let us given two pairs of initial data* $(\varphi_{0,1}, c_{0,1})$ *and* $(\varphi_{0,2}, c_{0,2})$, *satisfying* (3.4) *and* (3.6–3.7). *Moreover, let us assume that*

$$(c_{0,1})_\Omega = (c_{0,2})_\Omega. \tag{3.23}$$

Then, let us consider the system given by (3.13), (3.15–3.17), *and*

$$\partial_t c + Bw = 0 \qquad \text{in } V', \quad \text{a.e. in } (0,T). \tag{3.24}$$

and, for $i = 1, 2$, *let* $(\varphi_i, c_i, w_i, \xi_i)$ *be two solutions of such a system, related to the initial data* $(\varphi_{0,1}, c_{0,1})$ *and* $(\varphi_{0,2}, c_{0,2})$, *respectively. Moreover, let us assume that* $R > 0$ *is a constant such that*

$$\|c_1\|_{L^1(0,T;H^2(\Omega))} \le R. \tag{3.25}$$

Then, there exists a constant $C > 0$ *only depending on* Ω, T, R, *and* K, *such that*

$$\begin{aligned} &\|\varphi_1 - \varphi_2\|_{C^0([0,T];H)\cap L^2(0,T;V)} + \|c_1 - c_2\|_{L^\infty(0,T;V')\cap L^2(0,T;V)} \\ &\quad \le C\big(|\varphi_{0,1} - \varphi_{0,2}| + \|c_{0,1} - c_{0,2}\|_{V'}\big). \end{aligned} \tag{3.26}$$

In particular, the solution to the modified system provided by Theorem 3.1 *is unique.*

Finally, let us prove that more restrictive assumptions on F_1, F_2 ensure that the component φ of the solution admits a "physical" interpretation as a phase variable.

Theorem 3.6. *Under the hypotheses of Theorem 3.1, let us additionally assume that*

$$F_1(0) = F_1(1) = F_2(0) = F_2(1) = 0, \tag{3.27}$$

$$0 \le \varphi_0 \le 1 \qquad \text{a.e. in } \Omega. \tag{3.28}$$

Then, the function φ *whose existence is stated in Theorem 3.1 fulfills*

$$\varphi \in L^\infty(Q), \qquad \text{with } 0 \le \varphi \le 1, \quad \text{a.e. in } Q. \tag{3.29}$$

Remark 3.7. We observe that assumptions (3.5–3.7) actually generalize the natural *physical* assumptions on the graph β and the initial datum c_0, that we now report in a rigorous mathematical form for the sake of completeness:

$$0 \le c_0 \le 1 \quad \text{a.e. in } \Omega, \tag{3.30}$$

$$0 < c_\Omega < 1, \quad \text{where} \quad c_\Omega := (c_0)_\Omega, \tag{3.31}$$

$$\beta = \partial I_{[0,1]}. \tag{3.32}$$

Indeed, it is clear that, if (3.30–3.32) are fulfilled, then any solution of, e.g., Theorem 3.1 satisfies, in addition,

$$c \in L^\infty(Q), \qquad \text{with } 0 \le c \le 1 \quad \text{a.e. in } Q. \tag{3.33}$$

4 Proof of Theorem 3.1

4.1 Local existence

We aim to prove existence for (3.13–3.17) by exploiting a Faedo–Galerkin approximation. Indeed, the argument we use is rather standard and we prefer to sketch it just for the sake of completeness. First of all, we regularize the system, by replacing β with its Yosida approximation β_ε [9, p. 28]. Note that, in the physical case (3.32), β_ε is given by $\beta_\varepsilon(r) = \varepsilon^{-1}r$ if $r \leq 0$, $\beta_\varepsilon(r) = 0$ if $0 < r < 1$, and $\beta_\varepsilon(r) = \varepsilon^{-1}(r-1)$ if $r \geq 1$. We denote by ψ_ε the primitive of β_ε such that $\psi_\varepsilon(0) = 0$.

Then, we choose a special basis of test functions. Actually, for $i \in \mathbb{N}$, $i \geq 1$, we define $\{v_i\}$ as the complete system of eigenvectors of the eigenvalue problem

$$v_i \in V, \qquad Bv_i = \lambda_i v_i \quad \text{in } V'. \tag{4.1}$$

According to the general spectral theory, the eigenvalues λ_i can be increasingly ordered and counted according to their multiplicities in order to form a real divergent sequence. Moreover, the respective eigenvectors $\{v_i\}$ are assumed to be normalized in H and form an orthogonal basis both in V and in H. At this point, we can set $V_n := \operatorname{span}\{v_1, \dots, v_n\}$ and $V_\infty := \cup_{n=1}^\infty V_n$. Clearly, V_∞ is a dense subspace of V.

For any $n \in \mathbb{N}$, we consider a triplet of functions of the form

$$\varphi_\varepsilon^n = \sum_{i=1}^n \varphi_i(t) v_i, \qquad c_\varepsilon^n = \sum_{i=1}^n c_i(t) v_i, \qquad w_\varepsilon^n = \sum_{i=1}^n w_i(t) v_i, \tag{4.2}$$

where φ_i, c_i, and w_i are regarded as suitably regular real valued functions of time, whose dependence on ε, n is not stressed, for simplicity. We indicate by $\boldsymbol{\varphi}, \boldsymbol{c}, \boldsymbol{w}$ the vectors $\{\varphi_i\}_{i=1,\dots,n}$, $\{c_i\}_{i=1,\dots,n}$, $\{w_i\}_{i=1,\dots,n}$.

We now aim to write the finite-dimensional approximation of (3.13–3.16). Thus, fixing n, let us substitute the expressions given by (4.2) into that system and then choose $v = v_j$ for $j = 1, \dots, n$. Taking (4.1) into account, it is not difficult to arrive at the system

$$\boldsymbol{\varphi}' = -\Lambda \boldsymbol{\varphi} + \boldsymbol{F}(\boldsymbol{\varphi}, \boldsymbol{c}), \tag{4.3}$$

$$\boldsymbol{c}' = -M(\boldsymbol{\varphi}, \boldsymbol{c}) \boldsymbol{w}, \tag{4.4}$$

$$\boldsymbol{w} = \Lambda \boldsymbol{c} + \boldsymbol{\beta}_{\boldsymbol{\varepsilon}}(\boldsymbol{c}) + \boldsymbol{\gamma}(\boldsymbol{c}) + \boldsymbol{g}(\boldsymbol{\varphi}). \tag{4.5}$$

Here $\boldsymbol{F}, \boldsymbol{\beta}_{\boldsymbol{\varepsilon}}, \boldsymbol{\gamma}$, and $\boldsymbol{g}$ are locally Lipschitz continuous functions of their arguments that are naturally constructed starting from F_1 and F_2, β_ε, γ, and g, respectively. Moreover, Λ is the diagonal matrix of the first n eigenvalues of (4.1) and, finally, the matrix $M(\boldsymbol{\varphi}, \boldsymbol{c}) = (m_{ij}(\boldsymbol{\varphi}, \boldsymbol{c}))$ is defined as

$$m_{ij}(\boldsymbol{\varphi}, \boldsymbol{c}) := \int_\Omega \mu\Big(\sum_{k=1}^n \varphi_k v_k, \sum_{k=1}^n c_k v_k\Big) \nabla v_i \cdot \nabla v_j \, dx \tag{4.6}$$

and, by (2.4), is a locally Lipschitz function of $\boldsymbol{\varphi}, \boldsymbol{c}$. Of course, the above system has to be complemented with the boundary conditions obtained by approximating (3.17). Namely, we take $\varphi_{0n} \in V_n$, $c_{0n} \in V_n$, with

$$\varphi_{0n} \to \varphi_0 \quad \text{in } H, \qquad c_{0n} \to c_0 \quad \text{in } V \tag{4.7}$$

and require that the functions c_ε^n, φ_ε^n satisfy

$$\varphi_\varepsilon^n(0) = \varphi_{0n}, \quad c_\varepsilon^n(0) = c_{0n}, \qquad \text{a.e. in } \Omega. \tag{4.8}$$

Then, an application of Cauchy's Theorem for ODE's guarantees the local existence and uniqueness of the solution of the above system (4.3–4.5). Actually, plugging (4.5) into (4.4) and taking advantage of (3.1–3.2) and (3.8), we see that the right hand sides of the system given by the resulting relation and (4.3) depend on $\boldsymbol{\varphi}, \boldsymbol{c}$ in a locally Lipschitz continuous way. Hence, there exist $T_0 > 0$ and functions $\boldsymbol{\varphi}, \boldsymbol{c} \in C^1([0,T_0];\mathbb{R}^n)$, $\boldsymbol{w} \in C^0([0,T_0];\mathbb{R}^n)$, such that (4.3–4.5) hold $\forall\, t \in [0,T_0]$ together with (4.8).

4.2 A priori estimates

We now aim to prove some a priori estimates on the solution of the approximated system given by (4.3–4.5) and (4.8). Hence, let us rewrite (4.3–4.5) by replacing the expressions (4.2) therein,

$$\langle \partial_t \varphi_\varepsilon^n + B\varphi_\varepsilon^n, v\rangle = \langle F_1(\varphi_\varepsilon^n) + c_\varepsilon^n F_2(\varphi_\varepsilon^n), v\rangle \qquad \forall\, v \in V_n, \quad \forall\, t \in (0,T), \tag{4.9}$$

$$\langle \partial_t c_\varepsilon^n + B_{(\varphi_\varepsilon^n, c_\varepsilon^n)} w_\varepsilon^n, v\rangle = 0 \qquad \forall\, v \in V_n, \quad \forall\, t \in (0,T), \tag{4.10}$$

$$\langle w_\varepsilon^n, v\rangle = \langle B c_\varepsilon^n + \beta_\varepsilon(c_\varepsilon^n) + \gamma(c_\varepsilon^n) + g(\varphi_\varepsilon^n), v\rangle \qquad \forall\, v \in V_n, \quad \forall\, t \in (0,T). \tag{4.11}$$

In the sequel, k will be a positive constant, possibly varying even inside a single row, but allowed to depend only on $\Omega, \alpha, \mu_0, K, T$ and, in particular, not on T_0, n, ε. Symbols like, e.g., k_σ are intended to mean that the constant k_σ might also depend on one, or more, additional parameters (in this case, σ). The elementary Young inequality

$$rs \le \sigma r^2 + s^2/4\sigma \qquad \text{for any } r, s \in \mathbb{R},\ \sigma > 0 \tag{4.12}$$

will be repeatedly used in the following. Finally, we observe that, since our estimates do not depend on T_0, the limit solution will turn out to be defined up to the final time T. For this reason, we shall directly work in the interval $(0,T)$.

Energy estimate. Take $t \in (0,T]$, choose $v = w_\varepsilon^n$ in (4.10) and $v = \partial_t c_\varepsilon^n$ in (4.11), integrate over $(0,t)$ and sum together the results. Owing to (2.4), observing that two terms cancel, and integrating in time the term with β_ε, it is easy to infer

$$\begin{aligned} \alpha\|\nabla w_\varepsilon^n\|_{L^2(Q_t)}^2 + \frac{1}{2}|\nabla c_\varepsilon^n(t)|^2 + \int_\Omega \psi_\varepsilon(c_\varepsilon^n(t))\,dx \le \frac{1}{2}|\nabla c_{0n}|^2 \\ + \int_\Omega \psi_\varepsilon(c_{0n})\,dx - \int_0^t\int_\Omega \gamma(c_\varepsilon^n)\,\partial_t c_\varepsilon^n\,dx\,ds - \int_0^t\int_\Omega g(\varphi_\varepsilon^n)\,\partial_t c_\varepsilon^n\,dx\,ds. \end{aligned} \tag{4.13}$$

The last two integrals on the right hand side above can be estimated in several ways. For instance, splitting the former in the duality (V,V'), we have

$$\Big|\int_0^t\int_\Omega \gamma(c_\varepsilon^n)\,\partial_t c_\varepsilon^n\,dx\,ds\Big| \le \sigma\|\partial_t c_\varepsilon^n\|_{L^2(0,t;V')}^2 + k_\sigma + k_\sigma\|\nabla c_\varepsilon^n\|_{L^2(Q_t)}^2 \tag{4.14}$$

and, analogously,

$$\Big|\int_0^t\int_\Omega g(\varphi_\varepsilon^n)\,\partial_t c_\varepsilon^n\,dx\,ds\Big| \le \sigma\|\partial_t c_\varepsilon^n\|^2_{L^2(0,t;V')} + k_\sigma + k_\sigma\|\nabla\varphi_\varepsilon^n\|^2_{L^2(Q_t)}, \tag{4.15}$$

where, of course, σ is as in (4.12) and the above k_σ's also depend on the bound K to γ, g and their first derivatives.

Now, in order to estimate the norms of φ_ε^n appearing in (4.15), we choose $v = \varphi_\varepsilon^n$ in (4.9) and integrate again over $(0,t)$. Taking advantage of (3.1–3.2) again, it is immediate to infer

$$\frac{1}{2}|\varphi_\varepsilon^n(t)|^2 + \|\nabla\varphi_\varepsilon^n\|^2_{L^2(Q_t)} \le \frac{1}{2}|\varphi_{0n}|^2 + k\big(1 + \|\varphi_\varepsilon^n\|^2_{L^2(Q_t)} + \|c_\varepsilon^n\|^2_{L^2(Q_t)}\big). \tag{4.16}$$

We now have to estimate the norms of $\partial_t c_\varepsilon^n$ on the right hand side of (4.14–4.15). Then, note that $\lambda_1 = 0$, so that $1 \in V_n$ for every n and we can choose $v = 1$ in (4.10), obtaining

$$\langle \partial_t c_\varepsilon^n, 1\rangle = 0.$$

Thus, $\partial_t c_\varepsilon^n \in V_n \cap V_0$. Noting that $V_n \cap V_0 = \operatorname{span}\{v_2, \dots, v_n\}$ by orthogonality, one can readily see that the appropriate restriction of $\mathcal{N}$ is an isomorphism of $V_n \cap V_0$ onto itself (in coordinates it is the inverse of the diagonal matrix obtained by suppressing the first row and column of Λ). Hence, we can choose $v = \mathcal{N}\partial_t c_\varepsilon^n$ in (4.10). Integrating over $(0,t)$ and owing to Lemmas 2.2, 2.3 and to the inequalities (2.9) and (4.12), we get

$$\begin{aligned}\|\partial_t c_\varepsilon^n\|^2_{L^2(0,t;V')} &\le k\|B_{(\varphi_\varepsilon^n, c_\varepsilon^n)} w_\varepsilon^n\|^2_{L^2(0,t;V')} \le k\,\big\|B_{(\varphi_\varepsilon^n, c_\varepsilon^n)}\big(w_\varepsilon^n - (w_\varepsilon^n)_\Omega\big)\big\|^2_{L^2(0,t;V')}\\ &\le k\,\|w_\varepsilon^n - (w_\varepsilon^n)_\Omega\|^2_{L^2(0,t;V)} \le k\|\nabla w_\varepsilon^n\|^2_{L^2(Q_t)}.\end{aligned} \tag{4.17}$$

Before collecting the above computations, we still have to recover the full V-norm of c_ε^n on the left hand side of (4.13) and we do this by noting that

$$\begin{aligned}\frac{1}{2}|c_\varepsilon^n(t)|^2 &= \frac{1}{2}|c_{0n}|^2 + \int_0^t (\partial_t c_\varepsilon^n, c_\varepsilon^n)\,ds\\ &\le \frac{1}{2}|c_{0n}|^2 + \frac{1}{4}\|\partial_t c_\varepsilon^n\|^2_{L^2(0,t;V')} + k\|c_\varepsilon^n\|^2_{L^2(Q_t)} + k\|\nabla c_\varepsilon^n\|^2_{L^2(Q_t)}.\end{aligned} \tag{4.18}$$

Now, let us multiply (4.13) by $m_1 > 0$ and (4.16) by $m_2 > 0$, where m_1 and m_2 will be chosen later. Then, we take the sum of the resulting relations and add also (4.17) and (4.18). Taking (4.14) and (4.15) into account, we infer

$$\begin{aligned}&m_1\alpha\|\nabla w_\varepsilon^n\|^2_{L^2(Q_t)} + \frac{m_1}{2}|\nabla c_\varepsilon^n(t)|^2 + m_1\int_\Omega \psi_\varepsilon(c_\varepsilon^n(t))\,dx + \frac{m_2}{2}|\varphi_\varepsilon^n(t)|^2\\ &\quad + m_2\|\nabla\varphi_\varepsilon^n\|^2_{L^2(Q_t)} + \|\partial_t c_\varepsilon^n\|^2_{L^2(0,t;V')} + \frac{1}{2}|c_\varepsilon^n(t)|^2\\ &\le k_{\sigma,m_1,m_2} + \frac{m_1}{2}|\nabla c_{0n}|^2 + m_1\int_\Omega \psi_\varepsilon(c_{0n})\,dx + k_{\sigma,m_2}\|c_\varepsilon^n\|^2_{L^2(Q_t)}\\ &\quad + k_{\sigma,m_2}\|\varphi_\varepsilon^n\|^2_{L^2(Q_t)} + \Big(\frac{1}{4} + 2m_1\sigma\Big)\|\partial_t c_\varepsilon^n\|^2_{L^2(0,t;V')} + (m_1 k_\sigma + k)\|\nabla c_\varepsilon^n\|^2_{L^2(Q_t)}\\ &\quad + m_1 k_\sigma^*\|\nabla\varphi_\varepsilon^n\|^2_{L^2(Q_t)} + \frac{m_2}{2}|\varphi_{0n}|^2 + k^*\|\nabla w_\varepsilon^n\|^2_{L^2(Q_t)} + \frac{1}{2}|c_{0n}|^2,\end{aligned} \tag{4.19}$$

where k_σ^* is the constant k_σ multiplying the last norm in (4.15) and k^* is the constant k on the right hand side of (4.17).

Now, let us choose successively m_1, σ, and m_2, in order that

$$m_1 \geq \frac{2k^*}{\alpha}, \qquad \sigma \leq \frac{1}{4m_1}, \qquad m_2 \geq 2m_1 k_\sigma^*.$$

Then, all the terms on the right hand side of (4.19) are controlled. Indeed, (4.7) holds and we notice that, by (3.6) and [9, Prop. 2.11, p. 39],

$$\int_\Omega \psi_\varepsilon(c_{0n})\,dx \leq 1 + \int_\Omega \psi(c_0)\,dx \leq k,$$

at least for n large enough, depending on ε. Hence, we readily see that Gronwall's lemma can be applied to the function

$$t \mapsto \|c_\varepsilon^n(t)\|_V^2 + |\varphi_\varepsilon^n(t)|^2$$

in order to derive a bound. Since we need an estimate for the full V-norm of w_ε^n, we take $v = w_\varepsilon^n$ in (4.11) and integrate over $(0,t)$. Owing to the Lipschitz continuity of γ, g, and β_ε and noting that, due to the estimates given by (4.19),

$$\left|\int_0^t \langle Bc_\varepsilon^n, w_\varepsilon^n\rangle\,ds\right| \leq \frac{1}{2}\|\nabla w_\varepsilon^n\|_{L^2(Q_t)}^2 + \frac{1}{2}\|\nabla c_\varepsilon^n\|_{L^2(Q_t)}^2 \leq k,$$

we readily get

$$\|w_\varepsilon^n\|_{L^2(Q)}^2 \leq k_\varepsilon + k, \tag{4.20}$$

where the constant k_ε resulting from the term $\beta_\varepsilon(c_\varepsilon)$ is quadratically dependent on the Lipschitz constant of β_ε and explodes as $\varepsilon \searrow 0$.

4.3 Passage to the limit

Passage to the limit as $n \nearrow \infty$. We now aim to pass to the limit firstly as $n \nearrow \infty$ and then as $\varepsilon \searrow 0$. Up to the end of the paper, all the convergence relations will be meant to hold up to the extraction of suitable subsequences, generally not relabeled. Then, from relations (4.19–4.20), we see that there exists a triplet $(\varphi_\varepsilon, c_\varepsilon, w_\varepsilon)$, such that

$$\varphi_\varepsilon^n \to \varphi_\varepsilon \qquad \text{weakly star in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{4.21}$$

$$c_\varepsilon^n \to c_\varepsilon \qquad \text{weakly star in } H^1(0,T;V') \cap L^\infty(0,T;V), \tag{4.22}$$

$$w_\varepsilon^n \to w_\varepsilon \qquad \text{weakly in } L^2(0,T;V). \tag{4.23}$$

Then, standard interpolation and compact embedding results for vector-valued functions (see, e.g., [32, Sec. 8]) ensure that

$$c_\varepsilon^n \to c_\varepsilon \qquad \text{strongly in } C^0([0,T];H). \tag{4.24}$$

In order to derive some strong convergence for φ_ε^n we need an estimate of its time derivative. Due to the finite-dimensional setting, we cannot proceed by a direct comparison in (4.9); then, we choose $v = \mathcal{N}\big(\partial_t\varphi_\varepsilon^n - (\partial_t\varphi_\varepsilon^n)_\Omega\big) \in V_n$ in (4.9) and integrate over $(0,t)$, $t \le T$, deriving

$$\int_0^t \big\langle \partial_t\varphi_\varepsilon^n, \mathcal{N}\big(\partial_t\varphi_\varepsilon^n - (\partial_t\varphi_\varepsilon^n)_\Omega\big)\big\rangle\,ds + \int_0^t \big\langle B\varphi_\varepsilon^n, \mathcal{N}\big(\partial_t\varphi_\varepsilon^n - (\partial_t\varphi_\varepsilon^n)_\Omega\big)\big\rangle\,ds$$
$$= \int_0^t \big\langle F_1(\varphi_\varepsilon^n) + c_\varepsilon^n F_2(\varphi_\varepsilon^n), \mathcal{N}\big(\partial_t\varphi_\varepsilon^n - (\partial_t\varphi_\varepsilon^n)_\Omega\big)\big\rangle\,ds. \tag{4.25}$$

Note now that, since $\mathcal{N}\big(\partial_t\varphi_\varepsilon^n - (\partial_t\varphi_\varepsilon^n)_\Omega\big) \in V_0$ a.e. in $(0,T)$, we can rely on Lemma 2.2, obtaining

$$\int_0^t \big\langle \partial_t\varphi_\varepsilon^n, \mathcal{N}\big(\partial_t\varphi_\varepsilon^n - (\partial_t\varphi_\varepsilon^n)_\Omega\big)\big\rangle\,ds = \int_0^t \big\langle \partial_t\varphi_\varepsilon^n - (\partial_t\varphi_\varepsilon^n)_\Omega, \mathcal{N}\big(\partial_t\varphi_\varepsilon^n - (\partial_t\varphi_\varepsilon^n)_\Omega\big)\big\rangle\,ds$$
$$\ge k\big\|\partial_t\varphi_\varepsilon^n - (\partial_t\varphi_\varepsilon^n)_\Omega\big\|_{L^2(0,T;V')}^2. \tag{4.26}$$

Moreover, by (2.8), it is easy to check that

$$\int_0^t \big\langle B\varphi_\varepsilon^n, \mathcal{N}\big(\partial_t\varphi_\varepsilon^n - (\partial_t\varphi_\varepsilon^n)_\Omega\big)\big\rangle\,ds = |\varphi_\varepsilon^n(t) - (\varphi_\varepsilon^n)_\Omega|^2 - |\varphi_{0n} - (\varphi_{0n})_\Omega|^2,$$

and the latter norm is bounded, of course. Finally, the terms on the right hand side of (4.25) can be split in the duality (V', V) and estimated by taking account of (3.1–3.2), estimates (4.21–4.22), and the continuous embedding $V \subset H$. This allows us to derive from (4.25–4.26) the relation

$$\big\|\partial_t\varphi_\varepsilon^n - (\partial_t\varphi_\varepsilon^n)_\Omega\big\|_{L^2(0,T;V')}^2 \le k. \tag{4.27}$$

Next, we notice that, $(\partial_t\varphi_\varepsilon^n)_\Omega$ is constant in Ω everywhere in $[0,T]$. Thus, taking $v = \pm 1$ in (4.9) and noting that, by (3.3) and the second of (4.22),

$$\big|(\partial_t\varphi_\varepsilon^n)_\Omega(t)\big| \le k\big(1 + \|c_\varepsilon^n\|_{L^\infty(0,T;L^1(\Omega))}\big) \le k \tag{4.28}$$

for all $t \in [0,T]$. Then, (4.27) yields

$$\partial_t\varphi_\varepsilon^n \to \partial_t\varphi_\varepsilon \qquad \text{weakly in } L^2(0,T;V') \tag{4.29}$$

and, using (4.21) and [32, Sec. 8] again,

$$\varphi_\varepsilon^n \to \varphi_\varepsilon \qquad \text{strongly in } L^2(0,T;H) \cap C^0([0,T];V'). \tag{4.30}$$

Now, the boundedness and Lipschitz continuity of F_1, F_2, γ, g, and β_ε, together with relations (4.24) and (4.30), allow us to take the limits of the nonlinearities in (4.9) and (4.11). As for dealing with (4.10), we have to rewrite it as

$$\langle \partial_t c_\varepsilon^n, v\rangle + \int_\Omega \mu(\varphi_\varepsilon^n, c_\varepsilon^n)\nabla w_\varepsilon^n \cdot \nabla v\,dx \qquad \forall\, v \in V_n, \quad \text{a.e. in } (0,T).$$

Then, we note that, by (2.4) and Lebesgue's dominated convergence theorem,

$$\mu(\varphi_\varepsilon^n, c_\varepsilon^n) \to \mu(\varphi_\varepsilon, c_\varepsilon) \qquad \text{weakly star in } L^\infty(Q), \quad \text{and strongly in } L^p(Q) \tag{4.31}$$

for any $p < \infty$. Thus, recalling (4.23),

$$\mu(\varphi_\varepsilon^n, c_\varepsilon^n)\nabla w_\varepsilon^n \to \mu(\varphi_\varepsilon, c_\varepsilon)\nabla w_\varepsilon \qquad \text{weakly in } L^2(Q) \tag{4.32}$$

and this permits us to pass to the limit in (4.10). We still have to prove the H^2 regularity for c_ε. Then, it is enough to set $v = Bc_\varepsilon^n$ in (4.11), integrate in time, and note that, by monotonicity,

$$\int_0^t \langle \beta_\varepsilon(c_\varepsilon^n), Bc_\varepsilon^n \rangle \, ds = \int_0^t \int_\Omega \beta_\varepsilon'(c_\varepsilon^n)|\nabla c_\varepsilon^n|^2 \, dx \, ds \geq 0 \tag{4.33}$$

and that

$$\Big| \int_0^t \langle w_\varepsilon^n, Bc_\varepsilon^n \rangle \, ds \Big| \leq \|\nabla w_\varepsilon^n\|_{L^2(Q_t)} \|\nabla c_\varepsilon^n\|_{L^2(Q_t)} \leq k. \tag{4.34}$$

We point out that this estimate depends just on the L^2-norm of ∇w_ε^n and not on the full V-norm of w_ε^n; in particular, the k on the right hand side does not explode as $\varepsilon \searrow 0$.

Finally, we observe that, if $v \in V_\infty$ is fixed, system (4.9–4.11) surely makes sense for sufficiently large n. Then, by the density of V_∞ in V, in the limit we are allowed to take any $v \in V$ as a test function. As for the Cauchy conditions (4.8), they pass to the limit too, since (4.7), (4.24) and the second of (4.30) hold.

Passage to the limit as $\varepsilon \searrow 0$. We would like to repeat the procedure above; however, we can no longer take advantage of the Lipschitz continuity of β_ε and in particular of the bound (4.20). Hence, we have to perform a new estimate of $\beta_\varepsilon(c_\varepsilon)$, consisting in a variant of an argument devised in [23, Lemma 5.2]. Namely, we set $x_\varepsilon := (\beta_\varepsilon(c_\varepsilon))_\Omega$ and take $v = \beta_\varepsilon(c_\varepsilon) - x_\varepsilon$ in the n-limit of (4.11). Moreover, we choose $v = \mathcal{N}_{(\varphi_\varepsilon, c_\varepsilon)}\big(\beta_\varepsilon(c_\varepsilon) - x_\varepsilon\big)$ in (4.10), subtract from the above, and integrate over $(0,t)$, where $t \leq T$. Proceeding as in [16, Sec. 4], we can prove that

$$\|\beta_\varepsilon(c_\varepsilon) - x_\varepsilon\|_{L^2(Q)}^2 \leq k. \tag{4.35}$$

Note in particular that, by Lemma 2.3, (4.12), and the continuous embedding $V \subset H$,

$$\int_0^t \big\langle \partial_t c_\varepsilon, \mathcal{N}_{(\varphi_\varepsilon, c_\varepsilon)}\big(\beta_\varepsilon(c_\varepsilon) - x_\varepsilon\big)\big\rangle \, ds \leq \frac{1}{4}\|\beta_\varepsilon(c_\varepsilon) - x_\varepsilon\|_{L^2(Q_t)}^2 + k\|\partial_t c_\varepsilon\|_{L^2(0,t;V')}^2$$

and the latter quantity is bounded by (4.29).

The second part of the procedure consists in the estimation of the average x_ε, that can be performed exactly as in [7, Subsec. 5.3]. Observe that, at this step, the hypothesis (3.7) turns out to be crucial. This means that, in the physical case $\beta = \partial I_{[0,1]}$, we cannot start by concentrations c_0 a.e. equal to 0 or to 1. By this argument, it follows the existence of a function ξ such that

$$\beta_\varepsilon(c_\varepsilon) \to \xi \qquad \text{weakly in } L^2(0,T;H). \tag{4.36}$$

Now, we can improve the bound (4.20) by taking $v = w_\varepsilon$ in the n-limit of (4.11). We readily obtain

$$\|w_\varepsilon\|^2_{L^2(Q)} \le k, \tag{4.37}$$

where k does no longer depend on ε, thanks to (4.36). Now the passage to the limit as $\varepsilon \searrow 0$ can be performed as in the previous subsection and suitable limit functions (φ, c, w) are obtained as limits of $(\varphi_\varepsilon, c_\varepsilon, w_\varepsilon)$, respectively. The only difference concerns the identification of the function ξ. Indeed, by (4.36), the analogue of (4.24), namely

$$c_\varepsilon \to c \qquad \text{strongly in } C^0([0,T];H), \tag{4.38}$$

and the monotonicity argument of [2, Prop. 1.1, p. 42], one can actually prove (3.16). Finally, to conclude the proof of Theorem 3.1, it suffices to show (3.18) and, of course, it is enough to take $v = 1 \in V_1$ in (4.10) of the approximate statement, and pass to the limit in ε, n.

5 Regularity and uniqueness

Proof of Theorem 3.3. We derive additional a priori estimates, independent of ε, n, for the solutions of the system (4.9–4.11). We just give the highlights of this procedure, since it is rather similar to the analogous argument in [31, Sec. 3]. Thus, we first have to take $v = \partial_t \varphi^n_\varepsilon$ on the right hand side of (4.9) and integrate over $(0,t)$, for $t \le T$. Then, performing standard integrations by parts, using the uniform boundedness of F_1 and F_2 and (4.24), and taking advantage of (3.19), it is easy to arrive at

$$\|\varphi^n_\varepsilon\|_{H^1(0,T;H)} + \|\varphi^n_\varepsilon\|_{L^\infty(0,T;V)} \le k. \tag{5.1}$$

Of course, to make the procedure rigorous, we have to improve (4.7), by requiring

$$\varphi_{0n} \in V_n, \qquad \varphi_{0n} \to \varphi_0 \quad \text{in } V. \tag{5.2}$$

Finally, choosing $v = B\varphi^n_\varepsilon$ in (4.9) and working as above, we get the bound

$$\|B\varphi^n_\varepsilon\|_{L^2(0,T;H)} \le k. \tag{5.3}$$

The above relations can be taken into account in order to get the convergences yielding (3.20) at the limit. Indeed, the H^2-regularity follows from (5.3) and the standard elliptic regularity theorems, while the $C^0([0,T];V)$ regularity in (3.20) is a consequence of, e.g., [1, Lemma 6.3].

Proof of Theorem 3.4. Again, we proceed similarly to [31, Sec. 4]. Anyway, our conditions on c^n_ε are slightly different from those of [31]; so, we briefly detail the computations. Thus, we take $v = B^2\varphi^n_\varepsilon$ in (4.9) and integrate again over $(0,t)$. Thanks to (4.1), the Gauss-Green formula can be used to obtain

$$\frac{1}{2}|B\varphi^n_\varepsilon(t)|^2 + \|\nabla B\varphi^n_\varepsilon\|^2_{L^2(Q_t)} \le \frac{1}{2}|B\varphi_{0n}(t)|^2 + \int_0^t \langle F_1(\varphi^n_\varepsilon) + c^n_\varepsilon F_2(\varphi^n_\varepsilon), B^2\varphi^n_\varepsilon\rangle\, ds \tag{5.4}$$

and we have to estimate the right hand side above. The initial datum can be controlled just by postulating

$$\varphi_{0n} \in V_n, \qquad \varphi_{0n} \to \varphi_0 \quad \text{in } H^2(\Omega). \tag{5.5}$$

Furthermore, integrating by parts the latter term in (5.4) and using (3.2) and (4.12), we easily get

$$\begin{aligned}\int_0^t \langle F_1(\varphi_\varepsilon^n) + c_\varepsilon^n F_2(\varphi_\varepsilon^n), B^2\varphi_\varepsilon^n\rangle \, ds &\le \frac{1}{2}\|\nabla B\varphi_\varepsilon^n\|^2_{L^2(Q_t)} \\ &+ k\|\nabla\varphi_\varepsilon^n\|^2_{L^2(Q_t)} + k\|\nabla c_\varepsilon^n\|^2_{L^2(Q_t)} + k\int_0^t\int_\Omega |c_\varepsilon^n|^2|\nabla\varphi_\varepsilon^n|^2\,dx\,ds. \end{aligned}\tag{5.6}$$

Of course, recalling (4.21–4.22), we just have to control the last term. Thus, using some standard three dimensional embedding theorems, we get

$$\begin{aligned} k\int_0^t\int_\Omega |c_\varepsilon^n|^2|\nabla\varphi_\varepsilon^n|^2\,dx\,ds &\le k\int_0^t \|c_\varepsilon^n(s)\|^2_{L^4(\Omega)}\|\nabla\varphi_\varepsilon^n(s)\|^2_{L^4(\Omega)}\,ds \\ &\le k\int_0^t \|c_\varepsilon^n(s)\|^2_{H^1(\Omega)}\|\varphi_\varepsilon^n(s)\|^2_{H^2(\Omega)}\,ds \\ &\le k\int_0^t \|c_\varepsilon^n(s)\|^2_{H^1(\Omega)}|\varphi_\varepsilon^n(s)|^2\,ds + k\int_0^t \|c_\varepsilon^n(s)\|^2_{H^1(\Omega)}|B\varphi_\varepsilon^n(s)|^2\,ds. \end{aligned}\tag{5.7}$$

Now, the first integral on the right hand side is bounded by (4.21–4.22), while we can control the latter with the first term on the left hand side of (5.4) if we take advantage of the second of (4.22) and of the Gronwall lemma in the form of [9, Lemme A.4, p. 156].

Then, passing to the limit we derive the third and the fourth of (3.22). Now, we have to choose $v = \partial_t B\varphi_\varepsilon^n$ in (4.9) and integrate again in time. Proceeding as before, we obtain

$$\|\partial_t\nabla\varphi_\varepsilon^n\|^2_{L^2(Q_t)} + \frac{1}{2}|B\varphi_\varepsilon^n(t)|^2 \le \frac{1}{2}|B\varphi_{0n}(t)|^2 + \int_0^t \langle F_1(\varphi_\varepsilon^n) + c_\varepsilon^n F_2(\varphi_\varepsilon^n), \partial_t B\varphi_\varepsilon^n\rangle\,ds \tag{5.8}$$

and we readily see that the right hand side can be estimated as above. In particular, working as in (5.6–5.7), we arrive at

$$\begin{aligned}&\int_0^t\int_\Omega c_\varepsilon^n F_2'(\varphi_\varepsilon^n)\nabla\varphi_\varepsilon^n \cdot \partial_t\nabla\varphi_\varepsilon^n\,dx\,ds \\ &\qquad \le \frac{1}{4}\|\partial_t\nabla\varphi_\varepsilon^n\|^2_{L^2(Q_t)} + k\int_0^t \|c_\varepsilon^n(s)\|^2_{H^1(\Omega)}\big(|\varphi_\varepsilon^n(s)|^2 + |B\varphi_\varepsilon^n(s)|^2\big)\,ds. \end{aligned}\tag{5.9}$$

Then, (5.8) yields the second regularity in (3.22). To prove the first one, we differentiate in time (4.9) and test the result by $\partial_t\varphi_\varepsilon^n$. Note that this procedure is rigorous. Indeed, referring to the formulation (4.3), we see that the right hand side is at least locally $C^{0,1}$. Thus, we get

$$\begin{aligned}\frac{1}{2}|\partial_t\varphi_\varepsilon^n(t)|^2 + \|\partial_t\nabla\varphi_\varepsilon^n\|^2_{L^2(Q_t)} &\le \frac{1}{2}|\partial_t\varphi_\varepsilon^n(0)|^2 + \int_0^t \langle F_1'(\varphi_\varepsilon^n)\partial_t\varphi_\varepsilon^n, \partial_t\varphi_\varepsilon^n\rangle\,ds \\ &+ \int_0^t \langle c_\varepsilon^n F_2'(\varphi_\varepsilon^n)\partial_t\varphi_\varepsilon^n, \partial_t\varphi_\varepsilon^n\rangle\,ds + \int_0^t \langle \partial_t c_\varepsilon^n F_2(\varphi_\varepsilon^n), \partial_t\varphi_\varepsilon^n\rangle\,ds \end{aligned}\tag{5.10}$$

and we have to bound the four terms on the right hand side. As for the initial datum, we note that, by (4.9), (5.5), the second of (4.7), and the boundedness of F_1 and F_2,

$$|\partial_t\varphi_\varepsilon^n(0)|^2 \le k\big(|B\varphi_{0n}|^2 + |c_{0n}|^2 + 1\big) \le k. \tag{5.11}$$

By Hölder's inequality and the continuous embedding $H^2(\Omega) \subset L^\infty(\Omega)$, we see that the first couple of integrals $I_1 + I_2$ can be estimated as follows:

$$|I_1 + I_2| \le k \int_0^t \big(1 + \|c_\varepsilon^n(s)\|_{H^2(\Omega)}\big)|\partial_t\varphi_\varepsilon^n(s)|^2\,ds. \tag{5.12}$$

The latter integral I_3 on the right hand side of (5.10) has to be split in the duality between V' and V and gives

$$\begin{aligned}|I_3| &\le \frac{1}{2}\|\partial_t c_\varepsilon^n\|^2_{L^2(0,t;V')} + \frac{1}{2}\|F_2(\varphi_\varepsilon^n)\partial_t\varphi_\varepsilon^n\|^2_{L^2(0,t;V)}\\ &\le k + k\|\partial_t\varphi_\varepsilon^n\|^2_{L^2(0,t;V)} + k\int_0^t\int_\Omega |\nabla\varphi_\varepsilon^n|^2|\partial_t\varphi_\varepsilon^n|^2\,dx\,ds \qquad (5.13)\\ &\le k + k\int_0^t \|\varphi_\varepsilon^n\|^2_{H^3(\Omega)}|\partial_t\varphi_\varepsilon^n(s)|^2\,dx\,ds. \qquad (5.14)\end{aligned}$$

Thus, collecting (5.10–5.14) and using the third of (3.10) and the fourth of (3.22), we see that Gronwall's lemma applies to $t \mapsto |\partial_t\varphi_\varepsilon^n(t)|^2$, so that the proof of Theorem 3.4 turns out to be complete.

We now come to the "physical" interpretation of the solution and assume (3.27–3.28) in addition to (3.1–3.7). Then, we modify F_1 and F_2, by effecting a truncation. Namely, we set, for $i = 1, 2$,

$$\widetilde{F}_i(r) := 0 \quad \text{if } r < 0 \text{ or } r > 1, \qquad \widetilde{F}_i(r) := F_i(r) \quad \text{if } 0 \le r \le 1. \tag{5.15}$$

Thanks to (3.27), it is clear that $\widetilde{F}_1$, $\widetilde{F}_2$ still satisfy (3.1–3.2). Thus, Theorem 3.1 guarantees the existence of a solution to the system (3.13–3.17), where $\widetilde{F}_1$, $\widetilde{F}_2$ replace F_1, F_2 in (3.13). Now, we state a maximum principle argument ensuring that, under the regularity assumptions of Theorem 3.1, any solution to the modified system (3.13–3.17) is actually a solution also to the original one.

Lemma 5.1. *Under the assumptions* (3.1–3.7) *and* (3.27–3.28), *the component* φ *of any solution to* (3.13–3.17) – *with* $\widetilde{F}_1$, $\widetilde{F}_2$ *in place of* F_1, F_2 – *satisfies*

$$0 \le \varphi \le 1 \qquad \text{a.e. in } Q. \tag{5.16}$$

We do not report the proof of this lemma, that is rather standard and actually identical to the proof of [31, Thm. 3]. Anyway, we note that Theorem 3.6 follows as an easy consequence. Moreover, we observe that of course assumption (3.8) is not required in the above result, that holds for any g satisfying (3.3).

Remark 5.2. It is worthwhile to discuss an important consequence of the above property. Of course, we would like to prove the well-posedness of the system (3.13–3.17) in the physical case ensuring the bounds $0 \le \varphi, c \le 1$ a.e. in Q. The above

Lemma guarantees that any solution of the truncated system is a solution of the original one in the very general regularity setting of Theorem 3.1. Of course, if we were able to show the uniqueness of the solution to the original system, this would mean that this unique solution is physically meaningful, since it has to coincide with a solution of the truncated system, that surely exists. However, the uniqueness result provided by Theorem 3.5 holds just in case μ is a constant function. Hence, we cannot exclude that there exists some solution to (3.13–3.17), whose component φ attains its values also outside $[0,1]$.

Proof of Theorem 3.5. Let us set $\varphi := \varphi_1 - \varphi_2$, $c := c_1 - c_2$, $w := w_1 - w_2$, $\xi := \xi_1 - \xi_2$, $\varphi_0 := \varphi_{0,1} - \varphi_{0,2}$, $c_0 := c_{0,1} - c_{0,2}$. Then, write (3.13) firstly for φ_1, c_1 and then for φ_2, c_2, take the difference, multiply it by φ, and integrate over $(0,t)$, for $t \le T$. Then, owing to (3.1–3.2), it is easy to get

$$\frac{1}{2}|\varphi(t)|^2 + \|\nabla\varphi\|^2_{L^2(Q_t)} \le \frac{1}{2}|\varphi_0|^2 + \frac{3K}{2}\int_0^t |\varphi(s)|^2\,ds + \frac{K}{2}\int_0^t |c(s)|^2\,ds + \int_0^t\int_\Omega |c_1|\,|F_2(\varphi_1) - F_2(\varphi_2)|\,|\varphi|\,dx\,ds \tag{5.17}$$

and by the continuous embedding $H^2(\Omega) \subset L^\infty(\Omega)$, holding for $d \le 3$, the latter integral can be estimated as

$$\le K\int_0^t \|c_1(s)\|_{L^\infty(\Omega)}|\varphi(s)|^2\,ds \le k\int_0^t \|c_1(s)\|_{H^2(\Omega)}|\varphi(s)|^2\,ds, \tag{5.18}$$

where K is as in (3.2). Now, take the difference of (3.15) written for the two solutions, multiply it by c, and integrate again over $(0,t)$. Then, note that, by (3.23) and (3.18), $c(s) \in V_0$ for a.e. $s \in [0,T]$ and it makes sense to test the difference of the relations (3.14) by $\mathcal{N}c$. Moreover, by using Lemmas 2.1, 2.2 it is not difficult to prove that

$$\int_0^t \langle c_t, \mathcal{N}c\rangle \ge \frac{C_{\alpha,\Omega,\mu}}{2}\|c(t)\|^2_{V'} - \frac{C_{\alpha,\Omega}}{2}\|c_0\|^2_{V'}.$$

Thus comparing with the relation obtained from (3.15), exploiting the monotoniticy of β, and noting that two terms cancel, we readily obtain

$$\frac{C_{\alpha,\Omega,\mu}}{2}\|c(t)\|^2_{V'} + \|\nabla c\|^2_{L^2(Q_t)} \le \frac{C_{\alpha,\Omega}}{2}\|c_0\|^2_{V'} + k\|\varphi\|^2_{L^2(Q_t)} + k\|c\|^2_{L^2(Q_t)}. \tag{5.19}$$

Then, we note that, by the compact embedding $V \subset H$ and the inequality (2.9), for any $\sigma > 0$ there exists $k_\sigma > 0$ such that

$$\|c\|^2_{L^2(Q_t)} \le \sigma\|\nabla c\|^2_{L^2(Q_t)} + k_\sigma\|c\|^2_{L^2(0,t;V')}. \tag{5.20}$$

Then, summing together the expressions (5.17) and (5.19), taking (5.18) and (5.20) into account, choosing σ suitably small, and using hypothesis (3.25), we note that the Gronwall lemma in the form of, e.g., [9, Lemme A.4, p. 156] can be applied to the function

$$t \mapsto |\varphi(t)|^2 + \|c(t)\|^2_{V'},$$

so that relation (3.26) can be inferred by standard considerations.

Remark 5.3. In the physical case (3.32), or just assuming that $D(\beta)$ is bounded, assumption (3.25) can be avoided, since one can just take $R = \sup\{|r|,\ r \in D(\beta)\}$ to estimate the term in (5.18).

6 Sharp interface limit

We now perturb the system (3.13–3.17) and study a relate singular limit problem, whose physical relevance has been outlined in the Introduction. Henceforth, we assume that the function μ defined in (2.4) depends only on φ, as in the original thermodynamical setting, take a parameter $\lambda > 0$ that is supposed to go to 0 in the limit, and consider a solution $(\varphi_\lambda, c_\lambda, w_\lambda, \xi_\lambda)$ to the system

$$\partial_t \varphi_\lambda + B\varphi_\lambda = F_1(\varphi_\lambda) + c_\lambda F_2(\varphi_\lambda) \qquad \text{in } V', \quad \text{a.e. in } (0,T), \tag{6.1}$$

$$\partial_t c_\lambda + B_{\varphi_\lambda} w_\lambda = 0 \qquad \text{in } V', \quad \text{a.e. in } (0,T), \tag{6.2}$$

$$w_\lambda = \lambda B c_\lambda + \xi_\lambda + \lambda\gamma(c_\lambda) + g(\varphi_\lambda) \qquad \text{in } V', \quad \text{a.e. in } (0,T), \tag{6.3}$$

$$\xi_\lambda \in \beta(c_\lambda) \qquad \text{a.e. in } Q, \tag{6.4}$$

$$\varphi_\lambda(\cdot,0) = \varphi_0(\cdot), \quad c_\lambda(\cdot,0) = c_0(\cdot) \qquad \text{a.e. in } \Omega. \tag{6.5}$$

Here, φ_0, c_0, F_1, F_2, γ, g, and β are as in Theorem 3.1 that, of course, can be used to deduce the existence of such a solution. Moreover, the operator B_{φ_λ} is defined as in (2.5) with the only difference that it has to depend only on one function φ_λ. Hence, also the inverse map $\mathcal{N}_{\varphi_\lambda}$ of V_0' to V_0 retains the natural properties stated in Section 2. We have the following result,

Theorem 6.1. *Beyond the above stated hypotheses, let us assume that there exist* $k_1, k_2 > 0$ *such that*

$$\psi(r) \geq k_1 r^2 - k_2 \qquad \forall\, r \in D(\psi). \tag{6.6}$$

Then, there exists a quadruple (φ, c, w, ξ)*, such that, as* $\lambda \searrow 0$*, the following relations hold,*

$$\varphi_\lambda \to \varphi \qquad \text{weakly star in } H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{6.7}$$

$$c_\lambda \to c \qquad \text{weakly star in } H^1(0,T;V') \cap L^\infty(0,T;H), \tag{6.8}$$

$$\lambda c_\lambda \to 0 \qquad \text{strongly in } L^2(0,T;V) \quad \text{and weakly in } L^2(0,T;H^2(\Omega)), \tag{6.9}$$

$$w_\lambda \to w \qquad \text{weakly in } L^2(0,T;V), \tag{6.10}$$

$$\xi_\lambda \to \xi \qquad \text{weakly in } L^2(0,T;H). \tag{6.11}$$

Moreover, the quadruple (φ, c, w, ξ) *satisfies*

$$\partial_t \varphi + B\varphi = F_1(\varphi) + cF_2(\varphi) \qquad \text{in } V', \quad \text{a.e. in } (0,T), \tag{6.12}$$

$$\partial_t c + B_\varphi w = 0 \qquad \text{in } V', \quad \text{a.e. in } (0,T), \tag{6.13}$$

$$w = \xi + g(\varphi) \qquad \text{a.e. in } Q, \tag{6.14}$$

$$\xi \in \beta(c) \qquad \text{a.e. in } Q, \tag{6.15}$$

$$\varphi(\cdot,0) = \varphi_0(\cdot), \quad c(\cdot,0) = c_0(\cdot) \qquad \text{a.e. in } \Omega. \tag{6.16}$$

Remark 6.2. We notice that (6.6) is a standard assumption, that ensures the well-posedness of this kind of degenerate parabolic systems [17, 22]. Of course, it is verified in the physical case (3.32), where the second of (6.8) can be straightforwardly improved as

$$c_\lambda \to c \quad \text{weakly star in } L^\infty(Q) \qquad \text{and} \quad 0 \le c \le 1 \ \text{ a.e. in } Q. \tag{6.17}$$

PROOF. As usual, we proceed by compactness methods and start by deriving a new a priori estimate. To this aim, test (6.1) by φ_λ and integrate as usual over Q_t, $t \le T$, easily obtaining

$$\frac{1}{2}|\varphi_\lambda(t)|^2 + \|\nabla\varphi_\lambda\|^2_{L^2(Q_t)} \le \frac{1}{2}|\varphi_0|^2 + k + k\|\varphi_\lambda\|^2_{L^2(Q_t)} + k\|c_\lambda\|^2_{L^2(Q_t)}. \tag{6.18}$$

Now, test (6.2) by w_λ and (6.3) by $\partial_t c_\lambda$, integrate over Q_t and sum the results. Note that this procedure makes sense as the right hand side of (6.3) lies in $L^2(0,T;V)$. Then, we formally have

$$\begin{aligned} &\alpha\|\nabla w_\lambda\|^2_{L^2(Q_t)} + \frac{\lambda}{2}|\nabla c_\lambda(t)|^2 + \int_\Omega \psi(c_\lambda(t))\,dx \\ &\quad \le \frac{\lambda}{2}|\nabla c_0|^2 + \int_\Omega \psi(c_0)\,dx + \lambda\int_0^t \langle\partial_t c_\lambda, \gamma(c_\lambda)\rangle\,ds + \int_0^t \langle\partial_t c_\lambda, g(\varphi_\lambda)\rangle\,ds. \end{aligned} \tag{6.19}$$

Note anyway that the integration in time of the term with ξ_λ has to be justified, and this can be done by applying Lemma 2.5 (in particular, the integration formula (2.15)) with the choices of

$$J(v) := \int_\Omega \Big(\psi(v(x)) + \frac{\lambda}{2}|\nabla v(x)|^2\Big)\,dx, \qquad \text{for } v \in H, \tag{6.20}$$

$$u := c_\lambda, \qquad \eta := w_\lambda - \lambda\gamma(c_\lambda) - g(\varphi_\lambda). \tag{6.21}$$

Indeed, it is easy to show that $\eta \in L^2(0,T;V)$ for all λ and that

$$\eta(t) \in \partial J(u(t)) \qquad \text{for a.e. } t \in (0,T).$$

Finally, (2.14) is a consequence of (6.6), so that the lemma can be applied.

Now, proceeding as in (4.17), one readily sees that there exists a constant $k_\alpha > 0$ such that

$$\frac{\alpha}{2}\|\nabla w_\lambda\|^2_{L^2(Q_t)} \ge k_\alpha\|\partial_t c_\lambda\|^2_{L^2(0,t;V')}. \tag{6.22}$$

Then, we have to split the last two integrals in (6.19) w.r.t. the duality (V',V). Namely, for some k^* dependent on α, but not on λ, we have

$$\lambda\int_0^t \langle\partial_t c_\lambda, \gamma(c_\lambda)\rangle\,ds \le \frac{k_\alpha}{4}\|\partial_t c_\lambda\|^2_{L^2(0,t;V')} + k^*\lambda^2\|c_\lambda\|^2_{L^2(0,t;V)} + k\lambda^2 \tag{6.23}$$

and

$$\int_0^t \langle\partial_t c_\lambda, g(\varphi_\lambda)\rangle\,ds \le \frac{k_\alpha}{4}\|\partial_t c_\lambda\|^2_{L^2(0,t;V')} + k^*\|\varphi_\lambda\|^2_{L^2(0,t;V)} + k. \tag{6.24}$$

Now, let us multiply (6.18) by $m > 0$ (to be chosen at the end), and sum the result to (6.19), so that, collecting also (6.22–6.24), we infer

$$\frac{m}{2}|\varphi_\lambda(t)|^2 + (m-k^*)\|\nabla\varphi_\lambda\|^2_{L^2(Q_t)} + \frac{\alpha}{2}\|\nabla w_\lambda\|^2_{L^2(Q_t)} + \frac{k_\alpha}{2}\|\partial_t c_\lambda\|^2_{L^2(0,t;V')} + \frac{\lambda}{2}|\nabla c_\lambda(t)|^2$$
$$+ \int_\Omega \psi(c_\lambda(t))\,dx \le \frac{m}{2}|\varphi_0|^2 + k(1+m+\lambda^2) + \frac{\lambda}{2}|\nabla c_0|^2 + \int_\Omega \psi(c_0)\,dx$$
$$+ \big(mk+k^*\big)\|\varphi_\lambda\|^2_{L^2(Q_t)} + \big(mk+k^*\lambda^2\big)\|c_\lambda\|^2_{L^2(Q_t)} + k^*\lambda^2\|\nabla c_\lambda\|^2_{L^2(Q_t)}. \tag{6.25}$$

Now, using (6.6), we immediately have

$$\int_\Omega \psi(c_\lambda(t))\,dx \ge k_1|c_\lambda(t)|^2 - k_2, \tag{6.26}$$

so that, upon choosing $m \ge 2k^*$, we see that, at least for λ sufficiently small, Gronwall's lemma applies once more to

$$t \mapsto |\varphi_\lambda(t)|^2 + |c_\lambda(t)|^2 + \lambda|\nabla c_\lambda(t)|^2,$$

so that (6.8), the second and the third of (6.7), and the first of (6.9) readily follow from (6.25). Moreover, the first of (6.7) can be deduced by a direct comparison in (6.1), while for the other relations it is necessary to repeat the argument leading to the estimation of ξ_λ and this can be performed as in Section 4, with minor modifications. Note indeed that in this setting the function β is no longer regular; hence, to integrate by parts the term with Bc_λ, Lemma 2.6 has to be used. Actually, this procedure gives relation (6.11). Now, to deduce (6.10), it suffices to estimate the $L^2(0,T;H)$ norm of w_λ. Thus, test (6.3) by w_λ, integrate over $(0,t)$, and note that

$$\left|\int_0^t \lambda\langle Bc_\lambda, w_\lambda\rangle\,ds\right| \le k\|\nabla w_\lambda\|^2_{L^2(Q_t)} + \lambda^2\|\nabla c_\lambda\|^2_{L^2(Q_t)} \le k, \tag{6.27}$$

thanks to (6.25) and the first of (6.9). At this point, (6.10) is a consequence of the other convergence relations and the second of (6.9) can be proved by a comparison in (6.3).

Finally, we have to show that the limit functions (φ, c, w, ξ) fulfill system (6.12–6.16) and actually this can be performed similarly as in the proof of Theorem 3.1, with the complication that (6.8) by [32, Cor. 4, Sec. 8] just implies

$$c_\lambda \to c \qquad \text{strongly in } C^0([0,T];V'); \tag{6.28}$$

so, we do not have pointwise convergence for c_λ. However, (6.7) yields

$$\varphi_\lambda \to \varphi \qquad \text{strongly in } L^2(0,T;H) \quad \text{and pointwise.} \tag{6.29}$$

Thus, the continuity and boundedness of F_2, Lebesgue's dominated convergence theorem, and the second of (6.8) entail

$$c_\lambda F_2(\varphi_\lambda) \to cF_2(\varphi) \qquad \text{weakly in } L^p(Q) \quad \text{for any } p \in [1,2) \tag{6.30}$$

and this permits us to pass to the limit in (6.1). Moreover, the passage to the limit in (6.2–6.3) does not present difficulties, since of course $\lambda\gamma(c_\lambda)$ tends to 0, e.g., strongly in $L^\infty(Q)$ and the Cauchy conditions (6.16) are recovered as in Section 4.

Thus, to conclude the proof, we just have to identify ξ, i.e., to show (6.15). Note that we cannot proceed as before, since we do not have the strong convergence of c_λ in $L^2(Q)$. Hence, we have to test again (6.3) by c_λ and integrate over $(0,T)$, deriving

$$\begin{aligned}\int_0^T (\xi_\lambda(t), c_\lambda(t))\,dt &= \int_0^T \langle c_\lambda(t), w_\lambda(t)\rangle\,dt - \lambda\int_0^T |\nabla c_\lambda(t)|^2\,dt\\ &\quad - \lambda\int_0^T\int_\Omega c_\lambda\gamma(c_\lambda)\,dx\,dt - \int_0^T\int_\Omega c_\lambda g(\varphi_\lambda)\,dx\,dt. \end{aligned}\tag{6.31}$$

Then, we take the lim sup of the relation above as $\lambda \searrow 0$ and notice that, thanks to (6.28) and (6.10), it is

$$\lim_{\lambda\searrow 0}\int_0^T \langle c_\lambda(t), w_\lambda(t)\rangle\,dt = \int_0^T \langle c(t), w(t)\rangle\,dt.$$

Consequently, using the strong convergence in (6.29) and performing a comparison in the already deduced relation (6.14), we derive

$$\begin{aligned}\limsup_{\lambda\searrow 0}\int_0^T (\xi_\lambda(t), c_\lambda(t))\,dt &\le \int_0^T \langle c(t), w(t)\rangle\,dt - \int_0^T\int_\Omega c g(\varphi)\,dx\,dt\\ &= \int_0^T (\xi(t), c(t))\,dt,\end{aligned}\tag{6.32}$$

so that (6.15) is once more a consequence of [2, Prop. 1.1, p. 42]. ∎

References

[1] C. Baiocchi, *Sulle equazioni differenziali astratte lineari del primo e del secondo ordine negli spazi di Hilbert*, Ann. Mat. Pura Appl. (4), **76** (1967), 233–304.

[2] V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach Spaces", Noordhoff, Leyden, 1976.

[3] V. Barbu, P. Colli, G. Gilardi, and M. Grasselli, *Existence, uniqueness, and long-time behavior for a nonlinear Volterra integrodifferential equation*, Differential Integral Equations, **13** (2000), 1233–1262.

[4] J. F. Blowey and C. M. Elliott, *The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy. I. Mathematical analysis*, European J. Appl. Math., **2** (1991), 233–280.

[5] J. F. Blowey and C. M. Elliott, *The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy. II. Numerical analysis*, European J. Appl. Math., **3** (1992), 147–179.

[6] E. Bonetti, P. Colli, W. Dreyer, G. Gilardi, G. Schimperna, and J. Sprekels, *A solid-solid phase change model accounting for mechanical effects*, paper in preparation.

[7] E. Bonetti, W. Dreyer, and G. Schimperna, *Global solution to a viscous Cahn-Hilliard equation for tin-lead alloys with mechanical stresses*, paper in preparation.

[8] H. Brezis, *Intégrales convexes dans les espaces de Sobolev*, Israel J. Math., **13** (1972), 9–23.

[9] H. Brezis, "Opérateurs Maximaux Monotones et Sémi-groupes de Contractions dans les Espaces de Hilbert", North-Holland Math. Studies, 5, North-Holland, Amsterdam, 1973.

[10] H. Brezis, "Analyse Fonctionnelle", Masson, Paris, 1983.

[11] H. Brezis and W. A. Strauss, *Semi-linear second-order elliptic equations in* L^1, J. Math. Soc. Japan, **25** (1973), 565–590.

[12] J. W. Cahn and J. E. Hilliard, *Free energy of a nonuniform system. I. Interfacial free energy*, J. Chem. Phys., **28** (1958), 258–267.

[13] G. Caginalp and W. Xie, *Phase-field and sharp-interface alloy models*, Phys. Rev. E (3), **48** (1993), 1897–1909.

[14] J. Cahn, C. M. Elliott, and A. Novick-Cohen, *The Cahn-Hilliard equation with a concentration dependent mobility: motion by minus the Laplacian of the mean curvature*, Europ. J. Appl. Math., **7** (1996), 287–301.

[15] S. Clain, *Elliptic operators of divergence type with Hölder coefficients in fractional Sobolev spaces*, Rend. Mat. Appl. (7), **7** (1997), 207–236.

[16] P. Colli, G. Gilardi, M. Grasselli, and G. Schimperna, *The conserved phase-field system with memory*, Adv. Math. Sci. Appl., to appear.

[17] A. Damlamian, *Some results on the multi-phase Stefan problem*, Comm. Partial Differential Equations, **2** (1977), 1017–1044.

[18] W. Dreyer and W. H. Müller, *A study of the coarsening in tin/lead solders*, International Journal of Solids and Structures, to appear.

[19] C. M. Elliott and H. Garcke, *On the Cahn-Hilliard equation with degenerate mobility*, Siam J. Math. Anal., **27** (1996), 404–423.

[20] H. Garcke, On mathematical models for phase separation in elastically stressed solids, Habilitation Thesis, Friedrich-Wilhelms University, Bonn, 2000.

[21] D. Kessler, O. Krüger, and J.-F. Scheid, *Construction d'un modèle de champ de phase à température homogène pour la solidification d'un alliage binaire* (French), internal report n. 10/98, EPFL–Lausanne, 1998.

[22] N. Kenmochi, *Neumann problems for a class of nonlinear degenerate parabolic equations*, Differential Integral Equations, **3** (1990), 253–273.

[23] N. Kenmochi, M. Niezgódka, and I. Pawlow, *Subdifferential operator approach to the Cahn-Hilliard equation with constraint*, J. Differential Equations, **117** (1995), 320–356.

[24] J. L. Lions, “Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires”, Dunod, Gauthier-Villars, Paris, 1969.

[25] J. L. Lions and E. Magenes, “Non-homogeneous Boundary Value Problems and Applications”, Vol. I, Springer-Verlag, Berlin, 1972.

[26] F. Luterotti and U. Stefanelli, *One dimensional results to the full model of phase transitions*, paper in preparation.

[27] F. Luterotti, G. Schimperna, and U. Stefanelli, *Global solution to a phase-field model with irreversible and constrained phase evolution*, Quart. Appl. Math., to appear.

[28] A. Miranville, *Équations de Cahn-Hilliard généralisées dans un milieu deformable*, C. R. Acad. Sci. Paris Sér. I Math., **328** (1999), 1095–1100.

[29] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa (3), **13** (1959), 115–162.

[30] Y. Oono and S. Puri, *Study of the phase separation dynamics by use of cell dynamical systems. I. Modelling*, Phys. Rev. (A), **38** (1988), 434–453.

[31] J. Rappaz and J.-F. Scheid, *Existence of solutions to a phase-field model for the isothermal solidification process of a binary alloy*, Math. Meth. Appl. Sci., **23** (2000), 491–513.

[32] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4), **146** (1987), 65–96.

[33] J. A. Warren and W. J. Boettinger, *Prediction of dendritic growth and microsegregation patterns in a binary alloy using the phase-field model*, Acta Metall. Mater., **43** (1995), 689–703.

[34] A. A. Wheeler, W. J. Boettinger, and G. B. McFadden *Phase-field model for isothermal phase transitions in binary alloys*, Phys. Rev. A, **45** (1992), 7424–7439.

[35] E. Zeidler “Nonlinear Functional Analysis and its Applications. II/A. Linear Monotone Operators”, Springer-Verlag, New York-Berlin, 1990.