Magnetic control of magnetic shape-memory single crystals

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Abstract— **We present a phenomenological model for the magneto-mechanical evolution of shape-memory alloy single crystals. The existence of solutions for given magnetic field is commented and optimal control results are established.**

I. Introduction

Shape-memory alloys (SMAs) are *active* materials: comparably large strains can be induced by either thermal or mechanical stimuli [21]. Some of these alloys (Ni_2MnGa, NiMnInCo, NiFeGaCo, FePt, FePd, among others) are referred to as *magnetic* shape-memory alloys (MSMAs) as they feature a specific ferromagnetic character which entails a remarkable magnetostrictive behavior. For instance, a Ni_2MnGa single crystal can develop up to a 10% strain (at a 1-3 MPa activation stress under the effect of a 1 T magnetic field) whereas a *TerFeNOL-D* polycrystal, one of the most performing *giant* magnetostrictive materials, shows a maximal 0.2% strain (at 60 MPa stress and 0.2 T field).

The magnetically induced strains in MSMAs are the macroscopic effect of the orientation of the ferromagnetic martensitic variants of the material. In particular, the martensitic phase in MSMAs presents the classical ferromagnetic texture of magnetic domains. This mesostructure changes under the influence of an external field by magnetic-domain wall motion, magnetization vector rotation, and magnetic-field driven martensitic-variant reorientation. The first two effects above are present in all ferromagnetic materials whereas martensitic-variant reorientation is specific of MSMAs. The Engineering literature on MSMAs is already quite vast. The Reader shall be referred, with no claim of completeness, to [15], [24], [25], [31], [40], [46], see also the review in [26].

We shall be describing the microscopic martensitic phase-fraction distribution of a MSMA single crystal by the vector $\boldsymbol{p} \in \mathbb{R}^v$ taking values in the simplex $S := \{p_i \geq 0,\ p_1 + \ldots + p_v \leq 1\}$. In particular, $\boldsymbol{p} = \boldsymbol{0}$ stands for a purely austenitic specimen whereas $\boldsymbol{p} \in \{p_1 + \ldots + p_v = 1\}$ means pure martensite and $\boldsymbol{p}/(p_1 + \ldots + p_v)$ represents the local martensitic variants distribution. We have specifically in mind the cases $v = 3$ and $v = 6$ which correspond to cubic-to-tetragonal (3 variants) cubic-to-orthorombic (6 variants) austenite-martensite systems. Within this frame, we might assume each martensitic variant to show a specific so-called *easy axis* of magnetization. In particular, we assume that the linear relation $\boldsymbol{p} \mapsto \boldsymbol{A}\boldsymbol{p}$ for given $\boldsymbol{A} \in \mathbb{R}^{d\times v}$ gives the (directed) easy axis of the phase distribution $\boldsymbol{p}$ (and $|\boldsymbol{A}\boldsymbol{p}| = 1$ for pure phases). Additionally, the *orientation* of the variants with respect to the easy axis will be determined by the scalar $\alpha \in [-1, 1]$. Our theory will rely on the *ansatz* that the magnetization $\boldsymbol{M}$ on the material is given by

$$\boldsymbol{M} = m_{\rm sat}\alpha \boldsymbol{A}\boldsymbol{p} \tag{1}$$

where $m_{\rm sat} > 0$ is the saturation magnetization. In particular, we assume that the magnetic anisotropy of the material is sufficiently strong so that the magnetization stays rigidly attached to the easy axes of the martensitic variants and no magnetization rotation occurs. This assumptions are in agreement with experiments on Ni_2MnGa [40], [46]. Still, the Reader is referred to [11] for the analysis of a more general version of this model including magnetization rotations and to [45] for a related optimal control result.

In order to describe the complex thermo-magneto-mechanical behavior of a MSMA single crystal we shall rely on the modelization in [1], [2], [10] which corresponds to an extension of the celebrated Souza-Auricchio model for SMAs [4], [5], [6], [43]. The latter is formulated within the frame of generalized plasticity and is characterized by a remarkable *simplicity* (few easily identifiable material parameters suffice in order to the describe a full 3D situation) and a *variational structure* (which entails robustness with respect to approximations and discretizations). Extensions of the Souza-Auricchio model as well as analytical results have been obtained in [3], [7], [8], [9], [16], [17], [18], [19], [22], [29], [30], [35], [36], [37], [34]. In particular, the present constitutive model for MSMAs single crystals has been proved to admit weak solutions when coupled with quasi-static mechanics [10], [11].

The giant magnetostrictive behavior of MSMA

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gives the unprecedented possibility of activating SMA devices (sensors, actuator, etc.) *at a distance* by tuning an external magnetic field. The aim of this paper is that of providing a first optimal control result in this direction. Namely, after having recalled the basic features of the MSMA model from [1], [2], [10] in Section II, we address an optimal control problem where the admissible control is a time-dependent imposed magnetic field and controllable quantities are the displacement $\boldsymbol{u}$ from the reference configuration and the phase vector $\boldsymbol{p}$. We shall first recall some existence theory for the state problem in Section III and finally provide our optimal controllability statement and proof in Section IV.

II. Constitutive material model

We shall recall here the basic features of the modelization from [1], [2], [10]. In the following bold latin letters stand for vectors in $\mathbb{R}^d$, $(d = 2, 3)$ bold greek symbols are for $\mathbb{R}^{d\times d}$ tensors, and we use the standard notation for scalar and contraction products. The space of symmetric tensors is $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\mathbb{R}^{d\times d}_{\mathrm{dev}}$ denotes its subspace of *deviatoric* elements.

Let $\boldsymbol{u}$ be the displacement of the body the reference configuration. Moving within the small-strain regime, the linearized strain $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)/2$ is additively decomposed as

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathbb{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_0(\boldsymbol{p}). \tag{2}$$

Here $\mathbb{C}$ (symmetric, positive definite) is the isotropic elasticity 4-tensor (assumed to be constant for all variants), $\boldsymbol{\sigma} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is the stress, and the linear map $\boldsymbol{p} \mapsto \boldsymbol{\varepsilon}_0(\boldsymbol{p}) \in \mathbb{R}^{d\times d}_{\mathrm{dev}}$ represents the stress-free configuration corresponding to the phase distribution $\boldsymbol{p}$. In particular,

$$\boldsymbol{\varepsilon}_0(\boldsymbol{p}) := \boldsymbol{\varepsilon}_0^i p_i \tag{3}$$

(summation convention) where $\boldsymbol{\varepsilon}_0^i$ is the stress-free reference configuration of the i-th martensitic phase. In case $v = 3$ a standard choice for $d = 3$ is given by

$$\boldsymbol{\varepsilon}_0^i = \frac{\varepsilon_L}{\sqrt{6}}\Big(I - 3(\boldsymbol{e}^i \otimes \boldsymbol{e}^i)\Big) \tag{4}$$

where $\boldsymbol{e}^i$ is the unit vector of the i-th axis and $\varepsilon_L > 0$ represents the maximal strain modulus obtainable via martensitic variant reorientation. Note that, from relation (3), for every $\boldsymbol{p} \in S$ we have that $|\boldsymbol{\varepsilon}_0(\boldsymbol{p})| \le \varepsilon_L$.

Given the magnetic field $\boldsymbol{H}$ and the absolute temperature T, we shall prescribe the Gibbs free energy density of the material as

$$\begin{aligned} G(\boldsymbol{\sigma}, \boldsymbol{H}, T, \boldsymbol{p}, \alpha) &:= -\frac{1}{2}\boldsymbol{\sigma}{:}\mathbb{C}^{-1}\boldsymbol{\sigma} - \boldsymbol{\sigma}{:}\boldsymbol{\varepsilon}_0(\boldsymbol{p}) \\ &+ \beta(T)|\boldsymbol{\varepsilon}_0(\boldsymbol{p})| + \frac{h}{2}|\boldsymbol{\varepsilon}_0(\boldsymbol{p})|^2 + I_S(\boldsymbol{p}) \\ &+ \frac{1}{2\delta}\alpha^2 + I_{[-1,1]}(\alpha) - \mu_0 \boldsymbol{H}{\cdot}\alpha\, m_{\mathrm{sat}} \boldsymbol{A}\boldsymbol{p}. \end{aligned} \tag{5}$$

The first line in (5) correspond to classical linearized elasto-plasticity whereas the second line is the specific hardening choice of the Souza-Auricchio model. In particular, $T \mapsto \beta(T) \ge 0$ represents the critical yield stress for the austenite-martensite transition at temperature T, $h > 0$ is an isotropic hardening modulus, and I_S is the *indicator function* of the simplex S, namely $I_S(\boldsymbol{p}) = 0$ if $\boldsymbol{p} \in S$ and $I_S(\boldsymbol{p}) = \infty$ otherwise. From now on, we turn our attention to the isothermal situation by fixing the temperature to some value T^* and letting $\beta^* = \beta(T^*)$. We shall use the notation $F_{\mathrm{SA}}(\boldsymbol{p})$ for the whole second line in (5).

The third and final line in (5) describes the magnetic behavior of the material. The term $-\mu_0 \boldsymbol{H}{\cdot}\alpha\, m_{\mathrm{sat}} \boldsymbol{A}\boldsymbol{p}$ is the classical *Zeeman energy* term, namely $-\mu_0 \boldsymbol{H}{\cdot}\boldsymbol{M}$ (see (1)) and μ_0 is the magnetic permeability of vacuum. Note that $\boldsymbol{H}$ stands here for the *internal* magnetic field. Namely, $\boldsymbol{H}$ results from the sum of the applied external field and the corresponding induced demagnetization field. The indicator function $I_{[-1,1]}$ is constraining α to the interval $[-1, 1]$ and $1/\delta$ is a user-defined (dimensionalized in MPa) hardening parameter.

From the choice (5) of the Gibbs energy we derive the constitutive relations (1)-(2) as well as

$$\begin{aligned} \boldsymbol{\xi} \in -\partial_{\boldsymbol{p}} G &= \boldsymbol{\sigma}{:}\partial_{\boldsymbol{p}}\boldsymbol{\varepsilon}_0(\boldsymbol{p}) - \beta^* \partial_{\boldsymbol{p}}|\boldsymbol{\varepsilon}_0(\boldsymbol{p})| - \partial_{\boldsymbol{p}} I_S(\boldsymbol{p}) \\ &- h\boldsymbol{\varepsilon}_0(\boldsymbol{p}){:}\partial_{\boldsymbol{p}}\boldsymbol{\varepsilon}_0(\boldsymbol{p}) + \mu_0 \alpha m_{\mathrm{sat}}\, \boldsymbol{H}\boldsymbol{A}, \end{aligned} \tag{6}$$

$$\begin{aligned} \gamma \in -\partial_\alpha G &= -\alpha/\delta - \partial_\alpha I_{[-1,1]}(\alpha) \\ &+ \mu_0 m_{\mathrm{sat}} \boldsymbol{H}{\cdot}\boldsymbol{A}\boldsymbol{p} \end{aligned} \tag{7}$$

where $\boldsymbol{\xi} \in \mathbb{R}^v$ and $\gamma \in \mathbb{R}$ are the *thermodynamic forces* associated with the internal variables $\boldsymbol{p}$ and α, respectively (non-smooth but convex functions are subdifferentiated in the sense of Convex Analysis [12]). Note that we readily have that

$$(\partial_{\boldsymbol{p}}\boldsymbol{\varepsilon}_0(\boldsymbol{p}))_{ijk} = (\boldsymbol{\varepsilon}_0^k)_{ij}$$

$$(\partial_{\boldsymbol{p}}|\boldsymbol{\varepsilon}_0(\boldsymbol{p})|)_k = \frac{\boldsymbol{\varepsilon}_0(\boldsymbol{p}){:}\boldsymbol{\varepsilon}_0^k}{|\boldsymbol{\varepsilon}_0(\boldsymbol{p})|} \quad \text{for} \quad \boldsymbol{\varepsilon}_0(\boldsymbol{p}) \neq \boldsymbol{0}.$$

The evolution of the material is prescribed via a normality flow rule. We assume the behavior of $\boldsymbol{p}$ to be dissipative. In particular, we prescribe the von Mises-type yield function $F : \mathbb{R}^v \to \mathbb{R}$

$$F(\boldsymbol{\xi}) := |\boldsymbol{\xi}| - R$$

where $R > 0$ is the activation radius, and require $\dot{\boldsymbol{p}}$ to satisfy the flow rule and the complementary conditions

$$\dot{\boldsymbol{p}} = \dot{\zeta}\partial F(\boldsymbol{\xi}), \quad \dot{\zeta} \geq 0, \quad F \leq 0, \quad \dot{\zeta} F = 0.$$

The latter can be reformulated by means of the dissipation function $D(\dot{\boldsymbol{p}}) := R|\dot{\boldsymbol{p}}|$ as

$$\boldsymbol{\xi} \in \partial D(\dot{\boldsymbol{p}}). \tag{8}$$

On the other hand, we assume that α does not dissipate, namely $\gamma = 0$. This is of course disputable as the dissipation in α is the basic dissipative mechanism in ferromagnetic materials. Our assumption is however justified at the experimental level where it has been observed that the dissipation in α is negligible with respect to that in $\boldsymbol{p}$ [14], [27]. As α does not dissipate, it can be minimized out from the Gibbs energy (5) as

$$\alpha = \pi\big(\delta\mu_0 m_{\rm sat} \boldsymbol{H}\cdot\boldsymbol{A}\boldsymbol{p}\big)$$

where π stands for the projection on the interval $[-1, 1]$. In particular, by letting, for all $r \in \mathbb{R}$,

$$F_{\rm mag}(r) := \frac{1}{2\delta} \min\Big\{(\delta\mu_0 m_{\rm sat} r)^2, 2|\delta\mu_0 m_{\rm sat} r| - 1\Big\},$$

we can write the material constitutive relation (6)+(8) (now without α) as

$$\partial D(\dot{\boldsymbol{p}}) + \partial F_{\rm SA}(\boldsymbol{p}) - \partial_{\boldsymbol{p}} F_{\rm mag}(\boldsymbol{H}\cdot\boldsymbol{A}\boldsymbol{p}) \ni \boldsymbol{\sigma}{:}\partial_{\boldsymbol{p}}\boldsymbol{\varepsilon}_0(\boldsymbol{p}). \tag{9}$$

Let us now collect some remarks on the constitutive model (9). At first, note that, as $|\boldsymbol{A}\boldsymbol{p}| = 1$ for pure phases, we readily have from (1) that the natural constraint $|\boldsymbol{M}| \leq |m_{\rm sat}\boldsymbol{A}\boldsymbol{p}| = m_{\rm sat}|\boldsymbol{A}\boldsymbol{p}| \leq m_{\rm sat}$ is fulfilled for all $\boldsymbol{p} \in S$.

Secondly, one has to stress that in the purely martensitic phase $\boldsymbol{p} \in \{p_1 + \ldots + p_v = 1\}$, standard choices for $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{A}$ (see (3)-(4) for $v = 3$) entail that the model presents the so-called pairwise *magnetic compatibility* of energy wells [15]. In particular, in [1] we check that for all $1 \leq i, j \leq v$ there exist vectors $\boldsymbol{a}^{ij}$, $\boldsymbol{n}^{ij} \in \mathbb{R}^d$ such that

$$\boldsymbol{\varepsilon}_0^i - \boldsymbol{\varepsilon}_0^j = \frac{1}{2}(\boldsymbol{a}^{ij}\otimes\boldsymbol{n}^{ij} + \boldsymbol{n}^{ij}\otimes\boldsymbol{a}^{ij}), \tag{10}$$

$$\boldsymbol{A}\boldsymbol{p}_i - \boldsymbol{A}\boldsymbol{p}_j = \boldsymbol{n}^{ij}. \tag{11}$$

Condition (10) ensures that there exists a nontrivial continuous deformation such that $\boldsymbol{\varepsilon}(\boldsymbol{u})$ takes value in $\{\boldsymbol{\varepsilon}_0^i, \boldsymbol{\varepsilon}_0^j\}$. In particular, $\boldsymbol{a}^{ij}$, $\boldsymbol{n}^{ij}$ are the two possible normals to the discontinuity surface of the strain. On the other hand, condition (11) asserts that the interfaces with normal $\boldsymbol{n}^{ij}$ serve as pole-free surfaces of discontinuity of the magnetization.

From the purely mechanical viewpoint one has however to observe that the proposed model does not include the description of compatibility constraints between martensitic variants and austenite. In other words, the assumption $\boldsymbol{p} \in S$ is indeed a simplification as in the region $\{p_1 + \ldots + p_v < 1\}$ some phase proportions $\boldsymbol{p}$ are not accessible to real materials (this remark does not apply to the purely martensitic situation $p_1 + \ldots + p_v = 1$).

Finally, our model reproduces, although to some schematic extent, the *blocking stress* effect. Namely, no hard-axis magnetically-induced martensitic reorientation appears above a prescribed (and small) stress threshold [10].

Let us mention that different phenomenological models of internal-variable-type for MSMAs have been advanced by Hirsinger & Lexcellent [23] and Kiefer & Lagoudas [28]. Albeit basically informed by the same principles, these two models differ from the present one as they are essentially restricted to two dimensions (or two martensitic variants), assume the *scalar* local proportion of one martensitic variant with respect to the other as the relevant internal variable (whereas here we have a full vectorial description via $\boldsymbol{p}$), and rely on a considerably more complex choice of the Gibbs energy. In particular, the the referred models anisotropy is directly built in by means of the choice of specific anisotropic energy contributions.

Before closing this section let us motivate our interest for single-crystals modeling by remarking that MSMAs polycrystals, despite their relatively easier production process, have not been exploited yet in real devices. One can offer two possible motivations for this fact. First, a significant drop in the magnetostrictive behavior for polycrystals vs. single crystals is observed, see [13] for a discussion. This drop is probably the outcome of the relatively poor martensitic-variant structure of all MSMAs to date (tetragonal, orthorombic) and compatibility conditions for magnetizations at grain boundaries. Secondly, one shall observe that MSMA polycrystals of Ni_2MnGa developed so far turned out to be extremely brittle [44].

III. Existence for the state problem

We shall now assume to be given the internal magnetic field $(x, t) \mapsto \boldsymbol{H}(x, t) \in \mathbb{R}^d$ and solve the material constitutive relation (9) together with the quasistatic equilibrium system

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \ \text{ in } \ \Omega. \tag{12}$$

Here $\Omega \subset \mathbb{R}^d$ is a Lipschitz-bounded, connected open set and $\boldsymbol{f} : \Omega \to \mathbb{R}^d$ is a prescribed body force. Given $\Gamma_{\rm Dir} \cup \Gamma_{\rm tr} = \partial\Omega$ with $\Gamma_{\rm Dir}$ having positive surface measure, the system (12) is complemented by the boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \ \text{ on } \ \Gamma_{\rm Dir}, \quad \boldsymbol{\sigma n} = \boldsymbol{g} \ \text{ on } \ \Gamma_{\rm tr} \tag{13}$$

where $\boldsymbol{n}$ stands for the outward normal to $\partial\Omega$, and $\boldsymbol{g} : \Gamma_{\rm tr} \to \mathbb{R}^d$ is a given surface traction. The Dirichlet datum for $\boldsymbol{u}$ can be assumed non-homogeneous as well.

We shall define the state space $\mathcal{U}{\times}\mathcal{P} \ni (\boldsymbol{u}, \boldsymbol{p})$ as

$$\mathcal{U} := \{\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d) : \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_{\rm Dir}\}$$

and $\mathcal{P} = L^\infty(\Omega;\mathbb{R}^v)$. The *total load* $t \mapsto \boldsymbol{\ell}(t) \in \mathcal{U}'$ (dual) is given by

$$\langle \boldsymbol{\ell}(t), \boldsymbol{u}\rangle := \int_\Omega \boldsymbol{f}{\cdot}\boldsymbol{u} + \int_{\Gamma_{\rm tr}} \boldsymbol{g}{\cdot}\boldsymbol{u}.$$

We prescribe the *energy functional* $\mathcal{E} : [0,T]{\times}\mathcal{U}{\times}\mathcal{P}{\times} L^1(\Omega;\mathbb{R}^d) \to (-\infty,\infty]$ as

$$\mathcal{E}(t,\boldsymbol{u},\boldsymbol{p},\boldsymbol{H}) := \frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}){-}\boldsymbol{\varepsilon}_0(\boldsymbol{p}))^2 + \nu\,\mathrm{Var}(\boldsymbol{p}) + \int_\Omega \Big(F_{\rm SA}(\boldsymbol{p}){-}F_{\rm mag}(\boldsymbol{H}{\cdot}\boldsymbol{Ap})\Big) - \langle\boldsymbol{\ell}(t),\boldsymbol{u}\rangle.$$

Note that the energy contains an *interfacial* term where $\nu > 0$ is a scale parameter. In particular, the occurrence of such term penalizes phase interfaces. However, it still does not prevent $\boldsymbol{p}$ from possibly exhibiting jumps. This a particularly desirable feature in connection with shape-memory alloys where few-atoms-thick phase structures arise at the mesoscopic level. From the mathematical viewpoint, the interfacial energy term bears also a crucial compactifying effect.

Finally, the *dissipation* functional $\mathcal{D} : \mathcal{P}{\times}\mathcal{P} \to [0,\infty)$ is given by

$$\mathcal{D}(\boldsymbol{p}_1,\boldsymbol{p}_2) := R\int_\Omega |\boldsymbol{p}_1{-}\boldsymbol{p}_2|.$$

We are now in the position of presenting a weak formulation of the quasi-static evolution of system (9), (12)-(13) within the frame of *energetic solutions* à la Mielke [33], [39]. In particular, we define the set of *stable states* at time $t \in [0,T]$ and field $\boldsymbol{H} : \Omega \to \mathbb{R}^d$ as

$$\mathcal{S}(t,\boldsymbol{H}) := \big\{(\boldsymbol{u},\boldsymbol{p}) \in \mathcal{U}{\times}\mathcal{P} \ : \ \mathcal{E}(t,\boldsymbol{u},\boldsymbol{p},\boldsymbol{H}) < \infty,\ \mathcal{E}(t,\boldsymbol{u},\boldsymbol{p},\boldsymbol{H}) \le \mathcal{E}(t,\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}},\boldsymbol{H}) + \mathcal{D}(\boldsymbol{p},\widehat{\boldsymbol{p}})\ \ \forall(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}}) \in \mathcal{U}{\times}\mathcal{P}\big\}.$$

Hence, given the magnetic field $\boldsymbol{H} : \Omega\times[0,T] \to \mathbb{R}^d$ and the initial state $(\boldsymbol{u}^0,\boldsymbol{p}^0) \in \mathcal{U}{\times}\mathcal{P}$, an *energetic solution* is a trajectory $t \in [0,T] \mapsto (u(t),z(t)) \in \mathcal{U}{\times}\mathcal{P}$ such that $(u(0),z(0)) = (u^0,z^0)$, $t \mapsto \partial_t\mathcal{E}(t,\boldsymbol{u}(t),\boldsymbol{p}(t),\boldsymbol{H}(t))$ is integrable, and, for all $t \in [0,T]$, we have the two conditions:

Stability:

$$(\boldsymbol{u}(t),\boldsymbol{p}(t)) \in \mathcal{S}(t,\boldsymbol{H}(t)) \tag{14}$$

Energy balance:

$$\mathcal{E}(t,\boldsymbol{u}(t),\boldsymbol{p}(t),\boldsymbol{H}(t)) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{p},[0,t]) = \mathcal{E}(0,\boldsymbol{u}^0,\boldsymbol{p}^0,\boldsymbol{H}(0)) + \int_0^t \partial_t\mathcal{E}(s,\boldsymbol{u}(s),\boldsymbol{p}(s),\boldsymbol{H}(s))\mathrm{d}s \tag{15}$$

where the *total dissipation* of the process on the time interval $[s,t] \subseteq [0,T]$ be given by

$$\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{p},[s,t]) := \sup\sum_{i=1}^N \mathcal{D}(\boldsymbol{p}(t_{i-1}),\boldsymbol{p}(t_i)),$$

the sup being taken among all partitions $\{s = t_0 < t_1 < \cdots < t_N = t\}$.

We shall make the following assumptions on body force and traction:

$$(\boldsymbol{f},\boldsymbol{g}) \in W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^d){\times}L^2(\Gamma_{\rm tr};\mathbb{R}^d)) \tag{16}$$

Our result on the state problem reads as follows.

Theorem 1 (**Existence for the state problem)** Assuming (16), $\boldsymbol{H} \in W^{1,1}(0,T;L^1(\Omega;\mathbb{R}^d))$, and $(\boldsymbol{u}^0,\boldsymbol{p}^0) \in \mathcal{S}(0,\boldsymbol{H}(0))$, there exists an *energetic solution* $(\boldsymbol{u},\boldsymbol{p})$ of the state problem.

We shall not provide here a full proof of the latter result as the argument basically follows from the by-now classical existence analysis for energetic solutions [33]. In particular, an energetic solution may be obtained via passage to the limit within an implicit discretization procedure. Namely, by letting $\{0{=}t_0{<}t_1{<}\ldots{<}t_N{=}T\}$ be a given time-partition and defining $(\boldsymbol{u}_0,\boldsymbol{p}_0) = (\boldsymbol{u}^0,\boldsymbol{p}^0)$, we shall be interested in solving the following incremental problems

$$(\boldsymbol{u}_i,\boldsymbol{p}_i) \in \mathrm{Arg\,min}\big(\mathcal{E}(t_i,\boldsymbol{u},\boldsymbol{p},\boldsymbol{H}(t_i)) + \mathcal{D}(\boldsymbol{p}_{i-1},\boldsymbol{p})\big)$$

for all $i = 1,\ldots,N$, where the minimum is taken in $\mathcal{U}{\times}\mathcal{P}$. The (interpolant in time of the) time-discrete solution $\{(\boldsymbol{u}_i,\boldsymbol{p}_i)\}_{i=0}^N$ can then be proved to converge to a continuous energetic solution. Note however that the general method of [20] needs here a slight adaptation as the *power* of external actions

$$t \mapsto -\int_\Omega F'_{\rm mag}(\boldsymbol{H}(t){\cdot}\boldsymbol{Ap})\dot{\boldsymbol{H}}(t)\boldsymbol{A} - \langle\dot{\boldsymbol{\ell}},\boldsymbol{u}\rangle \tag{17}$$

need not be uniformly continuous. This modification is already mentioned on an abstract level on [41] and has been detailed for this model in [10].

We shall however give some detail on an a priori estimate on energetic solutions in terms of the magnetic field $\boldsymbol{H}$. Henceforth, C stands for a positive constant depending on data and may vary from line to line. From the energy balance (15) we have that, for all $t \in [0,T]$,

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}_0(\boldsymbol{p}(t)))^2 + \nu\,\mathrm{Var}(\boldsymbol{p}(t))\\
&+ \int_\Omega F_{\mathrm{SA}}(\boldsymbol{p}(t)) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{p},[0,t])\\
&= \mathcal{E}(t,\boldsymbol{u}(t),\boldsymbol{p}(t),\boldsymbol{H}(t)) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{p},[0,t])\\
&+ \int_\Omega F_{\mathrm{mag}}(\boldsymbol{H}(t)\cdot\boldsymbol{A}\boldsymbol{p}(t)) + \langle \boldsymbol{\ell}(t),\boldsymbol{u}(t)\rangle\\
&\overset{(15)}{=} \mathcal{E}(0,\boldsymbol{u}^0,\boldsymbol{p}^0,\boldsymbol{H}(0))\\
&- \int_0^t\int_\Omega F'_{\mathrm{mag}}(\boldsymbol{H}\cdot\boldsymbol{A}\boldsymbol{p})\dot{\boldsymbol{H}}\boldsymbol{A} - \int_0^t \langle \dot{\boldsymbol{\ell}},\boldsymbol{u}\rangle\\
&+ \int_\Omega F_{\mathrm{mag}}(\boldsymbol{H}(t)\cdot\boldsymbol{A}\boldsymbol{p}(t)) + \langle \boldsymbol{\ell}(t),\boldsymbol{u}(t)\rangle. \qquad (18)
\end{aligned}
$$

As F_{mag} is Lipschitz continuous and $\boldsymbol{p}(t) \in S$, the above right-hand side can be bounded, for all $\eta > 0$, by

$$
\begin{aligned}
&\eta\|\boldsymbol{u}(t)\|^2_{\mathcal{U}} + \frac{C}{\eta}\Big(1 + \|\boldsymbol{H}\|_{W^{1,1}(0,T;L^1(\Omega;\mathbb{R}^d))}\\
&\quad + \|\boldsymbol{\ell}(t)\|^2_{\mathcal{U}'} + \int_0^t \|\boldsymbol{\ell}\|_{\mathcal{U}'}\|\boldsymbol{u}\|_{\mathcal{U}}\Big).
\end{aligned}
$$

Hence, by choosing η small enough and applying Korn's inequality and Gronwall's lemma in (18), we deduce that

$$
\begin{aligned}
&\sup_{t\in[0,T]}\big(\|\boldsymbol{u}(t)\|^2_{\mathcal{U}} + \|\boldsymbol{p}(t)\|_{\mathcal{P}} + \mathrm{Var}(\boldsymbol{p}(t))\big)\\
&+ \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{p},[0,T])\\
&\le C\big(1 + \|\boldsymbol{H}\|_{W^{1,1}(0,T;L^1(\Omega;\mathbb{R}^d))}\big). \qquad (19)
\end{aligned}
$$

Before closing this section let us observe that, by letting the parameter $\delta \to 0$, one can rigorously prove that the present magnetic model reduces to the original non-magnetic Souza-Auricchio model [10]. This asymptotic limit argument can be ascertained via the Γ-convergence theory for rate-independent processes devised in [38].

IV. Optimal control

Theorem 1 ensures the energetic solvability of the quasi-static evolution problem for a given space- and time-dependent field $\boldsymbol{H}$. We shall denote by $\mathrm{Sol}(\boldsymbol{H})$ the set of all such energetic solutions. Now, let us assume to be able to control $\boldsymbol{H}$ in order to optimize a given cost functional. Note again that $\boldsymbol{H}$ is the *internal* magnetic field whereas some more natural control variable would be the *external* magnetic field instead. These two perspectives are indeed equivalent if we assume, given $(\boldsymbol{u},\boldsymbol{p})$, to be able to reconstruct in closed form the demagnetization field at every time. This is clearly too optimistic for the demagnetization tensor can be explicitly computed just in a few specific geometric situation and one would be forced to consider the coupling with the Maxwell system instead. We shall address this perspective in [45]. On the contrary, here we stick to this simplification by having in mind the case of relatively small displacements.

Given a set of admissible magnetic fields (controls) $\mathcal{H} \subset W^{1,1}(0,T;L^1(\Omega;\mathbb{R}^d))$, the *optimal control problem* consists in the minimization of a given cost functional

$$
\mathcal{J} : L^\infty(0,T;\mathcal{U}\times\mathcal{P}) \times \mathcal{H} \to (-\infty,\infty]
$$

which is depending on both the energetic solution and the control. Our problem is to find an *optimal control* $\boldsymbol{H}_* \in \mathcal{H}$ and a corresponding *optimal energetic solution* $(\boldsymbol{u}_*,\boldsymbol{p}_*) \in \mathrm{Sol}(\boldsymbol{H}_*)$ such that

$$
\begin{aligned}
(\boldsymbol{u}_*,\boldsymbol{p}_*) \in \mathrm{Arg\,Min}\Big\{&\mathcal{J}(\boldsymbol{u},\boldsymbol{p},\boldsymbol{H}) \ \text{ such that}\\
&(\boldsymbol{u},\boldsymbol{p}) \in \mathrm{Sol}(\boldsymbol{H}),\ \boldsymbol{H}\in\mathcal{H}\Big\}
\end{aligned}
$$

In order to possibly find optimal controls we shall consider the following standard requirements.

Compatibility of initial values and controls:

$$
(\boldsymbol{u}^0,\boldsymbol{p}^0) \in \mathcal{S}(0,\boldsymbol{H}(0)) \quad \forall \boldsymbol{H}\in\mathcal{H}. \qquad (20)
$$

Compactness of controls:

$$
\mathcal{H} \text{ is compact in } W^{1,1}(0,T;L^1(\Omega;\mathbb{R}^d)). \qquad (21)
$$

Lower semicontinuity of the cost functional:

$$
\begin{aligned}
&\Big(\boldsymbol{H}_n \to \boldsymbol{H} \text{ strongly in } W^{1,1}(0,T;L^1(\Omega;\mathbb{R}^d)),\\
&(\boldsymbol{u}_n,\boldsymbol{p}_n) \in \mathrm{Sol}(\boldsymbol{H}_n),\ (\boldsymbol{u}_n,\boldsymbol{p}_n) \to (\boldsymbol{u},\boldsymbol{p})\\
&\text{weakly-star in } L^\infty(0,T;\mathcal{U}\times\mathcal{P})\Big)\\
&\Rightarrow\ \mathcal{J}(\boldsymbol{u},\boldsymbol{p},\boldsymbol{H}) \le \liminf_{n\to\infty}\mathcal{J}(\boldsymbol{u}_n,\boldsymbol{p}_n,\boldsymbol{H}_n). \qquad (22)
\end{aligned}
$$

The compatibility condition in (20) was already presented in [41] and is just intended to ensure that the initial values are stable regardless of the choice of the control. In case all $\boldsymbol{H}\in\mathcal{H}$ share the same initial value (which is somehow natural in applications where $\boldsymbol{H}(0) = \boldsymbol{0}$ is usually taken) the compatibility condition (20) reduces to the purely mechanical stability of the initial state. The compactness of $\mathcal{H}$ from (21) is here chosen just for the sake of simplicity. In particular it can be relaxed by asking extra coercivity on the functional $\mathcal{J}$. The lower semicontinuity requirement in (22) is standard.

Let us now provide a first illustration of a possible *quadratic* cost functional covered by this theory. Indeed, we could consider

$$\mathcal{J}(\boldsymbol{u},\boldsymbol{p},\boldsymbol{H}) = \int_0^T |\boldsymbol{u}-\boldsymbol{u}_{\rm d}|^2 + \int_0^T |\boldsymbol{p}-\boldsymbol{p}_{\rm d}|^2 \\ + |\boldsymbol{u}(T)-\boldsymbol{u}_{\rm f}|^2 + |\boldsymbol{p}(T)-\boldsymbol{p}_{\rm f}|^2$$

where $(\boldsymbol{u}_{\rm d},\boldsymbol{p}_{\rm d}) \in L^2(0,T;L^2(\Omega;\mathbb{R}^d\times\mathbb{R}^v))$ are given displacement and phase-distribution profiles whereas $(\boldsymbol{u}_{\rm f},\boldsymbol{p}_{\rm f}) \in L^2(\Omega;\mathbb{R}^d\times\mathbb{R}^v)$ are given target states. Note that the latter functional is not lower semicontinuous with respect to the weak-star topology in $L^\infty(0,T;\mathcal{U}\times\mathcal{P})$. Still, it fulfills (22) as a result of the requirement $(\boldsymbol{u}_n,\boldsymbol{p}_n) \in \mathrm{Sol}(\boldsymbol{H}_n)$ which indeed provides additional compactness. In particular, as a consequence of the strong convergence $\boldsymbol{H}_k \to \boldsymbol{H}$ in $W^{1,1}(0,T;L^1(\Omega;\mathbb{R}^d))$ one has that (a suitable subsequence of) the corresponding solutions $(\boldsymbol{u}_k,\boldsymbol{p}_k) \in \mathrm{Sol}(\boldsymbol{H}_k)$ weakly star converge *pointwise* in $\mathcal{U}\times\mathcal{P}$.

Our optimal controllability statement reads as follows.

Theorem 2 (**Existence of optimal controls)** Under assumptions (16) and (20)-(22) there exists an optimal control.

We provide here a direct proof of this theorem. Still, one shall mention that the result could be derived as a consequence of the abstract theory by Rindler [41] as well.

Proof: Let $(\boldsymbol{u}_k,\boldsymbol{p}_k,\boldsymbol{H}_k)$ be a minimizing sequence for the functional $\mathcal{J}$. Namely $(\boldsymbol{u}_k,\boldsymbol{p}_k) \in \mathrm{Sol}(\boldsymbol{H}_k)$ and

$$\mathcal{J}(\boldsymbol{u}_k,\boldsymbol{p}_k,\boldsymbol{H}_k) \to \inf\{\mathcal{J}(\boldsymbol{u},\boldsymbol{p},\boldsymbol{H}) \mid (\boldsymbol{u},\boldsymbol{p}){\in}\mathrm{Sol}(\boldsymbol{H})\}.$$

Owing to the compactness (20) we extract a (not relabeled) subsequence such that $\boldsymbol{H}_k \to \boldsymbol{H}_*$ strongly in $W^{1,1}(0,T;L^1(\Omega;\mathbb{R}^d))$. Now, taking into account the estimate (19), one has that $(\boldsymbol{u}_k,\boldsymbol{p}_k)$ are uniformly bounded in $\mathcal{U}\times\mathcal{P}$ and that $\sup_k \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{p}_k,[0,T]) < \infty$. Hence, by Helly's selection theorem [32] we can extract again (still not relabeling) in such a way that

$$\begin{aligned}
\boldsymbol{p}_k(t) \to \boldsymbol{p}_*(t) \quad & \text{weakly in } \; BV(\Omega;\mathbb{R}^v)\\
& \text{and strongly in } \; L^1(\Omega;\mathbb{R}^v),\\
\boldsymbol{p}_k \to \boldsymbol{p}_* \quad & \text{strongly in } \; L^q(\Omega\times(0,T))\\
& \forall q\in[1,\infty),\\
\boldsymbol{u}_k(t) \to \boldsymbol{u}_*(t) \quad & \text{weakly in } \; H^1(\Omega;\mathbb{R}^d).
\end{aligned}$$

In particular, the pointwise convergence of $\boldsymbol{u}_k$ is obtained by stability (14) via the quadratic character of $\boldsymbol{u} \mapsto \mathcal{E}(t,\boldsymbol{u},\boldsymbol{p},\boldsymbol{H})$. In fact, there exists a linear operator $\mathcal{L}(t) : L^2(\Omega;\mathbb{R}^v) \to H^1(\Omega;\mathbb{R}^d)$ (independently of k) such that $\boldsymbol{u}_k(t) = \mathcal{L}(t)\boldsymbol{p}_k(t)$. In particular, $(\boldsymbol{u}_k,\boldsymbol{p}_k) \to (\boldsymbol{u}_*,\boldsymbol{p}_*)$ weakly-star in $L^\infty(0,T;\mathcal{U}\times\mathcal{P})$ so that, by the lower semicontinuity assumption (22), we have that

$$\mathcal{J}(\boldsymbol{u}_*,\boldsymbol{p}_*,\boldsymbol{H}_*) \le \liminf_{k\to\infty} \mathcal{J}(\boldsymbol{u}_k,\boldsymbol{p}_k,\boldsymbol{H}_k).$$

In order to conclude the proof we now aim at showing that $(\boldsymbol{u}_*,\boldsymbol{p}_*) \in \mathrm{Sol}(\boldsymbol{H}_*)$. Let us start from checking stability (14). For all given $t\in[0,T]$ and $(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}}) \in \mathcal{U}\times\mathcal{P}$ we have

$$\begin{aligned}
&\mathcal{E}(t,\boldsymbol{u}_*(t),\boldsymbol{p}_*(t),\boldsymbol{H}_*(t))\\
&\le \liminf_{k\to\infty} \big(\mathcal{E}(t,\boldsymbol{u}_k(t),\boldsymbol{p}_k(t),\boldsymbol{H}_k(t)\big)\\
&\le \liminf_{k\to\infty} \big(\mathcal{E}(t,\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}},\boldsymbol{H}_k(t)) + \mathcal{D}(\boldsymbol{p}_k(t),\widehat{\boldsymbol{p}})\big)\\
&= \mathcal{E}(t,\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}},\boldsymbol{H}_*(t)) + \mathcal{D}(\boldsymbol{p}_*(t),\widehat{\boldsymbol{p}})
\end{aligned}$$

where we have used the lower semicontinuity of $\mathcal{E}$ and the continuity of $\mathcal{D}$ in $(L^1(\Omega;\mathbb{R}^v))^2$. Hence $(\boldsymbol{u}_*(t),\boldsymbol{p}_*(t)) \in \mathcal{S}(t,\boldsymbol{H}_*(t))$.

By passing to the lim inf in the energy balance (15) we obtain that

$$\begin{aligned}
&\mathcal{E}(t,\boldsymbol{u}_*(t),\boldsymbol{p}_*(t),\boldsymbol{H}_*(t)) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{p}_*,[0,t])\\
&\le \liminf_{k\to\infty} \Big(\mathcal{E}(t,\boldsymbol{u}_k(t),\boldsymbol{p}_k(t),\boldsymbol{H}_k(t))\\
&+ \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{p}_k,[0,t])\Big) = \liminf_{k\to\infty} \Big(\mathcal{E}(0,\boldsymbol{u}^0,\boldsymbol{p}^0,\boldsymbol{H}_k(0))\\
&- \int_0^t \langle \dot{\boldsymbol{\ell}},\boldsymbol{u}_k\rangle - \int_0^t\int_\Omega F'_{\rm mag}(\boldsymbol{H}_k\cdot\boldsymbol{A}\boldsymbol{p}_k)\dot{\boldsymbol{H}}_k\cdot\boldsymbol{A}\boldsymbol{p}_k\Big)\\
&= \mathcal{E}(0,\boldsymbol{u}^0,\boldsymbol{p}^0,\boldsymbol{H}_*(0)) - \int_0^t \langle \dot{\boldsymbol{\ell}},\boldsymbol{u}_*\rangle\\
&- \int_0^t\int_\Omega F'_{\rm mag}(\boldsymbol{H}_*\cdot\boldsymbol{A}\boldsymbol{p}_*)\dot{\boldsymbol{H}}_*\cdot\boldsymbol{A}\boldsymbol{p}_*,
\end{aligned}$$

so that the *upper* energy estimate holds. In the latter we have exploited the pointwise almost everywhere convergence $F'_{\rm mag}(\boldsymbol{H}_k\cdot\boldsymbol{A}\boldsymbol{p}_k)\dot{\boldsymbol{H}}_k\cdot\boldsymbol{A}\boldsymbol{p}_k \to F'_{\rm mag}(\boldsymbol{H}_*\cdot\dot{\boldsymbol{H}}_*\cdot\boldsymbol{A}\boldsymbol{p}_*)\boldsymbol{A}\boldsymbol{p}_*$ (recall that $F'_{\rm mag}$ is Lipschitz continuous) and Dominated Convergence.

We shall now check the *lower* energy estimate. To this end fix $[s,t]\subseteq[0,T]$, a partition $\{s{=}s^0{<}s^1{<}\ldots{<}s^M{=}t\}$, define $(\boldsymbol{u}_*^j,\boldsymbol{p}_*^j,\boldsymbol{H}_*^j,\boldsymbol{\ell}^j) = (\boldsymbol{u}_*(s^j),\boldsymbol{p}_*(s^j),\boldsymbol{H}_*(s^j),\boldsymbol{\ell}(s^j))$, and let $\overline{\boldsymbol{u}}_*$, $\overline{\boldsymbol{p}}_*$ etc. be the corresponding piecewise-constant interpolants on the partition. Note in particular that $\overline{\boldsymbol{u}}_* \to \boldsymbol{u}_*$ weakly star in $L^\infty(0,T;\mathcal{U})$ and, for all $t\in[0,T]$, $\overline{\boldsymbol{p}}_*(t) \to \boldsymbol{p}_*(t)$ strongly in $L^q(\Omega;\mathbb{R}^v)$ for all $q\in[1,\infty)$. By exploiting $(\boldsymbol{u}_*^{j-1},\boldsymbol{p}_*^{j-1}) \in$

$\mathcal{S}(s^{j-1},\boldsymbol{H}_*^{j-1})$ we have

$$\begin{aligned}&\mathcal{E}(s^j,\boldsymbol{u}_*^j,\boldsymbol{p}_*^j,\boldsymbol{H}_*^j)-\mathcal{E}(s^{j-1},\boldsymbol{u}_*^{j-1},\boldsymbol{p}_*^{j-1},\boldsymbol{H}_*^{j-1})\\&+\mathcal{D}(\boldsymbol{p}_*^{j-1},\boldsymbol{p}_*^j)\\&\geq \mathcal{E}(s^j,\boldsymbol{u}_*^j,\boldsymbol{p}_*^j,\boldsymbol{H}_*^j)-\mathcal{E}(s^{j-1},\boldsymbol{u}_*^j,\boldsymbol{p}_*^j,\boldsymbol{H}_*^{j-1})\\&=-\langle\boldsymbol{\ell}^j-\boldsymbol{\ell}^{j-1},\boldsymbol{u}_*^j\rangle\\&-\int_\Omega F_{\rm mag}(\boldsymbol{H}_*^j\cdot\boldsymbol{A}\boldsymbol{p}_*^j)+F_{\rm mag}(\boldsymbol{H}_*^{j-1}\cdot\boldsymbol{A}\boldsymbol{p}_*^j)\end{aligned}$$

so that, by taking the sum for $j=1,\dots,M$, one obtains that

$$\begin{aligned}&\mathcal{E}(t,\boldsymbol{u}_*(t),\boldsymbol{p}_*(t),\boldsymbol{H}_*(t))-\mathcal{E}(s,\boldsymbol{u}_*(s),\boldsymbol{p}_*(s),\boldsymbol{H}_*(s))\\&+\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{p}_*,[s,t])\geq-\sum_{j=1}^M\int_\Omega\Bigg(F_{\rm mag}(\boldsymbol{H}_*^j\cdot\boldsymbol{A}\boldsymbol{p}_*^j)\\&-F_{\rm mag}(\boldsymbol{H}_*^{j-1}\cdot\boldsymbol{A}\boldsymbol{p}_*^j)\Bigg)-\sum_{j=1}^M\langle\boldsymbol{\ell}^j-\boldsymbol{\ell}^{j-1},\boldsymbol{u}_*^j\rangle.\end{aligned}\tag{23}$$

In order to conclude for the lower energy estimate one has to check that the above right-hand side converges to the integral on $[s,t]$ of the power of external actions (17) as the diameter of the given partition goes to 0. The treatment of the loading term is immediate as

$$\sum_{j=1}^M\langle\boldsymbol{\ell}^j-\boldsymbol{\ell}^{j-1},\boldsymbol{u}_*^j\rangle=\int_0^T\langle\dot{\boldsymbol{\ell}},\overline{\boldsymbol{u}}_*\rangle\to\int_0^T\langle\dot{\boldsymbol{\ell}},\boldsymbol{u}_*\rangle.$$

As for the remainder term in the right-hand side of (23) we argue as follows

$$\begin{aligned}&\sum_{j=1}^M\int_\Omega\big(F_{\rm mag}(\boldsymbol{H}_*^j\cdot\boldsymbol{A}\boldsymbol{p}_*^j)-F_{\rm mag}(\boldsymbol{H}_*^{j-1}\cdot\boldsymbol{A}\boldsymbol{p}_*^j)\big)\\&=\sum_{j=1}^M\int_\Omega\left(\int_0^1F'_{\rm mag}\big(\boldsymbol{H}_\theta^j\cdot\boldsymbol{A}\boldsymbol{p}_*^j)\mathrm{d}\theta\right)\left(\int_{s^{j-1}}^{s^j}\dot{\boldsymbol{H}}_*\right)\cdot\boldsymbol{A}\boldsymbol{p}_*^j\\&=\int_0^T\int_\Omega\left(\int_0^1F'_{\rm mag}(\overline{\boldsymbol{H}}_\theta\cdot\boldsymbol{A}\overline{\boldsymbol{p}}_*)\mathrm{d}\theta\right)\dot{\boldsymbol{H}}_*\cdot\boldsymbol{A}\overline{\boldsymbol{p}}_*\end{aligned}\tag{24}$$

where $\boldsymbol{H}_\theta^j=\theta\boldsymbol{H}_*^j+(1-\theta)\boldsymbol{H}_*^{j-1}$. Let us check that

$$\left(\int_0^1F'_{\rm mag}(\overline{\boldsymbol{H}}_\theta\cdot\boldsymbol{A}\overline{\boldsymbol{p}}_*)\mathrm{d}\theta\right)\to F'_{\rm mag}(\boldsymbol{H}_*\cdot\boldsymbol{A}\boldsymbol{p}_*)$$

pointwise almost everywhere in space-time. Indeed, for $t\in(s^{j-1},s^j]$ one has that

$$\begin{aligned}&\int_\Omega\left|\left(\int_0^1F'_{\rm mag}(\overline{\boldsymbol{H}}_\theta\cdot\boldsymbol{A}\overline{\boldsymbol{p}}_*)\mathrm{d}\theta\right)-F'_{\rm mag}(\boldsymbol{H}_*\cdot\boldsymbol{A}\boldsymbol{p}_*)\right|\\&=\int_\Omega\Bigg|\int_0^1F'_{\rm mag}(\overline{\boldsymbol{H}}_\theta\cdot\boldsymbol{A}\overline{\boldsymbol{p}}_*)-F'_{\rm mag}(\boldsymbol{H}_*\cdot\boldsymbol{A}\overline{\boldsymbol{p}}_*)\\&+F'_{\rm mag}(\boldsymbol{H}_*\cdot\boldsymbol{A}\overline{\boldsymbol{p}}_*)-F'_{\rm mag}(\boldsymbol{H}_*\cdot\boldsymbol{A}\boldsymbol{p}_*)\mathrm{d}\theta\Bigg|\\&\leq C\int_{s^{j-1}}^{s^j}\|\dot{\boldsymbol{H}}_*\|_{L^1}+C\int_\Omega|\boldsymbol{H}_*||\overline{\boldsymbol{p}}_*-\boldsymbol{p}_*|\to0\end{aligned}$$

by the Lipschitz continuity of $F'_{\rm mag}$ and the absolute continuity of $\boldsymbol{H}_*$. In particular, the space-time integrands in the last term of (24) converge pointwise almost everywhere to $F'_{\rm mag}(\boldsymbol{H}_*\cdot\boldsymbol{A}\boldsymbol{p}_*)\dot{\boldsymbol{H}}_*\cdot\boldsymbol{A}\boldsymbol{p}_*$. Hence, by Dominated Convergence, the respective integrals also converge. Eventually, the right-hand side of (23) converges to

$$-\int_0^T\int_\Omega F'_{\rm mag}(\boldsymbol{H}_*\cdot\boldsymbol{A}\boldsymbol{p}_*)\dot{\boldsymbol{H}}_*\cdot\boldsymbol{A}\boldsymbol{p}_*-\int_0^T\langle\dot{\boldsymbol{\ell}},\boldsymbol{u}_*\rangle$$

and the lower energy estimate and thus (15) follows. Hence, we have checked that $(\boldsymbol{u}_*,\boldsymbol{p}_*)\in\mathrm{Sol}(\boldsymbol{H}_*)$ This concludes the proof. ■

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