On Some Nonlocal Evolution Equations in Banach Spaces

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Abstract

This note is concerned with the initial value problem for the abstract nonlocal equation $(Au)' + (Bu) \ni f$ where A is a maximal monotone operator from a reflexive Banach space E to its dual E^*, while B is a nonlocal maximal monotone operator from $L^p(0,T;E)$ to $L^q(0,T;E^*)$ with $p^{-1} + q^{-1} = 1$, $p \in (1,\infty)$. Under proper boundedness and coercivity assumptions on the operators, a solution is achieved by means of a discretization argument. Uniqueness and continuous dependence are also discussed and we prove some estimates for the discretization error. Finally, we deal with the approximation of linear Volterra integrodifferential operators.

Keywords: doubly nonlinear evolution equations, nonlocal terms, abstract Cauchy problem, regularization, time discretization, convergence, error estimate, Volterra integrodifferential operators.

AMS (MOS) Subject Classification: 35K90, 45N05.

1 Introduction

Let E be a reflexive Banach space and denote by E^* its dual. Moreover, let $p, q \in (1,\infty)$ such that $p^{-1} + q^{-1} = 1$. We are given two possibly multivalued maximal monotone operators $A \subset E \times E^*$ and $B \subset L^p(0,T;E) \times L^q(0,T;E^*)$ where $T > 0$ is some reference time. This analysis is concerned with the following abstract Cauchy problem

$$(Au)' + (Bu) \ni f \qquad (Au)(0) \ni v^0, \tag{1.1}$$

where the prime denotes the derivative with respect to time, the equation is fulfilled in $L^q(0,T;E^*)$, and $f \in L^q(0,T;E^*)$, $v^0 \in R(A)$ (i.e., the range of A) are prescribed data.

The present contribution proves that the above problem admits at least a solution whenever the two operators fulfill some boundedness condition, A is a compact subdifferential, and B is causal and coercive.

As regards the compactness assumption on the operator A, we remark that, referring to possible applications to PDE, the order of A is asked to be strictly lower than that of B. On the other hand, the coercivity assumption on B is motivated by the fact that no restriction is imposed on the possible degeneracy of A. In particular, our result includes the (non interesting) case $A \equiv 0$. In the latter situation, by letting E be a finite dimensional space and B a subdifferential, it is easy to check that problem (1.1) has a solution if and only if B is coercive. Hence, some coerciveness assumption for B seems mandatory.

We shall mention some contribution to the analysis of doubly nonlinear evolution equations of the type of (1.1). First of all, a variety of papers have been devoted to the study of (1.1) in the case when B is a local in time operator. Among these papers one shall at least quote the important contributions [3, 10, 13, 16]. On the other hand, as regards nonlocal in time operators, we stress that some existence results for problems of the type of (1.1) are already present in the literature. A first result has been obtained in [8], where, nevertheless, some assumptions on the structure of B and on the regularity of f are needed. More recently, the paper [19] deals with some classes of nonlinear equations by assuming, in particular, B to be a time dependent subgradient. Furthermore, in [2] the authors achieve an existence result for some doubly nonlinear nonlocal equation by assuming that A is non degenerate on some intermediate space between E and E^* and B is a subdifferential. Finally, we aim to quote the paper [26], which represents a counterpart in Hilbert spaces to the present contribution.

The novelty of this paper is to be found both in the new existence result and in its proof. As regards the existence result, we extend the analysis of the former [26] to the Banach setting. The weaker functional framework is of course suitable of including a broader variety of applicative situations. On the other hand, the analysis in Banach spaces requires some additional care. In particular, the extension of the arguments of [26] it is not straightforward.

As for the proof, the novelty relies in the fact that we discuss a variable time-step discretization of (a properly regularized version of) problem (1.1). To this aim, we address the analysis of an approximation procedure for nonlocal in time operators detailing the solvability of the discrete problem and the convergence of the discrete solution to its continuous counterpart as the diameter of the partition of the time interval goes to zero.

We shall motivate our interest in problem (1.1) by pointing out its possible relation to *doubly nonlinear degenerate evolution systems* containing *nonlocal in time terms.* Let us stress from the very beginning that the following example is not intended to be the best possible in any sense and has been chosen for the sake of simplicity, in order to suggest a class of applications. Let us denote by Ω a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary Γ. Moreover, let α, β be two maximal monotone graphs of $\mathbb{R}^m \times \mathbb{R}^m$ $(m \geq 1)$ and $\mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$, respectively, and $K_0, K_1 : (0,T) \times (0,T) \longrightarrow M_{m\times m}(\mathbb{R})$ be two functions, where $M_{m\times n}(\mathbb{R})$ denotes the space of real $m \times n-$matrices.

Then, given a datum $f : Q := \Omega \times (0,T) \longrightarrow \mathbb{R}^m$, we look for a function $u : Q \longrightarrow \mathbb{R}^m$ such that the following system

$$\big(\alpha(u)\big)_t - \operatorname{div}\left(\beta(Du) + \int_0^t K_1(t,s)Du(s)\,ds\right) + \int_0^t K_0(t,s)u(s)\,ds \ni f, \quad (1.2)$$

is fulfilled at least in the sense of distributions, together with suitable initial and boundary conditions. In the latter relation Du stands for the matrix $(Du)_{ji} = \partial u_j/\partial x_i$ for $i = 1,\dots,n,\ j = 1,\dots,m$, and, for all functions M of Ω valued in $M_{m\times n}(\mathbb{R})$, we denote by $\operatorname{div} M$ the vector $(\operatorname{div} M)_j := \operatorname{div}(M_{j,\cdot})$, with obvious notations.

Of course some assumptions have to be made on data in order to possibly include (1.2) in our abstract tractation. In particular, as regards the graphs we have to require growth bounds of the type

$$\begin{aligned} \|v\|_{\mathbb{R}^m} &\le c_0\big(1 + \|u\|_{\mathbb{R}^m}^{p-1}\big) \quad \forall [u,v] \in \alpha, \\ \|W\|_{\mathbb{R}^{m\times n}} &\le c_0\big(1 + \|D\|_{\mathbb{R}^{m\times n}}^{p-1}\big) \quad \forall [D,W] \in \beta, \end{aligned}$$

where the latter are induced matrix norms, c_0 is some positive constant, and $p \in (1,\infty)$ is the same exponent introduced above. Moreover some coercivity is required. In particular, a possible sufficient condition is that

$$W_{ij}D_{ij} \ \ge\ c_1\|D\|_{\mathbb{R}^{m\times n}}^p \quad \forall [D,W] \in \beta,$$

for some positive constant c_1. Of course the latter assumption is just intended to point out a possible choice and may be weakened in many senses.

As for the kernels we require some kind of $L^p - L^q$ integrability (see the forthcoming (6.1)) and monotonicity. In particular, the main assumption of this analysis is that the functions K_0, K_1 are asked to be of *positive type*, i.e the following inequality

$$\int_0^{t_*}\left(\int_0^t K_\ell(t,s)v(s)\,ds\right)v(t)\,dt \ge 0, \quad (1.3)$$

(with obvious notation) has to be fulfilled for every $v \in (L^p(0,T))^m$, $t_* \in (0,T)$, and $\ell = 0,1$. Moreover, we recall that, taking into account the assumption (1.3), the latter kernels are referred as of *positive type* and are especially interesting within applications. Let us explicitly remark that the treatment of (1.2) with the above assumptions on the graphs and the kernels and $p = 2$ is among the aims of [26], whose analysis, however, cannot be directly extended to the case $p \neq 2$.

The problem (1.2) has a relevant interest within applications since it may arise in connection to nonlinear diffusion phenomena including nonlocal in time effects.

In particular, some degenerate parabolic problems with memory are included in our framework. In order to give an example of explicit modelization in this direction we shall discuss an approach to the theory of phase change with thermal memory [12]. Indeed, let us stress that the forthcoming example will not require all of the structure of (1.2) since we reduce ourselves, for the sake of clarity, to a single nonlinear equation. Namely, let us consider a substance that fills the region $\Omega \subset \mathbb{R}^3$ and may undergo a temperature driven phase transformation. We assume that our thermodynamic system is insulated from the exterior and fix as state variable the (relative) temperature θ of the medium. One introduces the energy balance relation

$$e_t + \operatorname{div} \boldsymbol{q} = g \quad \text{in} \quad \Omega \times (0,T),$$

where e is the internal energy of the system (enthalpy), $\boldsymbol{q}$ is the heat flux, and g is a given density of heat source. Hence we shall give the constitutive relations for e and $\boldsymbol{q}$. Following the general theory of phase transformation, we assume that e is monotone and jumps at the transition temperatures, in particular $e \subset \mathbb{R} \times \mathbb{R}$ is assumed to be a maximal monotone graph. We recall that the classical Stefan choice for the enthalpy e is $e(r) := r + \mathcal{H}(r)$ for $r \in \mathbb{R}$, where $\mathcal{H}$ is the Heaviside graph (i.e. $\mathcal{H}(r) = 0$ if $r < 0$, $\mathcal{H}(0) = [0,1]$, and $\mathcal{H}(r) = 1$ if $r > 0$). Of course other choices are possible. In particular, without referring to two-phase transitions, let us remark the two choices , $\alpha(r) = |r|^{\eta-1} r$ for some $\eta \in (0,1)$, and $\alpha(r) = \mathcal{H}(r)$ that are strictly connected to the porous media equation and the Hele-Shaw model, respectively.

As for the heat flux, we choose the following law

$$\boldsymbol{q}(t) = -k_0 |\nabla\theta(t)|^{p-2} \nabla\theta(t) - \int_0^t K_1(t,s) \nabla\theta(s)\, ds,$$

where the flux is the datum of an actual contribution (and $k_0 > 0$ represents an instantaneous heat conductivity) and an accumulated history averaged by means a suitable kernel K_1 (a standard example is $K_1(t,s) = (k_0/\mu)\exp(-(t-s)/\mu) I$ where k_0 is the same as above, $\mu > 0$ is a relaxation parameter, and I is the identity matrix). In the case $p = 2$ the latter position may be regarded to as a generalized law of Coleman-Gurtin type. Moreover, when $p = 2$ and $K_1 = 0$ we simply reduce to the standard Fourier law.

The latter class of problems fits of course into the general (1.2). For a full discussion about the underlying physics of models with memory the reader is referred to the papers [15, 18, 20, 21]. Moreover, we refer to [1, 9, 12] and the references therein for analytical results about degenerate Volterra integropartial differential equations.

Our work is organized as follows. Section 2 is devoted to the statement of our main existence result. The latter is proved in Section 3, while Section 4 contains some discussion about uniqueness of the solution and its possible continuous dependence on data. In Section 5 we prove that, under further assumptions on the operators, we have the strong convergence of our approximated solution to its continuous counterpart. Moreover, we deduce some estimates for the discretization error. Finally, Section 6 contains some details on our approximation in the case when B is an integrodifferential operator of positive Volterra type (see below).

2 Statement of the main result

Let us start by introducing some notation. Let E be a real Banach space, denote by E^* its dual, and by $\langle\cdot,\cdot\rangle$ the duality pairing between E^* and E. Given a possibly multivalued operator $A : E \longrightarrow 2^{E^*}$ we may regard it as a subset of the product space $E \times E^*$. In the latter sense, the notations $v \in Au$ and $[u, v] \in A$ are equivalent. We say that A is *monotone* if, for every $[u_1, v_1], [u_2, v_2] \in A$, one has that $\langle v_1 - v_2, u_1 - u_2\rangle \geq 0$. Moreover, we say that A is *maximal monotone* if it is maximal in the sense of inclusion of sets within the class of monotone operators (see [7] for details). Furthermore, we say that A is *strictly monotone* if, for all $[u_1, v_1], [u_2, v_2] \in A$, we have that $\langle v_1 - v_2, u_1 - u_2\rangle = 0$ implies $u_1 = u_2$.

In the forthcoming analysis the following notion will be extremely relevant. Let $\varphi : E \to (-\infty, +\infty]$ be a proper, convex, and lower semicontinuous function. Then we define the *subdifferential* of φ as the set

$$\partial\varphi := \big\{[u, v] \in E \times E^* \ : \ \langle v, w - u\rangle \leq \varphi(w) - \varphi(u), \quad \forall w \in D(\varphi)\big\}$$

where $D(\varphi) \subset E$ denotes the effective domain of φ. It is possible to prove that $\partial\varphi$ turns out to be a maximal monotone operator in the sense specified above [7, Thm. 2.1, p. 54]. For the sake of clarity, we introduce here the *conjugate* $\varphi^* : E^* \to (-\infty, +\infty]$ of the function φ by prescribing

$$\varphi^*(v) := \sup_{u\in E} \big\{\langle v, u\rangle - \varphi(u)\big\}. \tag{2.1}$$

We note that, under the above assumptions on φ, the conjugate φ^* is still a proper, convex, and lower semicontinuous function and that $v \in \partial\varphi(u)$ if and only if $u \in \partial\varphi^*(v)$ where

$$\partial\varphi^* := \big\{[u, v] \in E \times E^* \ : \ \langle w^* - v, u\rangle \leq \varphi^*(w^*) - \varphi^*(v), \quad \forall w^* \in D(\varphi^*)\big\}.$$

Before stating our result we stress that, letting E and F be locally convex spaces, $\varphi : F \to (-\infty, +\infty]$ be convex, $L : E \longrightarrow F$ be linear and continuous and φ be continuous at some point of $R(L)$ (i.e., the range of L) we have that the following *chain rule* holds (see, e.g., [24, Prop. 7.8, p. 82])

$$\partial\big(\varphi \circ L\big) = L^* \circ \partial\varphi \circ L, \tag{2.2}$$

where $L^* : F^* \longrightarrow E^*$ denotes the dual operator of L.

We are now able to state the main result of the paper.

Theorem 2.1. *Let E, F be two reflexive and strictly convex Banach spaces such that E^*, F^* are strictly convex and E is densely and compactly embedded in F by means of the injection $i : E \longrightarrow F$. Moreover, let $p, q \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, and the following hold*

(H1) $\varphi : F \longrightarrow (-\infty, +\infty]$ is a proper, convex, lower semicontinuous on F, and continuous at some point of E. Let $A = \partial\psi := \partial(\varphi \circ i) = i^* \circ \partial\varphi \circ i$. Moreover, for all $u \in L^p(0,T;E)$, let $\Phi(u) := \int_0^T \psi(u(t))\,dt$ if $\psi(u) \in L^1(0,T)$ and $\Phi(u) := +\infty$ otherwise, and $\tilde{A} := \partial\Phi$ map bounded sets into bounded sets of $L^q(0,T;F^*)$.

(H2) $B \subset L^p(0,T;E) \times L^q(0,T;E^*)$ is maximal monotone, causal, bounded on bounded sets, and coercive, i.e.

$$\lim_{\substack{\|u\|_{L^p(0,T;E)} \to +\infty \\ [u,w] \in B}} \frac{\int_0^T \langle w(t), u(t) \rangle dt}{\|u\|_{L^p(0,T;E)}} = +\infty. \tag{2.3}$$

Then, for every $f \in L^q(0,T;E^*)$ and $v^0 \in R(A)$, there exists a triplet $u \in L^p(0,T;E)$, $v \in W^{1,q}(0,T;E^*) \cap L^q(0,T;F^*)$, and $w \in L^q(0,T;E^*)$ such that

$$v'(t) + w(t) = f(t) \qquad \text{for a.e. } t \in (0,T), \tag{2.4}$$

$$[u(t), v(t)] \in A \qquad \text{for a.e. } t \in (0,T), \tag{2.5}$$

$$[u, w] \in B, \tag{2.6}$$

$$v(0) = v^0. \tag{2.7}$$

Before going on we shall explicitly remark that the function Φ in (H1) is actually proper, convex, and lower semicontinuous in $L^p(0,T;E)$ [7, Ex. 3, p. 61]. Hence, its subdifferential $\partial\Phi \subset L^p(0,T;E) \times L^q(0,T;E^*)$ is well-defined and maximal monotone.

Let us stress from the very beginning that $\tilde{A}$ is nothing but the realization of A as an operator on $L^p(0,T;E)$. Indeed, it is straightforward to check that $(\tilde{A}u)(t) = Au(t)$ for all $u \in L^p(0,T;E)$ and almost every $t \in (0,T)$. In the forthcoming of the paper we will use the same notation A for both the above cited operators.

For the sake of clarity we shall comment the notion of *causality* we are referring to. Letting $B \subset L^p(0,T;E) \times L^q(0,T;E^*)$ be a possibly multivalued operator, we say that B is *causal* if, for any $u_1, u_2 \in D(B)$ such that $u_1 = u_2$ on $(0, t_*)$ for some $t_* \in (0,T)$, the equality of sets $Bu_1 \equiv Bu_2$ holds up to time t_*. That is, letting $R_t : L^q(0,T;E^*) \longrightarrow L^q(0,t;E^*)$ be the usual restriction operator, the equality $R_t(Bu_1) \equiv R_t(Bu_2)$ is fulfilled for $t \in (0, t_*)$.

Let us stress that the assumption (H1) imply that $D(A) = E$. Since we have that $\psi(0) < +\infty$ we may assume with no loss of generality that $\psi(0) \leq 0$ so that ψ^* is everywhere non-negative (cf. (2.1)). Moreover, the boundedness of B yields $D(B) = L^p(0,T;E)$.

Remark 2.2. We point out that, since every reflexive Banach space E can be equivalently renormed in such a way that E and its dual E^* turn out to be strictly convex (this is the well known Asplund's result [4, 5]), the requirement about the strict convexity of the spaces in the statement of Theorem 2.1 may be avoided. However, we prefer to keep this assumption for the sake of clarity.

3 Existence

This is the plan of the proof of Theorem 2.1. First of all we address a regularized problem where B is replaced by a strictly monotone, *demicontinuous* operator B_λ, i.e. a *strongly-weakly* continuous operator (namely if u_n converges to u strongly in $L^p(0,T;E)$, then $B_\lambda u_n$ converge to $B_\lambda u$ weakly in $L^q(0,T;E^*)$). Then, we carefully analyze a variable time-step discretization of such a regularized problem. Finally, we pass to the limit both in the time discretization and in the regularization obtaining the existence result stated in Theorem 2.1.

3.1 Approximation

Given $\lambda > 0$, we shall ask for $B_\lambda : L^p(0,T;E) \longrightarrow L^q(0,T;E^*)$ everywhere defined, single-valued, demicontinuous, strictly monotone, coercive, and causal operator mapping bounded sets in bounded sets. Moreover, as $\lambda \longrightarrow 0$, we shall let B_λ converge to B in the following sense (cf. [7, Prop. 1.1, p. 42])

$$\text{for all } [u_\lambda, w_\lambda] \in B_\lambda \text{ weakly converging to } [u,w] \text{ in } L^p(0,T;E) \times L^q(0,T;E^*)$$
$$\text{such that } \limsup_{\lambda \to 0} \int_0^T \langle w_\lambda, u_\lambda \rangle \leq \int_0^T \langle w, u \rangle, \text{ one has that } [u,w] \in B. \quad (3.1)$$

Let us now discuss the possibility of finding such a regularization of the operator B. To this aim let us introduce the possibly multivalued map $\mathcal{G} \subset E \times E^*$ as

$$\mathcal{G}u := \{ v \in E^* : \langle v, u \rangle = \|u\|_E^p = \|v\|_{E^*}^q \}.$$

Of course, in the case $p = 2$ this is nothing but the *normalized duality mapping* $\mathcal{J} \subset E \times E^*$. On the other hand, in the case $p \neq 2$, the map $\mathcal{G}$ is usually referred to as *duality map with gauge* $\xi(r) := r^{p-1}$. Indeed, these two maps are obviously connected by the equality of sets $\mathcal{G}u = \|u\|_E^{p-2} \mathcal{J}u$, for any $u \in E \setminus \{0\}$. The map $\mathcal{G}$ enjoys most of the properties of $\mathcal{J}$. We have collected some of them for the reader's convenience in the following lemma that we state without proof [24, p. 91].

Lemma 3.1. *Let E be reflexive and $\mathcal{G}$ be defined as above. Then $\mathcal{G}$ is monotone, coercive, bounded on bounded sets, and fulfills*

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq \Big(\|u_1\|_E - \|u_2\|_E \Big) \Big(\|u_1\|_E^{p-1} - \|u_2\|_E^{p-1} \Big),$$

for all $[u_i, v_i] \in \mathcal{G}$, $i = 1, 2$. Moreover, if E^* is strictly convex, then $\mathcal{G}$ is single-valued and demicontinuous, and if E is strictly convex, then $\mathcal{G}$ is strictly monotone.

Let us now denote by $\mathcal{F} \subset L^p(0,T;E) \times L^q(0,T;E^*)$ the same construction of $\mathcal{G}$ based on the pair of spaces $L^p(0,T;E) - L^q(0,T;E^*)$, namely

$$\mathcal{F}u := \left\{ v \in L^q(0,T;E^*) : \int_0^T \langle v, u \rangle = \|u\|_{L^p(0,T;E)}^p = \|v\|_{L^q(0,T;E^*)}^q \right\}$$

for all $u \in L^p(0,T;E)$. It is a standard matter to check that the latter lemma applies to $\mathcal{F}$ as well. Moreover, one readily has that $\mathcal{F}$ is the $L^p(0,T;E) - L^q(0,T;E^*)$ realization of $\mathcal{G}$, i.e. $(\mathcal{F}u)(t) \equiv \mathcal{G}u(t)$ for any $u \in L^p(0,T;E)$ and almost every $t \in (0,T)$. In particular, the latter remark entail that $\mathcal{F}$ is a causal operator in the sense of Section 2. Finally, by arguing exactly as in [7, Thm. 1.2, p. 39], we readily prove the following result.

Lemma 3.2. *Let E and E^* be reflexive and strictly convex and $\mathcal{G}$ ($\mathcal{F}$ respectively) be defined as above. Moreover, let $A \subset E \times E^*$ ($B \subset L^p(0,T;E) \times L^q(0,T;E^*)$) be monotone. Then, A (B) is maximal monotone if and only if, for some $\lambda > 0$ (hence for all $\lambda > 0$) we have that $R(A + \lambda\mathcal{G}) = E^*$ ($R(B + \lambda\mathcal{F}) = L^q(0,T;E^*)$, respectively).*

Let us now introduce a possible approximation of B by suitably modifying the argument in [7, Sec. 1.2, p. 41]. We fix $u \in L^p(0,T;E)$ and $\lambda > 0$ and consider the problem

$$\mathcal{F}(v - u) + \lambda B v \ni 0, \tag{3.2}$$

which of course has a unique solution thanks to [7, Cor. 1.1, p. 39] and Lemma 3.1.

In the forthcoming analysis we will use the notation $J^\lambda u := v$, where u and v are defined above by means of (3.2). Let us set

$$B^\lambda u := -\lambda^{-1} \mathcal{F}(J^\lambda u - u) \qquad \forall u \in L^p(0,T;E), \ \lambda > 0. \tag{3.3}$$

In the case $p = 2$ the latter operator is usually called the *Yosida approximation* of B at level λ. Indeed, even in the case $p \neq 2$, all of the conclusions of [7, Prop.

1.1, p. 42] hold true. Of course, the operator B^λ is not a priori strictly monotone. On the other hand, the strict monotonicity of B_λ will turn out to be essential with respect to the well posedness of our discretized problem. In this respect we aim to consider a possible approximation of B of the form

$$B_\lambda := \lambda \mathcal{F} + B^\lambda, \tag{3.4}$$

where B^λ is defined as above. Taking into account the above analysis, it is a standard matter to prove that the latter operator B_λ is everywhere defined, single-valued, strictly monotone, demicontinuous, bounded on bounded sets, and converges to B in the sense of (3.1) [7, Prop. 1.1, p. 42 and Thm. 1.3, p. 40]. Moreover, B_λ turns out to be coercive as well, we prove this fact in the following lemma.

Lemma 3.3. *Under the assumptions of Lemma 3.2, let $B \subset L^p(0,T;E) \times L^q(0,T;E^*)$ be maximal monotone and coercive in the sense of (2.3) and let B_λ be defined as in (3.4). Then B_λ is coercive as well.*

Proof. First of all, we remark that the property (2.3) is equivalent to the following

$$\text{for any } \varepsilon > 0 \text{ there exists } c_\varepsilon > 0 \text{ such that}$$
$$\|u\|_{L^p(0,T;E)} \leq \varepsilon \int_0^T \langle w, u \rangle + c_\varepsilon \quad \text{for all } [u,w] \in B, \tag{3.5}$$

as it may be plainly checked. Next, owing to definitions (3.2)-(3.3), it is straightforward to deduce that

$$\begin{aligned} \int_0^T \langle B^\lambda u, u \rangle &= \int_0^T \langle z, J^\lambda u \rangle + \int_0^T \langle z, u - J^\lambda u \rangle \\ &= \int_0^T \langle z, J^\lambda u \rangle + \frac{1}{\lambda} \|u - J^\lambda u\|^p_{L^p(0,T;E)}, \end{aligned} \tag{3.6}$$

where $[J^\lambda u, z] \in B$. On the other hand, since B fulfills (2.3), for any fixed $\varepsilon > 0$ we find $c_\varepsilon > 0$ such that one has

$$\|J^\lambda u\|_{L^p(0,T;E)} \leq \varepsilon \int_0^T \langle z, J^\lambda u \rangle + c_\varepsilon.$$

Finally, the latter relation, (3.6), and easy calculations lead to the following

$$\begin{aligned} \|u\|_{L^p(0,T;E)} &\leq \|u - J^\lambda u\|_{L^p(0,T;E)} + \|J^\lambda u\|_{L^p(0,T;E)} \\ &\leq \frac{\varepsilon}{\lambda} \|u - J^\lambda u\|^p_{L^p(0,T;E)} + c^*_\varepsilon + \varepsilon \int_0^T \langle z, J^\lambda u \rangle + c_\varepsilon \\ = \varepsilon \int_0^T \langle B^\lambda u, u \rangle &+ (c_\varepsilon + c^*_\varepsilon) \leq \varepsilon \int_0^T \langle B_\lambda u, u \rangle + (c_\varepsilon + c^*_\varepsilon), \end{aligned} \tag{3.7}$$

where $c_\varepsilon^* := (\varepsilon p/\lambda)^{1-q} q^{-1}$. Hence, the operator B_λ fulfills a condition analogous to (3.5) and the assertion is proved. □

For the sake of later convenience we shall present a lemma on the uniform boundedness and coercivity of the operator B_λ with respect to $\lambda \in (0,1)$.

Lemma 3.4. *Under the assumption of Lemma 3.3, one has that*

$$\|B^\lambda u\|_{L^q(0,T;E^*)} \leq \|w\|_{L^q(0,T;E^*)} \quad \forall [u,w] \in B, \ \forall \lambda > 0. \tag{3.8}$$

Moreover, assume we are given a set $\{u_\lambda\}_{\lambda \in (0,1)} \subset L^p(0,T;E)$ *and a positive constant* c *such that*

$$\int_0^T \langle B_\lambda u_\lambda, u_\lambda \rangle \ \leq \ c\left(1 + \|u_\lambda\|_{L^p(0,T;E)}\right) \quad \forall \lambda \in (0,1). \tag{3.9}$$

Then $\{u_\lambda\}$ *is bounded uniformly with respect to* λ.

Proof. Let $[u,w] \in B$. Then, since B is monotone

$$0 \leq \int_0^T \langle w - B^\lambda u, u - J^\lambda u \rangle$$
$$\leq \|w\|_{L^q(0,T;E^*)} \|u - J^\lambda u\|_{L^p(0,T;E)} - \frac{1}{\lambda} \|u - J^\lambda u\|^p_{L^p(0,T;E)}.$$

Hence

$$\|B^\lambda u\|_{L^q(0,T;E^*)} = \frac{1}{\lambda} \|u - J^\lambda u\|^{p-1}_{L^p(0,T;E)} \leq \|w\|_{L^q(0,T;E^*)}.$$

As for the uniform boundedness of $\{u_\lambda\}$, it suffices to choose $\varepsilon = 1/(2c)$ in relation (3.7) where u is replaced by u_λ and exploit (3.9) to obtain

$$\|u_\lambda\|_{L^p(0,T;E)} \leq 1 + 2c_\varepsilon + \left(\frac{2\lambda c}{p}\right)^{q-1} \frac{2}{q}$$

and the assertion follows from the boundedness of λ. □

We conclude this discussion about our example of regularization observing that, whenever a causal operator B is taken into account, the regularized B_λ is causal as well. Namely, we have the following

Lemma 3.5. *Under the assumptions of Lemma 3.3, the operator* B_λ *is causal.*

Proof. Owing to relation (3.3) and to the causality of $\mathcal{F}$, it suffices to prove that the operator $J^\lambda : L^p(0,T;E) \longrightarrow L^p(0,T;E)$ is causal. We stress that J^λ is single-valued. Hence, the notion of causality for J^λ reduces to the standard one, i.e. for all $[u_1, v_1], [u_2, v_2] \in J^\lambda$ such that $u_1 = u_2$ in $(0, t)$ for some $t \in (0, T)$, one has that $v_1 = v_2$ in $(0, t)$ as well.

We define the *restriction operator* $R_t : L^r(0, T) \longrightarrow L^r(0, t)\,(r = p, q)$ and the *trivial extension operator* $E_t : L^r(0, t) \longrightarrow L^r(0, T)$ in the usual way and consider their respective vectorial valued realizations without changing symbols. Moreover, we let $\mathcal{F}_t : L^p(0, t, E) \longrightarrow L^q(0, t; E^*)$ be the realization on $(0, t)$ of $\mathcal{G} : E \longrightarrow E^*$ and $B_t \subset L^p(0, t, E) \times L^q(0, t; E^*)$ be the operator

$$B_t := R_t \circ B \circ E_t.$$

It is a standard matter to prove that B_t is monotone. Furthermore, letting $\mu > 0$ and $f \in L^q(0, t; E^*)$ and owing to Lemma 3.2, we find $u \in L^p(0, T; E)$ such that

$$\mu \mathcal{F} u + B u \ni E_t f.$$

Hence, applying R_t to both sides of the latter relation, one has that

$$\mu(\mathcal{F}_t \circ R_t) u + (R_t \circ B) u \ni f, \tag{3.10}$$

since $R_t \circ \mathcal{F} = \mathcal{F}_t \circ R_t$. Next, owing to the causality of B, we readily get that

$$R_t \circ B = R_t \circ B \circ E_t \circ R_t = B_t \circ R_t,$$

so that, looking back to (3.10) and choosing $w = R_t u$, we have proved that, for every $\mu > 0$ and all $f \in L^q(0, t; E^*)$, the equation

$$\mu \mathcal{F}_t w + B_t w \ni f,$$

has a solution in $L^p(0, t, E)$. Thus B_t is maximal monotone as well according to Lemma 3.2.

Finally, letting $u_* = R_t u_1$, from relations (see (3.2))

$$\mathcal{F}(v_i - u_i) + \lambda B v_i \ni 0 \qquad i = 1, 2,$$

we easily get that

$$\mathcal{F}_t\big(R_t v_i - u_*\big) + \lambda (B_t \circ R_t) v_i \ni 0 \qquad i = 1, 2,$$

and, since B_t is maximal monotone, the causality of J^λ follows from the uniqueness of the solution to the latter problem. $\square$

3.2 Discretization

Let us introduce a partition $\mathcal{P}$ of the time interval $[0, T]$, namely

$$\mathcal{P} := \big\{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\big\}$$

with variable time-step $\tau_i := t_i - t_{i-1}$. No constraints are imposed on the time-steps and $\tau := \max_{1\le i\le N} \tau_i$ denotes the diameter of the partition. Let us just remark that a constant time-step partition would perfectly serve for the sake of proving Theorem 2.1. Nevertheless, our discretization is naturally given in the framework of variable time-step partitions without any particular intricacy. Moreover, this possibility presents a relevant numerical interest since the time-steps could be tailored according to further numerical considerations such as error estimators (see also Section 6).

In the forthcoming analysis the following notation will be extensively used: let $\{u_i\}$, $i = 0, 1, \dots, N$ be a vector, we denote by $u_{\mathcal{P}}$ and $\overline{u}_{\mathcal{P}}$ two functions of the time interval $[0, T]$ which interpolate the values of the vector $\{u_i\}$ piecewise linearly and backward constantly on the partition $\mathcal{P}$, respectively. That is,

$$u_{\mathcal{P}}(0) := u_0, \quad u_{\mathcal{P}}(t) := \alpha_i(t) u_i + \big(1 - \alpha_i(t)\big) u_{i-1},$$
$$\overline{u}_{\mathcal{P}}(0) := u_0, \quad \overline{u}_{\mathcal{P}}(t) := u_i, \qquad \text{for } t \in (t_{i-1}, t_i], \quad i = 1, \dots, N$$

where $\alpha_i(t) := (t - t_{i-1})/\tau_i$, for $t \in (t_{i-1}, t_i]$, $i = 1, \dots, N$. Furthermore, we introduce the set of constant functions on $\mathcal{P}$ as

$$K_{\mathcal{P}} := \big\{ u : [0, T] \longrightarrow E \ : \ u \text{ is constant on } (t_{i-1}, t_i] \text{ for } i = 1, \dots, N \big\}.$$

Moreover, we regard $K_{\mathcal{P}}$ as a subspace of $L^p(0, T; E)$ and observe that $K_{\mathcal{P}}$ is isomorphic to a finite product of the space E. Furthermore, it is straightforward to check that the *Riesz representation isomorphism* between the dual of $L^p(0, T; E)$ and $L^q(0, T; E^*)$ ensures that $(K_{\mathcal{P}})^*$ is isomorphic to the analogue of $K_{\mathcal{P}}$ defined on the space E^*, that is $(K_{\mathcal{P}})^* \simeq K^*_{\mathcal{P}}$, where

$$K^*_{\mathcal{P}} := \big\{ u : [0, T] \longrightarrow E^* \ : \ u \text{ is constant on } (t_{i-1}, t_i] \text{ for } i = 1, \dots, N \big\}.$$

Finally, since $K_{\mathcal{P}}$ is finite dimensional, we have that both $K_{\mathcal{P}}$ and $K^*_{\mathcal{P}}$ are reflexive Banach spaces.

We shall introduce the *mean operator* $\Lambda_{\mathcal{P}} : L^r(0, T) \longrightarrow L^r(0, T)$ $(r \in [1, \infty))$ defined for every $w \in L^r(0, T)$ by

$$(\Lambda_{\mathcal{P}} w)(t) := \frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} w(s)\, ds \qquad \text{for } t \in (t_{i-1}, t_i],\ i = 1, \dots, N, \tag{3.11}$$

and stress that the latter operator turns out to be a linear contraction in $L^r(0, T)$ and that we will consider its vector-valued realization without changing symbols. Moreover, for all $u \in L^p(0, T)$, and $v \in L^q(0, T)$, one has that the following relation holds

$$\int_0^T \big(\Lambda_{\mathcal{P}} u\big)(s)\, v(s)\, ds = \int_0^T u(s)\, \big(\Lambda_{\mathcal{P}} v\big)(s)\, ds. \tag{3.12}$$

Let us fix a suitable approximation $\{f_i\}_{i=1}^N \in (E^*)^N$ for the function $f \in L^q(0,T;E^*)$ such that the strong convergence $\overline{f}_{\mathcal{P}} \longrightarrow f$ in $L^q(0,T;E^*)$ holds as the diameter τ of partition $\mathcal{P}$ goes to 0 (indeed, a straightforward choice would be $\overline{f}_{\mathcal{P}} := \Lambda_{\mathcal{P}} f$).

Taking into account the above notations, we are looking for $\{[u_i, v_i]\}_{i=0}^N \in (E \times E^*)^{N+1}$ solving the following scheme

$$v_0 = v^0, \tag{3.13}$$

$$\frac{v_i - v_{i-1}}{\tau_i} + \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} (B_\lambda \overline{u}_{\mathcal{P}})(s)\, ds = f_i \qquad \text{for } i = 1, \dots, N, \tag{3.14}$$

$$[u_i, v_i] \in A \qquad \text{for } i = 0, 1, \dots, N. \tag{3.15}$$

Let us observe that, the latter scheme is *fully implicit*. Furthermore, it is concerned with the function $B_\lambda \overline{u}_{\mathcal{P}} \in L^q(0,T;E^*)$ which is not a priori constant on the partition, i.e. $B_\lambda \overline{u}_{\mathcal{P}} \notin K^*_{\mathcal{P}}$. By making use of the above introduced notation we may conveniently reformulate the discrete problem as that of finding a solution $(\overline{u}_{\mathcal{P}}, \overline{v}_{\mathcal{P}}) \in K_{\mathcal{P}} \times K^*_{\mathcal{P}}$ to the problem

$$v'_{\mathcal{P}} + B_{\lambda\mathcal{P}}\, \overline{u}_{\mathcal{P}} = \overline{f}_{\mathcal{P}} \qquad \text{a.e. in } (0,T), \tag{3.16}$$

$$[\overline{u}_{\mathcal{P}}, \overline{v}_{\mathcal{P}}] \in A \qquad \text{a.e. in } (0,T), \tag{3.17}$$

$$v_{\mathcal{P}}(0) = v^0, \tag{3.18}$$

where

$$B_{\lambda\mathcal{P}} := \Lambda_{\mathcal{P}} \circ B_\lambda,$$

and we obviously denote by $v'_{\mathcal{P}}$ the *Euler's backward discretization* term $(\overline{v}_{\mathcal{P}} - \overline{v}_{\mathcal{P}}(\cdot - \overline{\tau}_{\mathcal{P}})/\overline{\tau}_{\mathcal{P}}$, and let $\tau_0 \neq 0$. Note that the discrete analogue of w in (2.4) does not come into play in (3.13)-(3.15) or (3.16)-(3.18). Indeed, since both $\Lambda_{\mathcal{P}}$ and B_λ are single-valued, we will simply define in the sequel

$$\overline{w}_{\mathcal{P}} := B_{\lambda\mathcal{P}}\, \overline{u}_{\mathcal{P}}. \tag{3.19}$$

Our next aim is that of proving that the above discretized problem (3.16)-(3.18) has a solution. To this end, letting $i = 1, \dots, N$ be fixed, we write (3.16)-(3.17) as

$$\Big(\big((\overline{\tau}_{\mathcal{P}})^{-1} A + B_{\lambda\mathcal{P}}\big)\, u\Big)(t) \ni \overline{f}_{\mathcal{P}}(t) + (\overline{\tau}_{\mathcal{P}})^{-1}(t)\, v(t - \overline{\tau}_{\mathcal{P}}) \quad \text{for a.e. } t \in (0, t_i), \tag{3.20}$$

where $[u, v] \in A$ almost everywhere in $(0, t_{i-1})$ and we collected in the above right hand side all the quantities known up to time t_i (letting also $v(r) = v^0$ for all $r < 0$).

Hence, since both A and $B_{\lambda\mathcal{P}}$ are causal on their respective domains, we shall check that

$$\text{for } i = 1, \dots, N \text{ the operator } R_{t_i} \circ \big((\overline{\tau}_{\mathcal{P}})^{-1} A + B_{\lambda\mathcal{P}}\big) \text{ is surjective on the space } R_{t_i}(K^*_{\mathcal{P}}). \tag{3.21}$$

Namely, the well-posedness of (3.16)-(3.18) follows directly from (3.20) and (3.21) by induction on i. As for the prove of the statement (3.21), it suffices to show that

$$\text{the operator } \ (\overline{\tau}_{\mathcal{P}})^{-1}A + B_{\lambda\mathcal{P}} \ \subset K_{\mathcal{P}} \times K_{\mathcal{P}}^{*} \ \text{ is surjective.} \tag{3.22}$$

Indeed, since $(\overline{\tau}_{\mathcal{P}})^{-1}A + B_{\lambda\mathcal{P}}$ is causal (we recall that $D(B_{\lambda\mathcal{P}}) \equiv K_{\mathcal{P}}$) property (3.21) turns out to be an easy consequence of (3.22). For the sake of clarity we split the proof of the latter statement (3.22) into two lemmas.

Lemma 3.6. *Under the above assumptions on* E, *let* $B_\lambda : L^p(0,T;E) \longrightarrow L^q(0,T;E^*)$ *be a single-valued, everywhere defined, demicontinuous, maximal monotone operator and define* $\Lambda_{\mathcal{P}}$ *as in (3.11). Then* $B_{\lambda\mathcal{P}} := \Lambda_{\mathcal{P}} \circ B_\lambda \subset K_{\mathcal{P}} \times K_{\mathcal{P}}^{*}$ *is everywhere defined, demicontinuous, and maximal monotone.*

Proof. Making use of (3.12) one proves that $B_{\lambda\mathcal{P}}$ is monotone. Moreover, we obviously have that B_λ is demicontinuous from $K_{\mathcal{P}}$ to $L^q(0,T;E^*)$ while $\Lambda_{\mathcal{P}}$ is weakly continuous from $L^q(0,T;E^*)$ to $K_{\mathcal{P}}^{*}$. Hence, $B_{\lambda\mathcal{P}}$ turns out to be demicontinuous and everywhere defined as well and the assertion follows from an application of [7, Thm. 1.3, p. 40]. □

Lemma 3.7. *Under assumptions of Lemma 3.6, let* $A \subset E \times E^*$ *be an everywhere defined maximal monotone operator and let* B_λ *be coercive in the sense of (2.3). Then one has that* $(\overline{\tau}_{\mathcal{P}})^{-1}A + B_{\lambda\mathcal{P}} \subset K_{\mathcal{P}} \times K_{\mathcal{P}}^{*}$ *is surjective.*

Proof. Since the interior of $D(A)$ (i.e. the domain of the $L^p(0,T;E)-L^q(0,T;E^*)$ realization of A) contains $K_{\mathcal{P}} \equiv D(B_{\lambda\mathcal{P}})$, we deduce from [7, Thm. 1.7, p. 46] that $(\overline{\tau}_{\mathcal{P}})^{-1}A + B_{\lambda\mathcal{P}} \subset K_{\mathcal{P}} \times K_{\mathcal{P}}^{*}$ is maximal monotone. Moreover, it is easily checked that, given $u \in K_{\mathcal{P}}$, one has

$$\int_0^T \langle (\overline{\tau}_{\mathcal{P}})^{-1} v, u\rangle + \int_0^T \langle B_{\lambda\mathcal{P}}\, u, u\rangle \geq \int_0^T \langle B_\lambda u, u\rangle,$$

for all $[u,v] \in Au$ almost everywhere in $(0,T)$. Hence, in particular, $(\overline{\tau}_{\mathcal{P}})^{-1}A + B_{\lambda\mathcal{P}}$ turns out to be coercive on $K_{\mathcal{P}}$ and the assertion follows from [7, Thm 1.3, p. 40]. □

We shall now exploit the induction procedure on i. First of all let us observe that the property (3.21) ensures in particular the existence of a solution $u \in R_{t_1}(K_{\mathcal{P}})$ of the equation

$$\left(\tau_1^{-1}A + B_{\lambda\mathcal{P}}\right) u(t) \ni f_1 + \tau_1^{-1} v^0 \quad \text{for a.e. } \ t \in (0,t_1).$$

Next, assume we are given $u \in R_{t_{i-1}}(K_{\mathcal{P}})$ and $v \in R_{t_{i-1}}(K_{\mathcal{P}}^{*})$ such that both $[u,v] \in A$ and relation (3.20) are fulfilled almost everywhere in $(0,t_{i-1})$. Then, taking into account the property (3.21), we may find $u_* \in R_{t_i}(K_\mathcal{P})$ and $v_* \in R_{t_i}(K^*_\mathcal{P})$ such that

$$(\overline{\tau}_\mathcal{P})^{-1} v_*(t) + B_{\lambda\mathcal{P}}\, u_*(t) = \overline{f}_\mathcal{P}(t) + (\overline{\tau}_\mathcal{P})^{-1}(t)\, v(t-\overline{\tau}_\mathcal{P}), \quad \text{and} \quad [u_*(t), v_*(t)] \in A,$$

for almost every $t \in (0, t_i)$. Now, in order to conclude the induction step, it suffices to extend v by v_* on $[t_{i-1}, t_i)$ and show that $R_{t_{i-1}} u_* \equiv u$. To this aim, let us take the difference between the latter relation restricted to $(0, t_{i-1})$ and equation (3.20). One obtains that

$$\big((\overline{\tau}_\mathcal{P})^{-1} v_* + B_{\lambda\mathcal{P}}\, u_*\big) - \big((\overline{\tau}_\mathcal{P})^{-1} v + B_{\lambda\mathcal{P}}\, u\big) = 0 \quad \text{a.e. in} \quad (0, t_{i-1}).$$

Hence, testing the above relation on $u_* - u$, integrating on $(0, t_{i-1})$, and taking into account the causality and strict monotonicity of $B_{\lambda\mathcal{P}}$, we deduce that $R_{t_{i-1}} u_* \equiv u$. Finally the induction step is complete and the existence of a discrete solution is established.

3.3 A priori estimates

Once we have proved the existence of a solution to the problem (3.16)-(3.18), our next aim is to establish some a priori bound on $\overline{u}_\mathcal{P}, \overline{v}_\mathcal{P}$, and $\overline{w}_\mathcal{P}$ independent of the partition $\mathcal{P}$ and of the regularization parameter $\lambda \in (0,1)$. To this end, let us test equation (3.16) by $\overline{u}_\mathcal{P}$ and integrate with respect to time. Owing to the subdifferential structure of A and the initial condition (3.18) we readily get

$$\psi^*\big(v_\mathcal{P}(T)\big) + \int_0^T \langle \overline{w}_\mathcal{P}, \overline{u}_\mathcal{P} \rangle \leq \psi^*\big(v^0\big) + \int_0^T \langle \overline{f}_\mathcal{P}, \overline{u}_\mathcal{P} \rangle. \tag{3.23}$$

Taking into account the positivity of ψ^* and relation (3.12), one has that the previous estimate entails, in particular,

$$\int_0^T \langle B_\lambda \overline{u}_\mathcal{P}, \overline{u}_\mathcal{P} \rangle \leq \psi^*\big(v^0\big) + \int_0^T \langle \overline{f}_\mathcal{P}, \overline{u}_\mathcal{P} \rangle. \tag{3.24}$$

Hence, owing to Lemma 3.4 and the uniform boundedness of $\{\overline{f}_\mathcal{P}\}$ in $L^q(0,T;E^*)$, we deduce that

$$\|\overline{u}_\mathcal{P}\|_{L^p(0,T;E)} \quad \text{is bounded independently of } \mathcal{P} \text{ and } \lambda \in (0,1). \tag{3.25}$$

Now, is it a standard matter to obtain from the above bound, (H1), and (3.8) that

$$\begin{aligned} &\|v_\mathcal{P}\|_{L^q(0,T;F^*)},\ \|B_\lambda \overline{u}_\mathcal{P}\|_{L^q(0,T;E^*)}, \text{ and } \|\overline{w}_\mathcal{P}\|_{L^q(0,T;E^*)} \\ &\quad \text{are bounded independently of } \mathcal{P} \text{ and } \lambda \in (0,1). \end{aligned} \tag{3.26}$$

Hence, a comparison in (3.16) and the boundedness of $\{\overline{f}_{\mathcal{P}}\}$ entail that

$$\|v_{\mathcal{P}}\|_{W^{1,q}(0,T;E^*)} \quad \text{is bounded independently of } \mathcal{P} \text{ and } \lambda \in (0,1). \tag{3.27}$$

For later purposes, let us set c_2 to be a positive constant that bounds the latter quantities uniformly with respect to $\mathcal{P}$ and $\lambda \in (0,1)$. In particular, we stress that c_2 just depends on $\|f\|_{L^q(0,T;E^*)}$, $\psi^*(v^0)$, and the boundedness and coercivity properties of the operators A and B.

Before passing to the limit we stress that a straightforward computation yields that

$$\|v_{\mathcal{P}} - \overline{v}_{\mathcal{P}}\|_{L^\infty(0,T;E^*)} \leq \tau^{1/p}\|v'_{\mathcal{P}}\|_{L^q(0,T;E^*)}, \tag{3.28}$$

where we recall that τ stands for the diameter of the partition $\mathcal{P}$.

3.4 Passage to the limit in the scheme

Taking into account the estimates (3.25)-(3.27), we find three functions u_λ, v_λ, and w_λ such that, possibly passing to a subsequence (not relabeled), the following convergences hold

$$\overline{u}_{\mathcal{P}} \longrightarrow u_\lambda \quad \text{weakly in} \quad L^p(0,T;E), \tag{3.29}$$

$$v_{\mathcal{P}} \longrightarrow v_\lambda \quad \text{weakly in} \quad W^{1,q}(0,T;E^*) \cap L^q(0,T;F^*), \tag{3.30}$$

$$\overline{w}_{\mathcal{P}} \longrightarrow w_\lambda \quad \text{weakly in} \quad L^q(0,T;E^*), \tag{3.31}$$

as the diameter τ of partition $\mathcal{P}$ goes to 0. Moreover, owing to the generalized Ascoli theorem (see, for instance, [25, Cor. 4]) and the estimate (3.28), we deduce from (3.30) that

$$v_{\mathcal{P}} \longrightarrow v_\lambda \quad \text{strongly in} \quad L^q(0,T;E^*), \tag{3.32}$$

$$\overline{v}_{\mathcal{P}} \longrightarrow v_\lambda \quad \text{strongly in} \quad L^q(0,T;E^*), \tag{3.33}$$

$$v_{\mathcal{P}}(t) \longrightarrow v_\lambda(t) \quad \text{weakly in} \quad F^*, \quad \forall t \in [0,T]. \tag{3.34}$$

The above convergences are sufficient to pass to the limit in the equation (3.16) and obtain

$$v'_\lambda(t) + w_\lambda(t) = f(t) \quad \text{a.e. in} \quad (0,T), \tag{3.35}$$

along with the initial condition. In order to conclude that the triplet $(u_\lambda, v_\lambda, w_\lambda)$ solves our regularized problem it remains to prove that

$$[u_\lambda(t), v_\lambda(t)] \in A \quad \text{for a.e. } t \in (0,T) \qquad \text{and} \qquad [u_\lambda, w_\lambda] \in B_\lambda. \tag{3.36}$$

First of all, we easily deduce from (3.29) and (3.33) that

$$\int_0^T \langle \overline{v}_{\mathcal{P}}, \overline{u}_{\mathcal{P}} \rangle \longrightarrow \int_0^T \langle v_\lambda, u_\lambda \rangle,$$

and the first relation of (3.36) is ensured by means of classical results on maximal monotone operators (see, e.g., [7, Lemma 1.3, p. 42]).

As regards the second inclusion of (3.36), taking the $\limsup$ in relation (3.23) as the diameter τ of partition $\mathcal{P}$ goes to 0 and making use of (3.29), (3.34)-(3.35), the lower semicontinuity of ψ^*, and the strong convergence of $\overline{f}_{\mathcal{P}}$, we obtain that

$$\limsup_{\tau\to 0}\int_0^T \langle \overline{w}_{\mathcal{P}}, \overline{u}_{\mathcal{P}}\rangle \le \int_0^T \langle w_\lambda, u_\lambda\rangle. \tag{3.37}$$

Now, let $[\xi,\eta] \in B_\lambda$ and define $\overline{\eta}_{\mathcal{P}} := \Lambda_{\mathcal{P}}\eta$ such that $[\xi,\overline{\eta}_{\mathcal{P}}] \in B_{\lambda\mathcal{P}} \subset L^p(0,T;E)\times K^*_{\mathcal{P}}$. Due to formula (3.12) and the monotonicity of B_λ we obtain

$$\begin{aligned}\int_0^T \Big\langle \overline{w}_{\mathcal{P}} - \overline{\eta}_{\mathcal{P}}, \overline{u}_{\mathcal{P}} - \xi\Big\rangle &= \int_0^T \Big\langle B_\lambda \overline{u}_{\mathcal{P}} - B_\lambda\xi, \overline{u}_{\mathcal{P}} - \Lambda_{\mathcal{P}}\xi\Big\rangle \\ = \int_0^T \Big\langle B_\lambda \overline{u}_{\mathcal{P}} - B_\lambda\xi, \overline{u}_{\mathcal{P}} - \xi\Big\rangle &+ \int_0^T \Big\langle B_\lambda \overline{u}_{\mathcal{P}} - B_\lambda\xi, \xi - \Lambda_{\mathcal{P}}\xi\Big\rangle \\ &\ge \int_0^T \Big\langle B_\lambda \overline{u}_{\mathcal{P}} - B_\lambda\xi, \xi - \Lambda_{\mathcal{P}}\xi\Big\rangle. \end{aligned}\tag{3.38}$$

Since we obviously have that $\xi - \Lambda_{\mathcal{P}}\xi$ strongly converges to 0 in $L^p(0,T;E)$ together with the diameter τ of the partition $\mathcal{P}$ and that $B_\lambda\overline{u}_{\mathcal{P}}$ is bounded in $L^q(0,T;E^*)$ independently of $\mathcal{P}$ and $\lambda \in (0,1)$, we conclude that the above right hand side goes to 0 with τ. Thus, passing to the $\limsup$ as $\tau\to 0$ in both sides of (3.38) and taking into account (3.29), (3.31), and (3.37), we achieve

$$\begin{aligned}0 \le \limsup_{\tau\to 0}\int_0^T \Big(\langle \overline{w}_{\mathcal{P}}, \overline{u}_{\mathcal{P}}\rangle - \langle \overline{w}_{\mathcal{P}}, \xi\rangle - \langle \overline{\eta}_{\mathcal{P}}, \overline{u}_{\mathcal{P}}\rangle + \langle \overline{\eta}_{\mathcal{P}}, \xi\rangle\Big) \\ \le \int_0^T \langle w_\lambda - \eta, u_\lambda - \xi\rangle\, ds.\end{aligned}$$

We have proved the above relation for all $[\xi,\eta]\in B_\lambda$. Thus, the maximality of B_λ yields that $[u_\lambda, w_\lambda]\in B_\lambda$ as well. Finally, the existence of a solution to the regularized problem (2.4)-(2.7) (where B is replaced by B_λ) is proved.

A technical point. We want to stress why the usual monotonicity tools (see, e.g., [6, Prop. 3.59, p. 361]) fail to work in our framework and we have to perform the above *ad hoc* identification procedure.

Let us denote by V a reflexive Banach space, by V^* its dual, and by $\langle\cdot,\cdot\rangle$ the respective duality pairing. Furthermore, let $B, B_n : V \longrightarrow V^*$ $(n\in\mathbb{N})$ be maximal monotone operators. We say that B_n *converges to* B *in the sense of G-convergence in* $V\times V^*$ if, for any $[u,w]\in B$, there exists a sequence $[u_n,w_n]\in B_n$ strongly converging to $[u,w]$ in $V\times V^*$ (the reader is referred to [6, Sect. 7.1.2, p. 360] for the details). The latter notion of convergence for operators is the natural one for maximal monotone nonlinearities. In particular, we have that it allows the identification of weak limits. Namely, if $[u_n, w_n] \in B_n$ converges weakly to $[u, w]$ in $V \times V^*$, B_n converges to B in the sense of G-convergence in $V \times V^*$ and

$$\liminf_{n \to +\infty} \langle w_n, u_n \rangle \le \langle w, u \rangle,$$

then $[u, w] \in B$.

In our particular framework $V = L^p(0,T;E)$ and $B_n := B_{\lambda \mathcal{P}_n}$ with the diameters τ_n of partitions $\mathcal{P}_n$ going to 0, we cannot apply the latter identification argument. In fact, we may prove the G-convergence of B_n to B_λ in $L^p(0,T;E) \times L^q(0,T;E^*)$ by choosing, for any $[u, w] \in B_\lambda$, $u_n := u$ and $w_n := \Lambda_{\mathcal{P}_n} w$, and we cannot prove the latter G-convergence by choosing $u_n \in K_{\mathcal{P}_n}$. On the other hand, the operator B_n is monotone on $K_{\mathcal{P}_n}$ and not on $L^p(0,T;E)$.

The situation is much simpler when E is *uniformly convex* (for instance when E is a Hilbert space). Indeed, in this situation one may check that B_λ is strongly continuous (see the analysis in [14]) and we can identify the limit w by exploiting the auxiliary operator $B_n \circ \Lambda_{\mathcal{P}_n}$. In fact, the latter operator is monotone on the whole space $L^p(0,T;E)$, agrees with B_n on $K_{\mathcal{P}_n}$, and G-converges to B_λ in $L^p(0,T;E) \times L^q(0,T;E^*)$.

However, in the general case of a strictly convex and reflexive space E, the regularization B_λ is not expected to be strongly continuous and the G-convergence of $B_n \circ \Lambda_{\mathcal{P}_n}$ to B_n in $L^p(0,T;E) \times L^q(0,T;E^*)$ fails to hold.

3.5 Passage to the limit in the approximation

In order to conclude the proof of Theorem 2.1 we shall now pass to the limit as $\lambda \in (0,1)$ goes to 0. Let us just sketch this procedure since it suffices to repeat the argument exploited above for the passage to the limit in the discrete scheme. Indeed, since (3.25)-(3.27) are independent of $\lambda \in (0,1)$, we find a triplet of functions (u, v, w) such that the following convergences hold for some (not relabeled) subsequences

$$\begin{array}{lll} u_\lambda \longrightarrow u & \text{weakly in} & L^p(0,T;E), \\ v_\lambda \longrightarrow v & \text{weakly in} & W^{1,q}(0,T;E^*) \ \text{ and strongly in } \ L^q(0,T;E^*), \\ w_\lambda \longrightarrow w & \text{weakly in} & L^q(0,T;E^*), \end{array}$$

and equation (2.4) holds. Finally, we prove (2.5)-(2.6) by arguing as in Subsection 3.4 and exploiting the latter convergences together with the operator convergence (3.1).

Remark 3.8. We shall observe that the necessity of a regularization for the operator B is just intended to ensure that the operator on which we perform our discretization is single-valued, strictly monotone, and demicontinuous. Of course, in the case of a single-valued, strictly monotone, and demicontinuous operator B, no regularization is actually needed.

4 Uniqueness and continuous dependence

As far as uniqueness is concerned, we just remark that problem (2.4)-(2.7) admits, in general, multiple solutions. First of all, note that the operators A and B may be multivalued and both v and w are actually selections. Hence, it is straightforward to find examples of non-uniqueness for v and w. Moreover, also counterexamples to uniqueness for u are known (see [13, Sec. 5]). Let us remark that non-uniqueness is not related to our functional setting nor to the nonlocality of B. In particular, the above referred counterexample to uniqueness for u applies to the case $E = F = \mathbb{R}$ and A, B local in time subdifferentials.

Nevertheless, for the sake of forthcoming numerical considerations, it is worth to provide a (very reductive) uniqueness and continuous dependence result relying on some further assumption on the operators A and B. In particular we ask A to be linear, continuous, and symmetric, and $A+B$ to be *strongly monotone.* That is, we assume that there exists a positive constant κ such that, for all $[u_1, z_1], [u_2, z_2] \in A + B$, one has

$$\kappa \|u_1 - u_2\|^p_{L^p(0,T;E)} \leq \int_0^T \langle z_1 - z_2, u_1 - u_2 \rangle, \tag{4.1}$$

(however, note that the choice of a constant $\gamma > 1$ instead of p in the latter relation brings to analogous results and that we reduce ourselves to the present situation just for the sake of simplicity). The reader should notice that the following lemma is nothing but the extension to nonlocal operators B and continuous dependence of the former uniqueness result [13].

Lemma 4.1. *Let $A = \partial\psi : E \longrightarrow E^*$ be linear, continuous, and symmetric, $B \subset L^p(0,T;E) \times L^q(0,T;E^*)$ be monotone and $A + B$ be strictly monotone. Then, for any $f : [0,T] \longrightarrow E^*$ and $v^0 \in R(A)$, there exists at most one solution to problem (1.1). Moreover, let (u_i, v_i, w_i) solve (2.4)-(2.7) for $(f_i, v_i^0) \in L^q(0,T;E^*) \times R(A)$, $i = 1, 2$, and $A + B$ be strongly monotone of constant κ. Then, there exists a positive constant c_3 depending only on κ, q, and T such that*

$$\|u_1 - u_2\|^p_{L^p(0,T;E)} \leq c_3 \left(\psi^*(v_1^0 - v_2^0) + \|f_1 - f_2\|^q_{L^q(0,T;E^*)} \right). \tag{4.2}$$

Proof. Let us take the difference between (2.4) written for (u_1, v_1, w_1) and the same equation written for (u_2, v_2, w_2). Then, we test the resulting equation by

$u_1 - u_2$ and integrate in time obtaining

$$\frac{1}{2}\langle A(u_1-u_2)(t),(u_1-u_2)(t)\rangle + \int_0^t \langle w_1 - w_2, u_1 - u_2\rangle$$
$$= \psi^*\left(v_1^0 - v_2^0\right) + \int_0^t \langle f_1 - f_2, u_1 - u_2\rangle, \qquad \forall t \in [0,T].$$

Thus, we have in particular that

$$\int_0^T \langle (Au_1 + w_1) - (Au_2 + w_2), u_1 - u_2\rangle$$
$$\leq (1+2T)\left(\psi^*\left(v_1^0 - v_2^0\right) + \|f_1 - f_2\|_{L^q(0,T;E^*)}\|u_1 - u_2\|_{L^p(0,T;E)}\right).$$

Hence, whenever $A+B$ is strictly monotone and $f_1 = f_2 = f$, $v_1^0 = v_2^0 = v^0$ (recall that we can fix with no loss of generality $\psi^*(0) = 0$), uniqueness of a solution to problem (1.1) follows. Moreover, the continuous dependence estimate (4.2) follows whenever $A+B$ is strongly monotone. In that case, it is straightforward to choose

$$c_3 := \max\left\{ q\left(\frac{1+2T}{\kappa}\right), \left(\frac{1+2T}{\kappa}\right)^q \right\}. \quad \square$$

5 Convergence and error estimates

In this section we prove that, taking into account the strong monotonicity of the operator B, we actually obtain the further convergence

$$\overline{u}_{\mathcal{P}} \longrightarrow u \quad \text{strongly in} \quad L^p(0,T;E), \tag{5.1}$$

which was not inferred by compactness (see (3.29)).

Let B be single-valued and demicontinuous. In this respect there is no need to regularize it further by means of B_λ and we perform our discretization directly on B (see Remark 3.8). For the sake of numerical purposes, we shall comment the possibility of approximating also the initial data. Indeed, this eventuality has not been exploited in Subsection 4.2. Nevertheless, letting $v_{\mathcal{P}}^0 \in R(A)$, we easily extend our discrete analysis to approximated initial data by replacing v^0 by $v_{\mathcal{P}}^0$ in (3.13) and (3.18). Thus, in the forthcoming convergence analysis, we are forced to assume that the approximated initial data $[u_{\mathcal{P}}^0, v_{\mathcal{P}}^0] \in A$ fulfill (see also (4.2))

$$\psi^*\left(v^0 - v_{\mathcal{P}}^0\right) \longrightarrow 0 \quad \text{as} \quad \tau \longrightarrow 0. \tag{5.2}$$

We have the following result.

Lemma 5.1. *Let assumptions (H1)-(H2) hold. Moreover, let A be linear, continuous, and symmetric and B be single-valued, demicontinuous, and strongly monotone. Moreover, let (u,v,w) be the solution of (2.4)-(2.7) and $(\overline{u}_{\mathcal{P}}, \overline{v}_{\mathcal{P}})$ be the solution on (3.16)-(3.18) (where B_λ and v^0 are replaced by B and $v^0_{\mathcal{P}}$ fulfilling (5.2), respectively). Then, the convergence (5.1) holds.*

Proof. Let us take the difference between (2.4) and (3.16), test it by $u - \overline{u}_{\mathcal{P}}$, and integrate on $(0,T)$. Recalling that $\overline{w}_{\mathcal{P}} = (\Lambda_{\mathcal{P}} \circ B)\overline{u}_{\mathcal{P}}$, we readily get that

$$\int_0^T \langle v' - v'_{\mathcal{P}}, u - \overline{u}_{\mathcal{P}} \rangle + \int_0^T \langle Bu - \overline{w}_{\mathcal{P}}, u - \overline{u}_{\mathcal{P}} \rangle = \int_0^T \langle f - \overline{f}_{\mathcal{P}}, u - \overline{u}_{\mathcal{P}} \rangle. \tag{5.3}$$

Since $\langle v'_{\mathcal{P}}, u_{\mathcal{P}} - \overline{u}_{\mathcal{P}} \rangle \le 0$ almost everywhere in $(0,T)$ and formula (3.12) holds, we simply deduce that

$$\begin{aligned} &\frac{1}{2}\langle A(u - u_{\mathcal{P}})(T), (u - u_{\mathcal{P}})(T) \rangle + \int_0^T \langle Bu - B\overline{u}_{\mathcal{P}}, u - \overline{u}_{\mathcal{P}} \rangle \le \psi^*\left(v^0 - v^0_{\mathcal{P}}\right) \\ &\quad + \int_0^T \langle f - \overline{f}_{\mathcal{P}}, u - \overline{u}_{\mathcal{P}} \rangle + \int_0^T \langle B\overline{u}_{\mathcal{P}}, \Lambda_{\mathcal{P}} u - u \rangle + \int_0^T \langle v', \overline{u}_{\mathcal{P}} - u_{\mathcal{P}} \rangle. \end{aligned} \tag{5.4}$$

The assertion follows from the strong monotonicity of B, the strong convergence of $\overline{f}_{\mathcal{P}}$ to f, bounds (3.26)-(3.27), convergences (3.29) and (5.2), the uniqueness of the limit ensured by Lemma 4.1, and the strong convergence

$$\Lambda_{\mathcal{P}} u \longrightarrow u \quad \text{strongly in } L^p(0,T;E),$$

as the diameter τ of partition $\mathcal{P}$ goes to zero. □

Moreover, whenever the solution u shows some regularity with respect to time we are in the position of deducing an estimate for the discretization error. Indeed, we have the following.

Lemma 5.2. *Let the assumptions of Lemma 5.1 hold. Then, there exists a positive constant c_4 such that*

$$\begin{aligned} \|u - \overline{u}_{\mathcal{P}}\|^p_{L^p(0,T;E)} &\le c_4\Big(\psi^*\left(v^0 - v^0_{\mathcal{P}}\right) + \|f - \overline{f}_{\mathcal{P}}\|^q_{L^q(0,T;E^*)}\Big) \\ &\quad + c_4\Big(\|u'\|_{L^p(0,T;E)} + \|u'_{\mathcal{P}}\|_{L^p(0,T;E)}\Big)\,\tau, \end{aligned} \tag{5.5}$$

where the constant c_4 depends on $c_2, q,$ and κ.

Proof. It suffices to recall (5.4) and observe that

$$\int_0^T \|\Lambda_{\mathcal{P}} u - u\|_E^p = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\| \frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} u(s)\, ds - u(t) \right\|^p dt$$

$$\le \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left(\int_{t_{i-1}}^{t_i} \|u'\|_E \right)^p \le \sum_{i=0}^N \int_{t_{i-1}}^{t_i} \tau_i^{p/q} \int_{t_{i-1}}^{t_i} \|u'\|_E^p dt \le \tau^p \|u'\|^p_{L^p(0,T;E)},$$

and

$$\|\overline{u}_{\mathcal{P}} - u_{\mathcal{P}}\|^p_{L^p(0,T;E)} = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \Big((1-\alpha_i(t))\, \|u_i - u_{i-1}\| \Big)^p dt$$

$$= \frac{1}{p+1} \sum_{i=1}^N \tau_i \|u_i - u_{i-1}\|^p \le \frac{\tau^p}{p+1} \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|u'_{\mathcal{P}}\|^p = \frac{\tau^p}{p+1} \|u'_{\mathcal{P}}\|^p_{L^p(0,T;E)},$$

in order to conclude for the validity of the lemma. □

Of course, recalling (3.25)-(3.27), we stress that the norms of u' and $u'_{\mathcal{P}}$ in $L^p(0,T;E)$ are not *a priori* bounded in our setting (i.e., bounded in terms of c_2). Nevertheless, let us remark that our analysis could be easily adapted to the case of problems (1.1) with $A = A_1 + A_2$, where A is a subdifferential, A_1 fulfills (H1), and A_2 is strongly monotone, linear, and continuous. The latter kind of operators arises in connection of so called *pseudo-parabolic* problems [11, 23], whose analysis is beyond the purposes of this paper (the reader is referred to [26] for some detail in this direction). In the latter case $A = A_1 + A_2$ one has that both the above cited norms are bounded in terms of data.

We conclude this section by proving the possibility of estimating the discretization error *a posteriori*, i.e. taking into account quantities depending on the discrete solution $u_{\mathcal{P}}$. To this aim and for the sake of clarity, we assume some regularity in time for f, fix its approximation, and ask for an order of convergence for $\psi^*(v^0 - v^0_{\mathcal{P}})$ as τ goes to 0.

Lemma 5.3. *Let the assumptions of Lemma 5.1 hold. Moreover let* $f \in W^{1,q}(0,T;E^*)$, $\overline{f}_{\mathcal{P}} := \Lambda_{\mathcal{P}} f$, *and* $\psi^*(v^0 - v^0_{\mathcal{P}}) \le c_{data}\, \tau^q$ *for a given constant* $c_{data} > 0$. *Then, there exists a positive constant* c_5 *such that*

$$\|u - \overline{u}_{\mathcal{P}}\|^q_{L^p(0,T;E)} \le c_5 \Big(1 + \|(B\overline{u}_{\mathcal{P}})'\|^q_{L^q(0,T;E^*)} \Big) \tau^q + c_5 \|u'_{\mathcal{P}}\|_{L^p(0,T;E)}\, \tau, \tag{5.6}$$

where c_5 *depends on* c_{data}, c_2, $\|f'\|_{L^q(0,T;E^*)}$, q, *and* κ.

Proof. Recalling (5.3) and exploiting the monotonicity of A we readily obtain that

$$\int_0^T \langle Bu - B\overline{u}_{\mathcal{P}}, u - \overline{u}_{\mathcal{P}} \rangle \leq \psi^*(v^0 - v^0_{\mathcal{P}}) + \int_0^T \langle f - \overline{f}_{\mathcal{P}}, u - \overline{u}_{\mathcal{P}} \rangle$$
$$+ \int_0^T \langle \overline{w}_{\mathcal{P}} - B\overline{u}_{\mathcal{P}}, u - \overline{u}_{\mathcal{P}} \rangle + \int_0^T \langle v', \overline{u}_{\mathcal{P}} - u_{\mathcal{P}} \rangle.$$

Since $\overline{w}_{\mathcal{P}} = (\Lambda_{\mathcal{P}} \circ B)\overline{u}_{\mathcal{P}}$, the assertion follows arguing as in the proof of Lemma 5.2. □

Let us stress that the latter estimate has, at the moment, a purely theoretical interest. Indeed, it seems to be not adequate for the purposes of a direct numerical implementation. In fact, while the norm of $u'_{\mathcal{P}}$ in $L^p(0,T;E)$ is *computable* in terms of the discrete solution (see the forthcoming Remark 5.4), it is not possible to evaluate efficiently the norm of $(B\overline{u}_{\mathcal{P}})'$ in $L^p(0,T;E)$. Of course, whenever the operator B maps bounded sets of $L^p(0,T;E)$ into bounded sets of $W^{1,q}(0,T;E^*)$ (this is indeed the case of a large class of Volterra integrodifferential operators, see below), the estimate (5.6) turns out to have an relevant numerical interest as well. Indeed, for the latter class of operators B, estimate (5.6) provides an *a posteriori* error bound of order $1/2$. Moreover, since no constraints are imposed on the time-steps, one may possibly choose the partition $\mathcal{P}$ according to some adaptive strategy. Finally, it is simple to give some explicit example for the constants c_4 and c_5 in terms of c_{data}, c_2, $\|f'\|_{L^q(0,T;E^*)}$, q, and κ. Namely, a straightforward choice is

$$c_4 := \max\left\{\frac{q}{\kappa}, \frac{1}{\kappa^q}, \frac{qc_2}{\kappa}\right\},$$
$$c_5 := \max\left\{\frac{qc_{data}}{\kappa} + \frac{2^{q-1}}{\kappa^q}\|f'\|^q_{L^q(0,T;E^*)}, \frac{2^{q-1}}{\kappa^q}, c_2\right\}. \tag{5.7}$$

Remark 5.4. We stress that the present formulation of estimates (5.5)-(5.6) is motivated by the sake of clarity. In particular the same estimates can be easily improved by replacing

$$\|u'_{\mathcal{P}}\|_{L^p(0,T;E)} \quad \text{by} \quad \left(\sum_{i=1}^N \tau_i \|u_i - u_{i-1}\|^p\right)^{1/p}.$$

This modification emphasizes the *a posteriori* character of estimates (5.5)-(5.6).

6 Approximation of Volterra operators

One of the interesting features of our abstract time discretization of problem (1.1) is that it turns out to be especially well tailored for Volterra integrodifferential operators. Throughout the remainder of this section we will consider the case of an operator $B := B_0 + B_1$ such that

$$B_0 \subset E \times E^* \quad \text{maximal monotone, bounded on bounded sets, and coercive,}$$

$$(B_1 u)(t) := \int_0^t k(t,s)\, Cu(s)\, ds \qquad \forall u \in L^p(0,T;E),\ t \in [0,T],$$

where $k : (0,T) \times (0,T) \longrightarrow \mathbb{R}$ is a given kernel and $C : E \longrightarrow E^*$ is a linear, continuous, and non-negative operator. Moreover, we ask B_1 to be bounded by prescribing

$$\sup_{\substack{\|u\|_{L^p(0,T)} \le 1 \\ \|v\|_{L^q(0,T)} \le 1}} \int_0^T \int_0^T |v(t)\, k(t,s)\, u(s)| ds\, dt < +\infty, \tag{6.1}$$

and the reader is referred to [17, Sec. 9.2] for details. Namely, it is straightforward to check that we include in our analysis kernels k of the type $k(t,s) = h(t-s)$ for some $h \in L^1(0,T)$. Thus, we possibly consider equations of convolution type. Let us stress that, in order to give an example of a possible application of our error estimate (5.6), we may simply ask the function h to be in $W^{1,1}(0,T)$. Indeed, the latter assumption ensures in particular that the operator B_1 is bounded from $L^p(0,T;E)$ into $W^{1,q}(0,T;E^*)$. Of course, this example has been chosen for the sake of simplicity and the latter boundedness is fulfilled also by a large class of general Volterra operators of non-convolution type.

Finally, we ensure the monotonicity of B_1 by asking k to be of *positive type*, that is

$$\int_0^{t_*} \left(\int_0^t k(t,s)\, v(s)\, ds \right) v(t)\, dt \ge 0 \quad \forall v \in L^p(0,T),\ t_* \in (0,T). \tag{6.2}$$

Note that, owing to (6.1), the function $t \mapsto \int_0^t k(t,s) v(s)\, ds$ belongs to $L^q(0,T)$ for all $u \in L^p(0,T)$ so that the integral above makes sense. The reader is referred to [17, Sec 20.2] for a full mathematical discussion on positive Volterra kernels. Finally, we readily check that the operator B fulfills (H2).

We now turn to the analysis of the approximation of B_1. Since the latter operator is single-valued and continuous, we will not perform its regularization by means of the Yosida approximation (recall Remark 3.8) and will limit ourselves to consider the discretized operator

$$\Lambda_{\mathcal{P}} \circ B_1|_{K_{\mathcal{P}}} : K_{\mathcal{P}} \longrightarrow K_{\mathcal{P}}^*.$$

Indeed, letting $\{u_j\}_{j=0}^N \in E^{N+1}$, we easily deduce that

$$(\Lambda_{\mathcal{P}} \circ B_1)(\overline{u}_{\mathcal{P}})(t) = \sum_{j=1}^{i} \tau_j k_{ij}\, Cu_j \qquad \forall t \in (t_{i-1}, t_i],\ i = 1, \ldots, N,$$

where the coefficients $k_{ij} \in \mathbb{R}$ are given by

$$k_{ij} := \frac{1}{\tau_i \tau_j} \int_{t_{i-1}}^{t_i} \left(\int_{t_{j-1}}^{\min\{t,t_j\}} k(t,s)\, ds \right) dt \quad \forall i,j = 1,\ldots,N. \tag{6.3}$$

Of course we shall stress that one really needs just those k_{ij} with $i \geq j$. Indeed, let us remark that the quantity $(\Lambda_{\mathcal{P}} \circ B_1)(\overline{u}_{\mathcal{P}})(t)$ for $t \in (t_{i-1}, t_i]$ is computable just in terms of $\{u_j\}_{j=0}^{i}$. This is nothing but the discrete analogue of causality.

Moreover, the existence of a solution to problem (3.13)-(3.15) turns out to be straightforward. Indeed, in the above framework we simply have that (3.14)-(3.15) at level i may be rewritten as

$$Au_i + \tau_i B_0 u_i + \tau_i^2 k_{ii} C u_i \ni \tau_i f_i + v_{i-1} - \tau_i \sum_{j=1}^{i-1} \tau_j k_{ij} C u_j, \tag{6.4}$$

where we collected in the right hand side all the quantities known up to time t_i. Now, for the sake of proving the existence of a solution u_i to the latter equation it suffices to prove that $\tau_i^2 k_{ii} \geq 0$. In fact, we may let $v(t) = 1$ if $t \in (t_{i-1}, t_i)$, $v(t) = 0$ otherwise and exploit the relation (6.2) obtaining

$$0 \leq \int_{t_{i-1}}^{t_i} \left(\int_{t_{i-1}}^{t} k(t,s)\, ds \right) dt = \tau_i^2 k_{ii}.$$

Hence, since the operators A, B_0, and C are all bounded on bounded sets and monotone, and B_0 is coercive it turns out that the operator $A + \tau_i B_0 + \tau_i^2 k_{ii} C$ is maximal monotone and coercive as well. Finally, it suffices to apply [7, Thm 1.3, p. 40] in order to prove that the latter operator is surjective and equation (6.4) has a solution in E, for all $i = 1, \ldots, N$. Note that, in this case, no strict monotonicity is required for B.

We conclude this discussion by remarking that the possibility of considering quadrature formulas taking into account coefficients as in (6.3) was already mentioned in [22] where the authors detail some numerical considerations about some related discretization methods.

Acknowledgment

The main part of this research was performed during a visit to Germany under the sponsorship of the Programma Vigoni 1999-2000: *Phase transitions, Stefan type problems, and minimizing movements.* The kind hospitality of the Weierstrass Institut für Angewandte Analysis und Stochastik - Berlin is gratefully acknowledged. Moreover, the author is indebted to Prof. Giuseppe Savaré for some fruitful discussion about Lemma 3.3 and to the Referee for her/his careful reading of the manuscript.

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