On a Class of Doubly Nonlinear Nonlocal Evolution Equations*

Ulisse Stefanelli
Dipartimento di Matematica, Università di Pavia,
via Ferrata 1, 27100 Pavia, Italy
e-mail: ulisse@dimat.unipv.it

Abstract

This note deals with the initial value problem for the abstract nonlinear nonlocal equation $(\mathcal{A}u)' + (\mathcal{B}u) \ni f$, where $\mathcal{A}$ is a possibly degenerate maximal monotone operator from the Hilbert space V to its dual space V^*, while $\mathcal{B}$ is a nonlocal maximal monotone operator from $L^2(0,T,V)$ to $L^2(0,T;V^*)$. Assuming suitable boundedness and coerciveness conditions and letting $\mathcal{A}$ be a subgradient, existence of a solution is established by making use of an approximation procedure. Applications to various classes of degenerate nonlinear integrodifferential equations are discussed.

Key words: nonlocal evolution equations, abstract Cauchy problem, existence, uniqueness, approximation, time discretization.

AMS (MOS) Subject Classification: 35K55, 45N05.

1 Introduction

Let V be an Hilbert space and denote by V^* its dual. Moreover, let T be some reference time. We are given two possibly multivalued maximal monotone operators $\mathcal{A} : V \to V^*$ and $\mathcal{B} : L^2(0,T;V) \to L^2(0,T;V^*)$ and we consider the Cauchy problem

$$(\mathcal{A}u)' + (\mathcal{B}u) \ni f, \quad \mathcal{A}u(0) \ni v^0 \tag{1.1}$$

where the prime stands for the derivative with respect to time and $f \in L^2(0,T;V^*)$, $v^0 \in V^*$ are given data. We shall investigate the above problem under the assumption that both $\mathcal{A}$ and $\mathcal{B}$ are linearly bounded. Moreover, $\mathcal{A}$ is asked to be a compact subdifferential from V to V^*. As regards the latter assumption, let us stress that, referring to possible applications to PDE problems, we are requiring that the order of the operator $\mathcal{A}$ is strictly lower than that of $\mathcal{B}$. Moreover, some coerciveness has to

*This work has been partially supported by the Istituto di Analisi Numerica del CNR, Pavia, Italy

be assumed at least on one of the two operators. Indeed, as far as we are interested in possibly degenerate operators $\mathcal{A}$ and $\mathcal{B}$, we shall consider the (non interesting) situation in which one of them is identically zero. Thus, (1.1) would reduce to a stationary problem of the form $\mathcal{C}u \ni g$ with $\mathcal{C} = \mathcal{A}$ $(\mathcal{B})$ a.e. in $(0,T)$ (in $L^2(0,T;V)$, respectively). Note that, if $\mathcal{C}$ is a subdifferential on a finite dimensional space, then it is surjective if and only if it is coercive. In this concern, some coerciveness assumption on the operators seems mandatory.

The problem (1.1) has a relevant interest within applications to some classes of nonlinear integrodifferential equations and systems. Namely, by reducing ourselves to the local in time case (i.e. letting $\mathcal{B}$ be the realization in $L^2(0,T;V)$ of a suitable maximal monotone operator $B : V \to V^*$), problem (1.1) occurs in a large number of physical models including elastic viscoplasticity, thermodynamics, and dissipation phenomena [5, 10, 21, 23, 33]. Indeed, in our approach, $\mathcal{B}$ may be nonlocal in time and this feature ensures the possibility of taking into account memory effects in the above referred models [22, 25, 28, 29].

We mention some related work on (1.1). First of all, let us remark that the literature on the subject of local in time doubly nonlinear evolution equations is rather wide. Indeed, the local in time version of (1.1) has been studied under a large number of different assumptions (see, for instance, [5, 8, 9, 19, 32, 33]). Among these contributions, we refer the reader to the fundamental paper [21] since the present analysis moves exactly in that framework extending those results to the nonlocal in time case.

As regards nonlocal in time equations, we remark that a large amount of work has been done in the case when $\mathcal{A}$ or $\mathcal{B}$ is linear. A comprehensive list of existence result may be found in the papers [5, 12, 20] and [1, 7, 16, 17, 18, 14, 15, 36], respectively.

Concerning the study of (1.1) when both $\mathcal{A}$ and $\mathcal{B}$ are nonlinear, a first result has been obtained in [6], where, nevertheless, some assumptions on the structure of $\mathcal{B}$ and on the regularity of f are needed. More recently, we aim to quote the paper [26], which deals with some special classes of doubly nonlinear equations involving nonlocal in time terms. Finally, in [2] the authors achieve an existence result for some doubly nonlinear nonlocal equation by assuming that $\mathcal{A}$ is nondegenerate on some intermediate space and $\mathcal{B}$ is a subdifferential.

The main novelty of this paper is that of dealing with a completely arbitrary, nonlocal, linearly bounded, maximal monotone, causal operator $\mathcal{B}$. By not requiring the latter to be a subdifferential, we obtain a significantly larger class of applications, especially to integrodifferential equations and systems.

In order to prove our existence result we exploit an approximation procedure of independent interest. First of all, we address the regularized problem (see also [6, 8, 21])

$$\big((\varepsilon\mathcal{R} + \mathcal{A})(u)\big)' + (\mathcal{B}u) \ni f, \quad (\varepsilon\mathcal{R} + \mathcal{A})u(0) \ni \varepsilon\mathcal{R}u^0 + v^0 \tag{1.2}$$

where $\mathcal{R} : V \to V^*$ stands for the Riesz isomorphism and $u^0 \in V$. A variable time step discretization of the above problem in the case of a Lipschitz continuous operator $\mathcal{B}$ is carefully detailed. Then, existence of a solution to our continuous problem (1.1) is achieved by means of an abstract approximation approach.

This is the plan of the paper. Section 2 contains the statement of our main results and a preliminary existence proof for (1.2) is obtained in Section 3. Then, our main approximation argument is developed in Section 4, while Section 5 brings to some application to nonlinear integrodifferential equations.

2 Main results

Before stating our main results, let us introduce some notation and recall some basic facts about maximal monotone operators which will be intensively used throughout the paper. The reader is referred to [5, 11, 12] for details and proofs.

Let V be an Hilbert space endowed with the scalar product $(\cdot,\cdot)$ and let A be a subset of the product space $V \times V$. Indeed, A may be regarded as a multivalued mapping $A : V \to 2^V$ and the expression $v \in Au$ means that $[u, v] \in A$. We say that A is *monotone* if, for every $[u_1, v_1], [u_2, v_2] \in A$, one has that $(u_1 - u_2, v_1 - v_2) \geq 0$. Moreover, we call A *maximal monotone* if it is maximal in the sense of inclusion of graphs within the class of monotone operators. Maximal monotone operators may be characterized by the existence of some (hence, all) $\lambda > 0$ such that $R(I + \lambda A) = V$ where R indicates the range of the operator and I is the identity in V. For any maximal monotone operator A and any $\lambda > 0$ we are in the position of defining the *resolvent* $J_\lambda := (I + \lambda A)^{-1}$ which turns out to be a one-to-one contraction mapping on V. Moreover, we define the *Yosida approximation* A_λ of A by letting $A_\lambda := \lambda^{-1}(I - J_\lambda)$. The latter approximation is an everywhere defined Lipschitz continuous mapping with Lipschitz constant equal to λ^{-1}. We shall also use some basic arguments for convergence of functions and operators. Referring to [4] for full discussion, we just define the two convergence notions that will be used in the sequel. Namely, let A, A_n be maximal monotone operators and let φ, $\varphi_n : V \to (-\infty, +\infty]$ be proper, convex, and lower semicontinuous functions. We say that A_n *converges to* A *in the sense of G-convergence in* V if for all $[u, v] \in A$ there exists a sequence $[u_n, v_n] \in A_n$ such that $[u_n, v_n]$ converges to $[u, v]$ strongly in $V \times V$. On the other hand, we say that φ_n *converges to* φ *in the sense of Mosco on* V if

$$\forall u_n \to u \ \text{ weakly in } V, \ \ \varphi(u) \leq \liminf_{n\to+\infty} \varphi_n(u_n) \ \text{ and}$$
$$\forall u \in V \ \exists \{u_n\} \ \text{ such that } \ u_n \to u \ \text{ strongly in } V \text{ and } \ \varphi(u) = \lim_{n\to+\infty} \varphi_n(u_n).$$

In the forthcoming analysis will be extremely relevant the notion of *subdifferential*. Let $\varphi : V \to (-\infty, +\infty]$ be a proper, convex, and lower semicontinuous function, then we term subdifferential of φ the set

$$\partial\varphi := \big\{[u, v] \in V \times V : (v, w - u) \leq \varphi(w) - \varphi(u), \ \ \forall w \in D(\varphi)\big\},$$

where $D(\varphi)$ stands for the effective domain of φ. It is a standard matter to prove that $\partial\varphi$ turns out to be a maximal monotone operator in the sense above (see, e.g., [12, Ex.

2.3.4., p. 25]). For the sake of clarity, we introduce here the notion of *conjugate* function of φ by prescribing $\varphi^* : V \to (-\infty, +\infty]$ as

$$\varphi^*(u) := \sup_{v \in V} \big\{ (u, v) - \varphi(v) \big\}. \tag{2.1}$$

We have that, with the above assumptions on φ, the conjugate φ^* is again a proper, convex, and lower semicontinuous function and that $v \in \partial\varphi(u)$ if and only if $u \in \partial\varphi^*(v)$ (see, e.g., [12]).

All the definitions and results recalled above have a natural extension to the framework of two Hilbert spaces in duality, namely V and V^* (see [5] for this particular setting). Indeed, note that these two settings are equivalent by means of the Riesz isomorphism $\mathcal{R} : V \to V^*$ since $\mathcal{A} : V \to V^*$ is maximal monotone if and only if $\mathcal{R}^{-1} \circ \mathcal{A} : V \to V$ is maximal monotone. The notion of subdifferential may also be given in the framework of two Banach spaces in duality W and W^*, namely $\partial\varphi$ will be defined by

$$\partial\varphi := \big\{ [u, v] \in W \times W^* : \langle v, w - u \rangle \le \varphi(w) - \varphi(u), \quad \forall w \in D(\varphi) \big\}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between W^* and W and φ is now defined on $D(\varphi) \subset W$.

Before stating our results and owing to the above definition we stress that, letting V and W be locally convex spaces, $\varphi : W \to (-\infty, +\infty]$ be convex, $L : V \longrightarrow W$ be linear and continuous and φ be continuous at some point of $R(L)$ we have that the following *chain rule* holds (see, e.g., [33, Prop. 7.8, p. 82])

$$\partial\big(\varphi \circ L\big) = L^* \circ \partial\varphi \circ L, \tag{2.2}$$

where $L^* : W^* \longrightarrow V^*$ denotes the dual operator of L.

We are now able to state the main results of our paper.

Theorem 2.1. *Let W be a reflexive Banach space and V be an Hilbert space densely and compactly embedded in W through the injection $i : V \to W$. Moreover, let the following hold*

(H1) $\varphi : W \to (-\infty, +\infty]$ is a proper, convex, and lower semicontinuous function on W, continuous at some point of V, $\partial\varphi \circ i : V \to W^*$ is linearly bounded, and $\mathcal{A} := i^* \circ \partial\varphi \circ i$.

(H2) $\mathcal{B} : L^2(0, T; V) \to L^2(0, T; V^*)$ is maximal monotone, causal, and linearly bounded.

Then, for every $f \in L^2(0, T; V^*)$ and $[u^0, v^0] \in \mathcal{A}$, there exist a triplet $u \in H^1(0, T; V)$, $v \in H^1(0, T; V^*)$, and $w \in L^2(0, T; V^*)$ such that

$$\big(\mathcal{R}u(t) + v(t)\big)' + w(t) = f(t) \qquad \textit{for a.e. } t \in (0, T), \tag{2.3}$$

$$[u(t), v(t)] \in \mathcal{A} \qquad \textit{for a.e. } \ t \in (0, T), \tag{2.4}$$

$$[u, w] \in \mathcal{B}, \tag{2.5}$$

$$\mathcal{R}u(0) + v(0) = \mathcal{R}u^0 + v^0. \tag{2.6}$$

Theorem 2.2. *Let the assumption (H1) of Theorem 2.1 and the following hold*

(H3) $\mathcal{B} : L^2(0,T;V) \to L^2(0,T;V^*)$ *is maximal monotone, causal, linearly bounded, and coercive, i.e.*

$$\lim_{\substack{\|u\|_{L^2(0,T;V)}\to+\infty \\ [u,w]\in\mathcal{B}}} \frac{\int_0^T \big(w(t),u(t)\big)\,dt}{\|u\|_{L^2(0,T;V)}} = +\infty. \tag{2.7}$$

Then, for every $f \in L^2(0,T;V^*)$ *and* $v^0 \in R(\mathcal{A})$ *there exists a triplet* $u \in L^2(0,T;V)$, $v \in H^1(0,T;V^*)$, *and* $w \in L^2(0,T;V^*)$ *such that*

$$v'(t) + w(t) = f(t) \qquad \text{for a.e. } t \in (0,T), \tag{2.8}$$

$$[u(t),v(t)] \in \mathcal{A} \qquad \text{for a.e. } t \in (0,T), \tag{2.9}$$

$$[u,w] \in \mathcal{B}, \tag{2.10}$$

$$v(0) = v^0. \tag{2.11}$$

Let us stress that the assumption (H1) and formula (2.2) imply that $\mathcal{A} = \partial(\varphi \circ i)$ and $D(\mathcal{A}) = V$. Since we have that $\varphi(0) < +\infty$ we may assume with no loss of generality that $\varphi(0) \le 0$ so that φ^* is everywhere nonnegative (cf. (2.1)). Moreover, $\mathcal{A}$ turns out to be compact from V to V^* and the boundedness assumption on $\mathcal{B}$ yields $D(\mathcal{B}) = L^2(0,T;V)$. For the sake of completeness we recall that $\mathcal{B}$ is *causal* if, for any $[u_i,w_i] \in \mathcal{B}$, $i = 1,2$, and any $t \in (0,T)$,

$$u_1 = u_2 \ \text{ on } \ (0,t) \quad \Longrightarrow \quad w_1 = w_2 \ \text{ on } \ (0,t). \tag{2.12}$$

Moreover, we stress that $\mathcal{B}$ is linearly bounded whenever there exists a positive constant $C_{\mathcal{B}}$ such that

$$\|w\|_{L^2(0,T;V)} \le C_{\mathcal{B}}\big(1 + \|u\|_{L^2(0,T;V)}\big) \quad \forall [u,w] \in \mathcal{B}.$$

Furthermore, we remark that the choice of the exponent 2 in the above spaces is motivated by the sake of simplicity. Indeed, the same result holds in a pair of spaces in duality $L^p(0,T;V) - L^{p'}(0,T,V^*)$ where $p,p' \in (1,\infty)$, $1/p + 1/p' = 1$.

It is well known that strong nonuniqueness may occur for both problems (2.3)-(2.6) and (2.8)-(2.11). Indeed, we are aware of counterexamples showing nonuniqueness not only for v and w, which are actually selected, but also for u. In this concern, we refer the reader to [21, Sec. 5] where nonuniqueness for u have been proved even for $V = \mathbb{R}$ and $\mathcal{A},\mathcal{B}$ monotone subdifferentials. Of course, whenever $\mathcal{A}$ is linear continuous and symmetric and the sum $\mathcal{A}+\mathcal{B}$ to be strictly monotone, we easily extend to our nonlocal in time case the former uniqueness result [21, Thm. 4].

3 A Preliminary Existence Result

This section is devoted to the proof of an existence result for the system (2.3)-(2.6) under further assumptions on the operator $\mathcal{B}$. In particular, the following lemma will be the starting point of an approximation procedure leading to the proof of both Theorems 2.1 and 2.2.

From now on we leave the $V - V^*$ setting of problems and reduce ourselves to completely equivalent problems in the space V by applying to all operators and data in (2.3)-(2.6) (and (2.8)-(2.11)) the inverse of the Riesz map $\mathcal{R}^{-1}$. In this respect, we replace $A := \mathcal{R}^{-1} \circ \mathcal{A}$, $B := \mathcal{R}^{-1} \circ \mathcal{B}$, etc. to the corresponding terms and denote by the symbol $\|\cdot\|$ the norm in V.

Namely, fixed $[u^0, v^0] \in A$ and $f \in L^2(0,T;V)$, relations (2.3)-(2.6) may be rewritten as follows

$$u'(t) + v'(t) + w(t) = f(t) \quad \text{for a.e. } t \in (0,T), \tag{3.1}$$

$$[u(t), v(t)] \in A \quad \text{for a.e. } t \in (0,T), \tag{3.2}$$

$$[u, w] \in B, \tag{3.3}$$

$$u(0) + v(0) = u^0 + v^0, \tag{3.4}$$

where u', v', and w are now functions of $L^2(0,T;V)$ and (3.1)-(3.2), (3.4) are considered in V. In this framework (see (H1)), A turns out to be linearly bounded into $\mathcal{R}^{-1}(W^*)$, which is compactly embedded into V. The remainder of this section is devoted to the proof of the following result

Lemma 3.1. *Under the assumptions of Theorem 2.1, let* $A := \mathcal{R}^{-1} \circ \mathcal{A}$, $B := \mathcal{R}^{-1} \circ \mathcal{B}$, *etc. Moreover, let* B *be Lipschitz continuous. Then, there exist* $u \in H^1(0,T;V)$ *and* $v \in H^1(0,T;V)$ *fulfilling (3.1)-(3.4).*

Note that in the above statement w doesn't come into play. Indeed, B is *single-valued* and w is simply determined by (3.3).

As regards the proof of Lemma 3.1, we aim to investigate a variable time step discretization of the problem (3.1)-(3.4). To this end, let $\mathcal{P}$ be a partition of the time interval $[0,T]$, namely

$$\mathcal{P} := \{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\},$$

with variable step $\tau_i := t_i - t_{i-1}$. No constraints are imposed on the time steps and $\tau := \max_{1 \le i \le N} \tau_i$ denotes the diameter of the partition. Let us just remark that a constant time step partition would perfectly serve for the sake of proving the forthcoming existence result. Nevertheless, our approximation is naturally given in the framework of variable time step partitions without any particular intricacy. Moreover, this possibility presents a relevant numerical interest since the time steps could be adaptively tailored with respect to other numerical considerations such as error estimators.

In the forthcoming analysis we will extensively use the following notation: let $\{u_i\}$, $i = 0, 1, \ldots, N$ be a vector, we denote by u_τ and $\overline{u}_\tau$ two functions of the time interval $[0,T]$ which interpolate the values of the vector piecewise linearly and backward constantly on the partition $\mathcal{P}$, respectively. That is,

$$u_\tau(0) := u_0, \quad u_\tau(t) := \alpha_i(t)u_i + \big(1-\alpha_i(t)\big)u_{i-1},$$
$$\overline{u}_\tau(0) := u_0, \quad \overline{u}_\tau(t) := u_i, \qquad \text{for } \ t \in (t_{i-1}, t_i], \quad i = 1, \dots, N$$

where $\alpha_i(t) := (t - t_{i-1})/\tau_i$, for $t \in (t_{i-1}, t_i]$, $i = 1, \dots, N$. The notation above refers to τ and not to $\mathcal{P}$ as one would expect. Let us note that this apparently misleading choice causes no confusion and is motivated by the sake of clarity. Indeed, we will later pass to the limit as the diameter τ of the partition goes to 0.

Let us fix a suitable approximation $\{f_i\}_{i=1}^N \in V^N$ of the function f such that the strong convergence $\overline{f}_\tau \longrightarrow f$ in $L^2(0,T;V)$ holds as τ goes to 0. Indeed, a possible choice is

$$f_i := \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} f(s)\,ds \in V, \quad \text{for } \ i = 1, \dots, N.$$

Then, we look for two vectors $\{u_i\}$, $\{v_i\} \in V$, $i = 0, 1, \dots, N$ fulfilling the following conditions

$$\frac{u_i - u_{i-1}}{\tau_i} + \frac{v_i - v_{i-1}}{\tau_i} + \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} (B\overline{u}_\tau)(s)\,ds = f_i, \qquad \text{for } \ i = 1, \dots, N, \tag{3.5}$$

$$[u_i, v_i] \in A, \qquad \text{for } \ i = 1, \dots, N, \tag{3.6}$$

$$u_0 + v_0 = u^0 + v^0. \tag{3.7}$$

This scheme is fully implicit and involves the function $(B\overline{u}_\tau) \in L^2(0,T;V)$ which is not a priori constant on the partition.

We shall prove that, whenever the diameter τ of the partition $\mathcal{P}$ is small enough, the system above admits a unique solution. Let us begin by fixing a tentative solution $\widetilde{u} \in L^2(0,T;V)$ (possibly not constant on the partition $\mathcal{P}$) and solving the equation

$$\frac{u_i - u_{i-1}}{\tau_i} + \frac{v_1 - v_{i-1}}{\tau_i} + \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} (B\widetilde{u})(s)\,ds = f_i, \qquad \text{for } \ i = 1, \dots, N, \tag{3.8}$$

together with (3.6)-(3.7). By letting

$$\widetilde{f}_i := \tau_i f_i + u_{i-1} + v_{i-1}$$

the system above reads

$$\big(I + A\big)(u_i) \ni \widetilde{f}_i - \int_{t_{i-1}}^{t_i} (B\widetilde{u})(s)\,ds, \qquad \text{for } \ i = 1, \dots, N,$$

where we recall that I stands for the identity in V. Indeed, the right hand side above is known at level i and the operator $I + A : V \to V$ is maximal monotone and coercive, hence surjective (see, e.g., [12, Cor. 2.4, p. 31]). This consideration ensures that, for every given $\widetilde{u} \in L^2(0,T;V)$, we find a function $S(\widetilde{u}) \in L^2(0,T;V)$ which is constant on the partition and solves (3.8) in the sense that one has

$$S(\widetilde{u})_i = (I+A)^{-1}\left(\widetilde{f}_i - \int_{t_{i-1}}^{t_i} (B\widetilde{u})(s)\,ds\right) \qquad \text{for } i = 1,\ldots,N, \tag{3.9}$$

with the obvious notation $S(\widetilde{u})_i := S(\widetilde{u})(t)$ for $t \in (t_{i-1}, t_i]$, $i = 1,\ldots,N$.

Our next aim is to show that S has a fixed point in $L^2(0,T;V)$ by proceeding by induction. In particular, we consider to know $\overline{u}_\tau$ up to time t_{i-1} for some $i = 1,\ldots,N$ and prove the existence and uniqueness of a solution $\overline{u}_\tau$ of problem (3.5)-(3.7) on $[0,t_i]$. Let us set the space

$$\mathcal{S}_i := \big\{u \in L^2(0,t_i;V) \ \text{ such that } \ u = \overline{u}_\tau \ \text{ on } \ [0,t_{i-1}]\big\}.$$

We obviously have that $\mathcal{S}_i$ is closed in $L^2(0,t_i;V)$. Moreover, owing to the causality of B and thus of S, one readily has that

$$S(\mathcal{S}_i) := \big(R_i \circ S \circ E_i\big)(\mathcal{S}_i) \subseteq \mathcal{S}_i,$$

where $R_i : L^2(0,T;V) \longrightarrow L^2(0,t_i;V)$ and $E_i : L^2(0,t_i;V) \longrightarrow L^2(0,T;V)$ stand for the usual *restriction* and *trivial extension* operators, respectively. We aim to prove that, upon choosing the diameter τ of partition $\mathcal{P}$ small enough, it turns out that $S_i := R_i \circ S \circ E_i$ is a strict contraction on $\mathcal{S}_i \subseteq L^2(0,t_i;V)$. To this end we choose $\widetilde{u}_1, \widetilde{u}_2 \in \mathcal{S}_i$ and take the difference between (3.9) written for $\widetilde{u}_1$ and the same equation written for $\widetilde{u}_2$. Since $(I+A)^{-1}$ is a contraction on V, we easily check that

$$\|S_i(\widetilde{u}_1)_i - S_i(\widetilde{u}_2)_i\|^2 \leq \tau_i \int_{t_{i-1}}^{t_i} \big\|\big((B\widetilde{u}_1) - (B\widetilde{u}_2)\big)(s)\big\|^2\,ds,$$

so that one in particular obtains

$$\|S_i(\widetilde{u}_1) - S_i(\widetilde{u}_2)\|^2_{L^2(0,t_i;V)} \leq \tau^2 \|(B\widetilde{u}_1) - (B\widetilde{u}_2)\|^2_{L^2(0,t_i;V)}. \tag{3.10}$$

Now we exploit the causality of B. In fact, we have that

$$R_i \circ B = R_i \circ B \circ E_i \circ R_i =: B_i.$$

Thus, it suffices to go back to (3.10) and recall the Lipschitz continuity of B in order to obtain that

$$\begin{aligned}
\|S_i(\widetilde{u}_1) - S_i(\widetilde{u}_2)\|^2_{L^2(0,t_i;V)} &\leq \tau^2 \|(B_i\widetilde{u}_1) - (B_i\widetilde{u}_2)\|^2_{L^2(0,t_i;V)} \\
&\leq \tau^2 \|B\big((E_i \circ R_i)(\tilde{u}_1)\big) - B\big((E_i \circ R_i)(\tilde{u}_2)\big)\|^2_{L^2(0,T;V)} \\
\leq (\tau\lambda_B)^2 \|(E_i \circ R_i)(\tilde{u}_1 - \tilde{u}_2)\|^2_{L^2(0,T;V)} &= (\tau\lambda_B)^2 \|\tilde{u}_1 - \tilde{u}_2\|^2_{L^2(0,t_i;V)},
\end{aligned}$$

where λ_B stands for the Lipschitz constant of B. Finally, choosing $\tau < (\lambda_B)^{-1}$, the contraction character of S_i is established and a unique solution $\overline{u}_\tau \in \mathcal{S}_i$ is achieved.

By making use of the previous introduced notation we may rewrite the scheme (3.5)-(3.7) as

$$u'_\tau + v'_\tau + \overline{w}_\tau = \overline{f}_\tau \qquad \text{a.e. in } (0,T), \tag{3.11}$$

$$[\overline{u}_\tau, \overline{v}_\tau] \in A \qquad \text{a.e. in } (0,T), \tag{3.12}$$

$$\overline{w}_\tau(t) = \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} (B\overline{u}_\tau)(s)\,ds \qquad \text{for } t \in (t_{i-1}, t_i], \quad i = 1, \dots, N, \tag{3.13}$$

$$u_\tau(0) + v_\tau(0) = u^0 + v^0. \tag{3.14}$$

In order to prove the existence result of Lemma 3.1, we establish some *a priori* estimates on the approximating solution introduced above. We test (3.11) by u'_τ and integrate on $(0, t_i)$ for $t_i \in \mathcal{P}$. Due to the monotonicity of A one infers

$$\begin{aligned}\|u'_\tau\|^2_{L^2(0,t_i;V)} &\le \int_0^{t_i} \left(\|\overline{f}_\tau(s)\| + \|\overline{w}_\tau(s)\|\right)\|u'_\tau(s)\|\,ds \\ &\le \frac{1}{2}\|u'_\tau\|^2_{L^2(0,t_i;V)} + \int_0^{t_i} \left(\|\overline{f}_\tau(s)\|^2 + \|\overline{w}_\tau(s)\|^2\right)ds.\end{aligned}$$

Taking into account assumption (H2) and relation (3.13), we easily exploit the causality of $\mathcal{B}$ and deduce that

$$\begin{aligned}\|\overline{w}_\tau\|^2_{L^2(0,t_i;V)} \le \|B\overline{u}_\tau\|^2_{L^2(0,t_i;V)} &\le 2C_B^2\left(1 + \|\overline{u}_\tau\|^2_{L^2(0,t_i;V)}\right) \\ 2C_B^2&\left(1 + 2T\|u^0\|^2 + 2T\sum_{j=1}^{i}\tau_j\|u'_\tau\|^2_{L^2(0,t_j;V)}\right)\end{aligned}$$

where C_B is the linear boundedness constant of B. Upon choosing τ small enough (that is $\tau < (8TC_B^2)^{-1}$), we are in the position of applying the discrete Gronwall lemma (see, e.g., the version reported in [27, Prop. 2.2.1, p. 52]) and get

$$\|u'_\tau\|_{L^2(0,T;V)}, \le C,$$

for a proper C depending on data but independent of τ.

Once we have established the bound above we easily deduce from assumptions (H1), (H2), and a comparison in (3.11) that the following estimate holds as well

$$\begin{aligned}&\|u_\tau\|_{H^1(0,T;V)} + \|\overline{u}_\tau\|_{L^\infty(0,T;V)} + \|v_\tau\|_{H^1(0,T;V)} \\ &+\|\mathcal{R}\overline{v}_\tau\|_{L^\infty(0,T;W^*)} + \|\overline{w}_\tau\|_{L^2(0,T;V)} \le C,\end{aligned} \tag{3.15}$$

for a proper constant C depending on data but independent of τ.

Before passing to the limit, let us claim that

$$\|v_\tau - \overline{v}_\tau\|_{L^\infty(0,T;V)} \leq \sqrt{\tau}\|v'_\tau\|_{L^2(0,T;V)}, \tag{3.16}$$

as a direct check provides.

Taking into account the estimate (3.15), we may find three functions u, v, and w such that, possibly passing to a subsequence (not relabeled),

$$u_\tau \longrightarrow u \quad \text{weakly in } H^1(0,T;V), \tag{3.17}$$
$$\overline{u}_\tau \longrightarrow u \quad \text{weakly star in } L^\infty(0,T;V), \tag{3.18}$$
$$v_\tau \longrightarrow v \quad \text{weakly in } H^1(0,T;V), \tag{3.19}$$
$$\mathcal{R}\overline{v}_\tau \longrightarrow \mathcal{R}v \quad \text{weakly star in } L^\infty(0,T;W^*), \tag{3.20}$$
$$\overline{w}_\tau \longrightarrow w \quad \text{weakly in } L^2(0,T;V), \tag{3.21}$$

whenever τ goes to zero. Due to the generalized Ascoli theorem [35, Cor. 4], the two convergences (3.19) and (3.20) entail the further convergence

$$v_\tau \longrightarrow v \qquad \text{strongly in } C^0([0,T];V). \tag{3.22}$$

Referring to the estimate (3.16), it is now a standard matter to deduce that

$$\overline{v}_\tau \longrightarrow v \qquad \text{strongly in } L^\infty(0,T;V), \tag{3.23}$$

so that the latter together with (3.18) yields

$$\int_0^T \big(\overline{u}_\tau(t), \overline{v}_\tau(t)\big)\,dt \longrightarrow \int_0^T \big(u(t), v(t)\big)\,dt, \tag{3.24}$$

and the inclusion (3.2) follows by standard arguments on maximal monotone operators (see the forthcoming Lemma 3.2).

We now need to discuss briefly a technical argument concerning convergence of operators. Indeed, let us denote by $\Lambda_\tau : L^2(0,T;V) \longrightarrow L^2(0,T;V)$ the *mean operator* defined by

$$(\Lambda_\tau u)(t) := \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} u(s)\,ds$$
$$\text{for } \ t \in (t_{i-1}, t_i], \quad i = 1, \ldots, N, \quad \forall u \in L^2(0,T;V).$$

We stress that Λ_τ is a linear contraction on $L^2(0,T;V)$. Moreover, we set

$$K_\tau := \big\{u : [0,T] \longrightarrow V \ \text{ such that } u \text{ is piecewise constant on } \mathcal{P}\big\},$$

and consider the monotone operators $\Lambda_\tau \circ B : K_\tau \longrightarrow L^2(0,T;V)$. Letting C be a maximal monotone operator with domain $D(C)$, we denote by $ext(C)$ any maximal extension within the class of monotone operators on the domain $D(C)$. Of course, if $[u,v] \in C$, we have that $[u,v] \in ext(C)$ as well. Now, recalling the notion of G-convergence for maximal monotone operators, we easily prove that

$$ext(\Lambda_\tau \circ B) \longrightarrow B \text{ in the sense of G-convergence in } L^2(0,T;V).$$

Indeed, for any $[u,w] \in B$ (that is $u \in L^2(0,T;V)$, $w = (Bu)$), we may define $u^\tau := \Lambda_\tau u$, $w^\tau := (\Lambda_\tau \circ B)(u^\tau)$. Thus, we obviously have that $[u^\tau, w^\tau] \in ext(\Lambda_\tau \circ B)$ and $u^\tau \to u$ strongly in $L^2(0,T;V)$ as $\tau \to 0$. As regards w, one has

$$\begin{aligned} \|w - w^\tau\|_{L^2(0,T;V)} &\leq \|w - \Lambda_\tau w\|_{L^2(0,T;V)} + \|\Lambda_\tau(Bu - (B \circ \Lambda_\tau)u)\|_{L^2(0,T;V)} \\ &\leq \|w - \Lambda_\tau w\|_{L^2(0,T;V)} + \lambda_B \|u - \Lambda_\tau u\|_{L^2(0,T;V)} \end{aligned}$$

and the above right hand side vanishes as $\tau \longrightarrow 0$. Let us stress that the above G-convergence still holds when the operator B is just continuous from $L^2(0,T;V)$ to $L^2(0,T;V^*)$ as may be easily checked.

We will use the latter proved convergence in order to pass to the limit in the term $\overline{w}_\tau$. To this aim, we recall here a result which turns out to be crucial in the sequel

Lemma 3.2. *Let H be an Hilbert space, $C, C_n : H \longrightarrow H$ be maximal monotone operators. Moreover, let $x, y \in H$, $[x_n, y_n] \in C_n$, and the following hold*

$$\begin{aligned} &x_n \longrightarrow x, \quad y_n \longrightarrow y \quad \textit{weakly in } H, \\ &C_n \longrightarrow C \quad \textit{in the sense of G-convergence in } H, \\ &\liminf_{n \to +\infty} (x_n, y_n)_H \leq (x,y)_H. \end{aligned}$$

Then, $[x,y] \in C$ and $(x_n, y_n)_H \longrightarrow (x,y)_H$.

The proof of the previous result is omitted and is essentially to be found in [4, Prop. 3.59, p. 361].

Testing now (3.11) by $\overline{u}_\tau$ and integrating on $(0,T)$ we easily achieve

$$\begin{aligned} &\int_0^T \big(\overline{w}_\tau(s), \overline{u}_\tau(s)\big)\, ds \leq -\frac{1}{2}\|\overline{u}_\tau(T)\|^2 + \frac{1}{2}\|u^0\|^2 \\ &-\varphi^*\big(\overline{v}_\tau(T)\big) + \varphi^*\big(v^0\big) + \int_0^T \big(\overline{f}_\tau(s), \overline{u}_\tau(s)\big)\, ds. \end{aligned}$$

Hence, taking the $\limsup$ as τ goes to 0 in both sides of the above relation, owing to (3.17), (3.22), the lower semicontinuity of φ^*, and the convergence of $\overline{f}_\tau$ to f, we get that

$$\limsup_{\tau \to 0} \int_0^T \big(\overline{w}_\tau(s), \overline{u}_\tau(s)\big)\, ds \leq \int_0^T \big(w(s), u(s)\big)\, ds \tag{3.25}$$

Finally, also (3.3) is achieved according to Lemma 3.2 and the proof of Lemma 3.1 is complete.

4 Approximation

In this section we aim to present an abstract approximation procedure which entails, in particular, the proof of both Theorems 2.1 and 2.2. Indeed, the forthcoming approximation result essentially extends to the nonlocal in time case the former [3, Thm. 3.2].

Suppose we are given A, A_n $(n \in \mathbb{N})$ maximal monotone operators on V and B, B_n maximal monotone operators on $L^2(0,T;V)$. Moreover, let A, A_n be subdifferentials. Namely, let φ, $\varphi_n : V \to (-\infty, +\infty]$ be proper, convex, and lower semicontinuous functions and consider $A = \partial\varphi$, $A_n = \partial\varphi_n$. Finally, we introduce the data f, $f_n \in L^2(0,T;V)$ and $v^0 \in R(A)$, $v_n^0 \in R(A_n)$.

We are interested in the approximation of a solution

$$[u, v, w] \in L^2(0,T;V) \times H^1(0,T;V) \times L^2(0,T;V),$$

of the following Problem (P)

$$v'(t) + w(t) = f(t) \quad \text{for a.e. } t \in (0,T), \tag{4.1}$$

$$[u(t), v(t)] \in A \quad \text{for a.e. } t \in (0,T), \quad [u, w] \in B, \tag{4.2}$$

$$v(0) = v^0, \tag{4.3}$$

by means of solutions

$$[u_n, v_n, w_n] \in L^2(0,T;V) \times H^1(0,T;V) \times L^2(0,T;V),$$

of the Problem (P_n)

$$v_n'(t) + w_n(t) = f_n(t) \quad \text{for a.e. } t \in (0,T), \tag{4.4}$$

$$[u_n(t), v_n(t)] \in A_n \quad \text{for a.e. } t \in (0,T), \quad [u_n, w_n] \in B_n, \tag{4.5}$$

$$v_n(0) = v_n^0, \tag{4.6}$$

under proper convergence hypotheses for operators, data, and solutions. Namely, we shall prove the following theorem

Theorem 4.1. *Referring to the above notation, we assume that*

$$\varphi_n \longrightarrow \varphi \quad \textit{in the sense of Mosco in } V, \tag{4.7}$$

$$B_n \longrightarrow B \quad \textit{in the sense of G-convergence in } L^2(0,T;V), \tag{4.8}$$

$$f_n \longrightarrow f \quad \textit{strongly in } L^2(0,T;V). \tag{4.9}$$

Moreover, assume that the sequence (u_n, v_n, w_n) *of solutions to Problem* (P_n) *fulfills*

$$u_n \longrightarrow u \quad \textit{weakly in } L^2(0,T;V), \tag{4.10}$$

$$v_n \longrightarrow v \quad \textit{weakly in } H^1(0,T;V) \textit{ and strongly in } L^2(0,T;V), \tag{4.11}$$

$$w_n \longrightarrow w \quad \textit{weakly in } L^2(0,T;V), \tag{4.12}$$

$$\limsup_{n \to +\infty} \int_\Omega \varphi_n^*\big(v_n(0)\big) \leq \int_\Omega \varphi^*\big(v(0)\big), \tag{4.13}$$

for a suitable triplet of functions $[u, v, w]$. *Then* $[u, v, w]$ *solves Problem* (P).

Remark 4.2. Let us stress that condition (4.13) is fulfilled in a quite large number of interesting situations. Indeed, one can consider the case when φ_n^* is a pointwise increasing approximation of φ^* (such as the Yosida approximation) and $v_n^0 = v^0$. On the other hand, we may easily prove that, if $v_n^0 \to v^0$ strongly in V and the set $\{\partial\varphi_n^*(v_n^0); \partial\varphi_n^*(v^0)\}$ is bounded in V^*, one has that (4.13) is satisfied. In particular, the supremum is indeed a limit and the equality holds.

Proof of Theorem 4.1. It suffices to prove the inclusions (4.2). As regards the first relation, we may use [4, Thm. 3.66, p. 373] to obtain that

$$A_n \longrightarrow A \quad \text{in the sense of G-convergence in } V.$$

Thus, the convergences (4.10)-(4.11) and an application of Lemma 3.2 (with $H = V$, $C = A$, $C_n = A_n$) ensures that

$$[u(t), v(t)] \in A \quad \text{for a.e. } t \in (0, T).$$

Now, by testing equation (4.4) by u_n and integrating on $(0, T)$ we easily obtain that

$$\int_0^T (w_n, u_n) = -\varphi_n^*(v_n(T)) + \varphi_n^*(v_{0n}) + \int_0^T (f_n, u_n).$$

Our next aim is that of passing to the $\limsup$ as $n \longrightarrow +\infty$ in both sides of the above equality. First of all, let us point out that the convergence (4.7) entails (indeed, is equivalent to) the following [4, Thm 3.18, p. 295]

$$\varphi_n^* \longrightarrow \varphi^* \quad \text{in the sense of Mosco in } V.$$

Hence, relations (4.9), (4.11), and (4.13) ensure that

$$\limsup_{n\to+\infty} \int_0^T (w_n, u_n) \le -\varphi^*(v(T)) + \varphi^*(v_0) + \int_0^T (f, u) = \int_0^T (w, u).$$

Once again, owing to (4.8) and Lemma 3.2 (applied with $H = L^2(0, T; V)$, $C = B$, $C_n = B_n$), one has

$$[u, w] \in B$$

and the proof of Theorem 4.1 is complete. □

Remark 4.3. Let us remark that the assumption of Theorem 4.1 may be properly weakened in various directions. Indeed, we have chosen (4.7)-(4.12) just for the sake of simplicity as they easily fit into the framework of Theorems 2.1 and 2.2.

We are now in the position of proving both Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Replace B by its Yosida approximation at level $1/n$ and recall that B_n converges to B in the sense of G-convergence in $L^2(0,T;V)$ as $n \longrightarrow +\infty$. Moreover, owing to (H2), it is a standard matter to prove that the operator B_n turns out to be monotone, Lipschitz continuous, and causal. Then, such a regularized problem admits at least a solution thanks to Lemma 3.1. Taking into account (H1)-(H2) it is straightforward to reproduce the argument developed in the proof of Lemma 3.1 and establish (an analogous of) the estimate (3.15) independently of n. Thus, taking into account well-known compactness results and the Ascoli theorem (see, e.g., [35, Cor. 4]), we find a triplet $[u, v, w]$ such that, up to a (not relabeled) subsequence, the convergences (4.10)-(4.12) hold (actually much more is true). Finally, the proof of Theorem 2.1 follows from an application of Theorem 4.1. □

Proof of Theorem 2.2. Replace B by B_n as in the proof of Theorem 2.1 and φ by φ_n defined by

$$\varphi_n(u) := \frac{1}{2n}\|u\|^2 + \varphi(u) \qquad \forall\, u \in V.$$

Let us remark that we may easily prove that φ_n converges to φ in the sense of Mosco in V. Then, for any n, the regularized problem is solved due to Lemma 3.1 and we denote by $[u_n, v_n, w_n]$ its solution. Now, test the regularized equation by u_n. Owing to the coercivity of B it is a standard matter to deduce that

$$\|u_n\|_{L^2(0,T;V)} \quad \text{is bounded independently of } n.$$

As a consequence of the latter estimate and the boundedness assumptions on A and B, we easily infer that

$$\|v_n\|_{H^1(0,T;V)}, \quad \|\mathcal{R}v_n\|_{L^2(0,T;W^*)}, \quad \|w_n\|_{L^2(0,T;V)}$$
$$\text{are bounded independently of } n.$$

Thanks to the above stated bounds and owing to well-known compactness results, one finds a triplet $[u, v, w]$ such that, possibly taking subsequences (not relabeled), the convergences (4.10)-(4.12) hold true. Moreover we obviously have that $\varphi_n^*(v^0) \le \varphi^*(v^0)$, hence relation (4.13) is fulfilled. Once again we are in the position of applying Theorem 4.1 and conclude the proof. □

5 Applications

We shall now present some applications of our existence results to classes of nonlinear degenerate evolution problems taking into account nonlocal dynamics. Let us stress that these examples have been chosen just in order to suggest a variety of applications.

Let Ω denote a bounded domain in $\mathbb{R}^n \,(n \ge 1)$ with a sufficiently smooth boundary Γ. We shall let V be any closed subspace of $H^1(\Omega)$ containing $H^1_0(\Omega)$ and denote by

$\gamma : V \to L^2(\Gamma)$ the corresponding trace operator. For the details about this functional setting we refer the reader to [31].

We introduce some construction of operators $\mathcal{A}$ and $\mathcal{B}$ fitting our requirements. Moreover, as boundary evolution might be taken into account in our examples, we indicate a decomposition of these operators according to the analysis of [33, Prop. II.5.3, p. 65]. The latter decomposition is intended to resolve an abstract functional equation in (1.1) as a *formal part*, which in the examples will stand for an integropartial differential equation, and a *boundary part* accounting for the boundary condition in $R(\gamma)$, i.e. the range of γ. Let us stress that the latter conditions are in addition to those imposed directly on the space V.

For the sake of clarity, we prefer to sketch briefly this argument. Let $G = R(\gamma)$ (thus, G turns out to be a subspace of $H^{1/2}(\Gamma)$), let W fulfill the assumption of Theorem 2.1, and denote the set $V_0 := \ker \gamma = H^1_0(\Omega)$. It is a standard matter to prove that the dual operator γ^* defined by $\gamma^*(g) := g \circ \gamma$ for all $g \in G^*$ is an isomorphism between G^* and the *annihilator* $V_0^\perp$ of V_0 into V^*, namely

$$V_0^\perp := \{v \in V^* \text{ such that } v|_{V_0} = 0\}$$

where we use a standard notation for the restriction. Next, assume we are given an operator $\mathcal{D} : V \to V^*$. We will define its *formal part* as the pointwise restriction $\mathcal{D}_0 := \mathcal{D}|_{V_0}$. Then, for any $[u, v] \in \mathcal{D}$, we have that $v - v|_{V_0} \in V_0^\perp$. Hence, there exists a unique $g \in G^*$ such that $v - v|_{V_0} = \gamma^*(g)$. Taking into account the previous considerations, we denote by $\mathcal{D}_\Gamma : V :\to G^*$ the *boundary operator* defined as

$$\mathcal{D}_\Gamma := \{[u, g] \in V \times G^* \text{ such that } \gamma^*(g) = v - v|_{V_0} \text{ for some } v \in \mathcal{D}(u)\}.$$

Finally, we may resolve $\mathcal{D}$ as

$$\mathcal{D} = \mathcal{D}_0 + \gamma^* \circ \mathcal{D}_\Gamma, \tag{5.1}$$

which is an abstract generalization of the well-known *Green formula*. Furthermore, the extension of the above referred argument to the case of nonlocal in time operators is straightforward. Referring to [21, 33] for the details, we just remark that, by assuming some regularity for the ingredients of problem (1.1), it may be proved that the solution to (1.1) solves indeed two problems. In particular, a *formal problem* taking into account the above introduced formal operators is solved in V_0^*, and a *boundary problem* in terms of boundary operators is solved in G^*.

As regards operator $\mathcal{A}$, we will follow the analysis of [21] letting $W = H^s(\Omega)$ with $s \in (1/2, 1)$. It is well known that the injection $i : V \to W$ is compact and dense and that the trace operator $\gamma : W \to L^2(\Gamma)$ is continuous. Assume we are given $\alpha_0, \alpha_\Gamma \subset \mathbb{R} \times \mathbb{R}$ maximal monotone graphs in $\mathbb{R}$ with at most linear growth at infinity, i.e. fulfilling the estimate

$$|\eta| \le C\big(1 + |\xi|\big) \quad \forall [\xi, \eta] \in \alpha_0, \alpha_\Gamma, \tag{5.2}$$

for some positive constant C. Then, let us denote by $\widehat{\alpha}_0$ and $\widehat{\alpha}_\Gamma$ two convex and continuous primitives of α_0 and α_Γ, respectively. We define a functional $\varphi : W \to$

$(-\infty, +\infty]$ as

$$\varphi(u) := \int_\Omega \widehat{\alpha}_0\big(u(x)\big)\,dx + \int_\Gamma \widehat{\alpha}_\Gamma\big(\gamma u(\xi)\big)\,d\xi \quad u \in W.$$

The latter functional turns out to be convex and continuous on W. Moreover, its subdifferential $\partial\varphi$ is bounded into W^* owing to (5.2). We will consider operators $\mathcal{A}$ of the form

$$\mathcal{A} := i^* \circ \partial\varphi \circ i. \tag{5.3}$$

Referring to [21, Example 7.a] for the details, we just recall that the *formal part* and *boundary part* of the above introduced operator are simply given by

$$\mathcal{A}_0(u) = \alpha_0(u) \quad \text{a.e. in } \Omega, \qquad \mathcal{A}_\Gamma(u) = \alpha_\Gamma(\gamma u) \quad \text{a.e. in } \Gamma.$$

As regards the operator $\mathcal{B}$, we shall present two classes of operators fitting in our framework. The first class is obtained as the subdifferential of a convex functional on V and was already considered in [21, Sec. 6]. The second one includes some *Volterra type* integral operators (see, e.g., [24, Chapter 9, p. 225])

Given $u \in L^2(0,T;V)$, we let $\partial_i u := \partial u/\partial x_i \in L^2(\Omega \times (0,T))$ for $i = 1,\dots,n$. Moreover, let $\beta_\Gamma, \beta_0, \beta_1, \dots, \beta_n \subset \mathbb{R} \times \mathbb{R}$ be maximal monotone graphs in $\mathbb{R}$ which are at most linear at infinity in the sense of (5.2). Thus, we denote by $\widehat{\beta}_\Gamma, \widehat{\beta}_0, \widehat{\beta}_1, \dots, \widehat{\beta}_n$ their respective convex and continuous primitives. Let us define the functional $\psi : V \to (-\infty, +\infty]$ as

$$\psi(u) := \sum_{i=1}^n \int_\Omega \widehat{\beta}_i\big(\partial_i u(x)\big)\,dx + \int_\Omega \widehat{\beta}_0\big(u(x)\big)\,dx + \int_\Gamma \widehat{\beta}_\Gamma\big(\gamma u(\xi)\big)\,d\xi \quad \forall u \in V.$$

The functional above turns out to be convex and continuous on V. We may consider a first class of operators $\mathcal{B}$ of the form

$$\mathcal{B}_1 := \partial\psi, \tag{5.4}$$

where, of course, we consider its realization in $L^2(0,T;V)$. Indeed, it may be easily checked that, taking into account the sublinear growth at infinity of the graphs, $\mathcal{B}_1$ is linearly bounded from $L^2(0,T;V)$ to $L^2(0,T;V^*)$. Moreover, it is not difficult to find sufficient conditions on $\widehat{\beta}_\Gamma, \widehat{\beta}_0, \widehat{\beta}_1, \dots, \widehat{\beta}_n$ in order to ensure the coercivity property stated in (2.7). Referring to [21] for the details and applying (2.2), we recall that the above introduced operator $\mathcal{B}_1$ fulfills

$$\mathcal{B}_1(u) = \sum_{i=1}^n \partial_i^* \beta_i(\partial_i u) + \beta_0(u) + \gamma^* \circ \beta_\Gamma(\gamma u) \quad \text{a.e. } \forall u \in V,$$

where ∂_i^* are the dual operators of ∂_i. Thus, whenever some additional regularity is assumed on the ingredients, its *formal part* $\mathcal{B}_{1,0}$ and *boundary part* $\mathcal{B}_{1,\Gamma}$ are given respectively by the equality of sets

$$\mathcal{B}_{1,0}(u) = -\sum_{i=1}^{n} \partial_i\big(\beta_i(\partial_i u)\big) + \beta_0(u),$$

$$\mathcal{B}_{1,\Gamma}(u) = \sum_{i=1}^{n} \beta_i(\partial_i u)\nu_i + \beta_\Gamma(\gamma u),$$

where we have denoted by $(\nu_1,\dots,\nu_n)$ the outward unit normal vector to Γ. In particular, if $[u,v] \in \mathcal{B}_1$ and $v_i \in \beta_i(\partial_i u)$ a.e. for $i = 0,1,\dots,n$, $v_\Gamma \in \beta_\Gamma(\gamma u)$ a.e. are the functions fulfilling

$$\langle v, w\rangle = \sum_{i=0}^{n} \int_\Omega v_i\, \partial_i w \ + \int_\Gamma v_\Gamma\, \gamma w \qquad \forall w \in V,$$

we have that the above equalities of sets are a consequence of the classical Green theorem whenever $\partial_i v \in L^2(\Omega)$ for $i = 0,1,\dots,n$ (see [21] for a full discussion of this topic).

The second class of operators $\mathcal{B}$ shall account for some nonlocal in time dynamics. Namely, given $\lambda \geq 0$ and $k : (0,T) \times (0,T) \longrightarrow \mathbb{R}$ fulfilling suitable integrability assumptions, we aim to consider the following *Volterra operator* $\mathcal{K} : L^2(0,T) \to L^2(0,T)$ defined by

$$(\mathcal{K}u)(t) = \lambda u(t) + \int_0^t k(t,s)u(s)\,ds \quad \forall u \in L^2(0,T0,\ t \in (0,T). \tag{5.5}$$

Now, under proper assumptions on k (see, e.g., [24, Sec. 20.2]) the operator $\mathcal{K}$ fulfills some positivity property. Namely, $\mathcal{K}$ the following inequality might be satisfied

$$\int_0^T (\mathcal{K}u)(s)\, u(s)\,ds \geq \omega\|u\|^2_{L^2(0,T)} \qquad \forall u \in L^2(0,T), \tag{5.6}$$

either for $\omega = 0$ (in this case we say that $\mathcal{K}$ is of *positive type*) of for $\omega > 0$ (*strictly positive type*, respectively). Let us just remark that, in order to obtain an operator $\mathcal{K}$ of strictly positive type it is mandatory for λ in (5.5) to be positive.

Let us introduce now $\mathcal{E} : V \longrightarrow V^*$ as the realization of a linear, continuous, and symmetric elliptic operator and consider two *Volterra operators* $\mathcal{K}_1$, $\mathcal{K}_2$ defined as above. Moreover, we ask $\mathcal{E}$ to be coercive on V. Then, we may set an equivalent inner product on V as $((u,v)) := (\mathcal{E}u, v)$ for all $u, v, \in V$. Thus, $\mathcal{E}$ stands for the Riesz isomorphism between V and V^*. Finally, another possible choice for the operator $\mathcal{B}$ is

$$\mathcal{B}_2 := \mathcal{K}_1 \circ \mathcal{E} + \gamma^* \circ \mathcal{K}_2 \circ \gamma. \tag{5.7}$$

Clearly, the operator $\mathcal{K}_2$ above accounts for some nonlocal in time boundary condition. Indeed, we obviously have that, for any $u \in V$, the formal and boundary part of $\mathcal{B}_2$ read as follows

$$\mathcal{B}_{2,0}(u) = (\mathcal{K}_1 \circ \mathcal{E}|_{V_0})(u), \quad \mathcal{B}_{2,\Gamma}(u) = \mathcal{K}_2(\gamma u).$$

The proof of the following lemma is straightforward (see also [5, Thm. 1.3, p. 40]).

Lemma 5.1. *According to the above notations, the operator $\mathcal{B}_2$ is maximal monotone, causal, and bounded whenever $\mathcal{K}_1$, $\mathcal{K}_2$ are continuous and of positive type. Moreover $\mathcal{B}_2$ is coercive iff $\mathcal{K}_1$ is of strictly positive type.*

We are now in a position of providing our examples

1. Degenerate nonlinear integrodifferential equations.
With the notation of the previous section we consider the problem

$$(\mathcal{A}u)' + (\mathcal{B}u) \ni f, \quad \mathcal{A}u(0) \ni v^0 \tag{5.8}$$

where $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$ is coercive. Indeed, taking into account $f \in L^2(0,T;V^*)$ and $v^0 \in R(\mathcal{A})$, the problem (5.8) admits at least a solution according to Theorem 2.2.

In particular, the equation (5.8) stands for a suitable abstract formulation of the coupled system of an equation and a *dynamic boundary condition* of the form

$$\big(\mathcal{A}_0(u)\big)_t + \mathcal{B}_{1,0}(u) + (\mathcal{K}_1 \circ \mathcal{R})(u) \ni g \quad \text{in} \quad L^2(0,T,H^{-1}(\Omega)), \tag{5.9}$$

$$\big(\mathcal{A}_\Gamma(\gamma u)\big)_t + \mathcal{B}_{1,\Gamma}(\gamma u) + (\mathcal{K}_2 \circ \gamma)(u) \ni h \quad \text{in} \quad L^2(0,T,(\gamma(V))^*), \tag{5.10}$$

where we used the above introduced notations and assumptions. Moreover, $\mathcal{B}_1$ is asked to be coercive, $g \in L^2(0,T,H^{-1}(\Omega))$, and $h \in L^2(0,T,(\gamma(V))^*)$. Once again, let us stress that, in the case $V = H^1_0(\Omega)$, no boundary condition is to be considered since $H^1_0(\Omega) = \ker\gamma$. On the other extremal case $V = H^1(\Omega)$, we may account for a variety of different boundary conditions by properly choosing the graphs. In particular, nonlinear and nonlocal dynamic Neumann and Robin conditions may be considered. Let us just remark that, whenever we take into account operators of type $\mathcal{B}_2$, we are indeed making use of the fact that the operator $\mathcal{B}$ is not required to be a subdifferential in our framework.

The problem above has a relevant interest within applications. Indeed, owing to the choice of the graph α_0, equation (5.9) may rise in a variety of different situations as the implicit formulation of an initial boundary value problem connected for instance with the Stefan problem, the porous media equation or the Hele-Shaw model. Letting $\mathcal{H}$ denote the Heaviside graph, the choices $\alpha_0(\xi) = \xi + \mathcal{H}(\xi)$, $\alpha_0(\xi) = \xi\,|\xi|^{\eta-1}$ for some $\eta \in (0,1)$, and $\alpha_0(\xi) = \mathcal{H}(\xi)$ correspond to the above mentioned models, respectively.

Furthermore, we are in the position of including in the above referred models both multiple nonlinearities and nonlocal in time dynamics such as memory effects.

Let us remark that, in particular, the equation (5.8) stands for a suitable variational formulation of the equation

$$\big(\alpha_0(u)\big)_t - \Delta(u + \mathcal{K}_1 u) \ni f \tag{5.11}$$

where $\mathcal{K}_1$ is of positive Volterra type and the graph α_0 is sublinear. Of course, the latter boundedness assumption on α_0 can be relaxed by means of well-known Sobolev embeddings theorems [31]. In particular, in the case $\Omega \in \mathbb{R}^3$, the growth $|y| \leq C(1 + |x|^\eta)$, $\forall [x,y] \in \alpha_0$ with $\eta \leq 36/5$ (indeed $\eta = p/p'$ with $p \in [1,6)$, $1/p + 1/p' = 1$) can be handled by carefully extending our results. In the equation above a possible choice for $\mathcal{K}_1$ would be $\mathcal{K}_1 u = h * u$ where the kernel h belongs to $L^1(0,T)$ and is of positive type, i.e.

$$\int_0^t (h * u)(s) u(s)\, ds \geq 0 \quad \forall u \in L^2(0,T),\ \forall t \in [0,T].$$

Taking into account our analysis, this equation (properly complemented with initial and boundary conditions) turns out to have a weak solution in the sense of Theorem 2.2. The reader is asked to compare this existence result with those proved in [1, 17] where the authors face an analogous problem for equation (5.11) by reducing to the case of an operator $\mathcal{K}$ of convolution type. Nevertheless, in the referred papers the positivity assumption on the kernel h is avoided in spite of a stronger assumption on the regularity of the kernel, namely $h \in W^{1,1}(0,T)$.

2. Degenerate pseudoparabolic integrodifferential equations.

Referring again to the above introduced notations we aim to exploit Theorem 2.1 in order to obtain the existence of a solution to the problem

$$(\mathcal{A} + \mathcal{E})(u)' + (\mathcal{B}u) \ni f, \quad (\mathcal{A} + \mathcal{E})u(0) \ni v^0 + \mathcal{E}u^0 \tag{5.12}$$

where $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$. Now, no coercivity is assumed on $\mathcal{B}$. In particular, $\mathcal{K}_1$ is asked to be just of positive Volterra type, i.e. λ may possibly vanish in (5.5).

The equation (5.12) may rise as an abstract formulation of various diffusion problems so called *pseudoparabolic* [13, 30]. Applying the above detailed analysis we establish an existence results for pseudoparabolic problems taking into account nonlocal in time dynamics which include, possibly, memory effects [25].

Let us now give a concrete example of an equation which, along with suitable initial and boundary conditions, may be formulated as (5.8). Indeed, we may consider

$$\big(\alpha_0(u) - \Delta u\big)_t - \Delta(\mathcal{K}_1 u) \ni f$$

Once again we possibly choose $\mathcal{K}u = h * u$ where h is an integrable positive kernel. The above equation represents a viscous version of the so-called hyperbolic Stefan problem. In connection to this particular problem, the reader is referred to the papers [15, 34] where some results on hyperbolic Stefan problems are achieved and its connection to nonlocal in time degenerate parabolic equations is detailed. According to our analysis, a proper variational version of the above integrodifferential equation of Volterra type turns out to have a weak solution due to Theorem 2.1.

3. Second order equations I.

We shall now discuss an application of our results to some classes of degenerate second order integrodifferential equations. In particular, taking into account the above introduced notations, we aim to address the equation

$$\big((\mathcal{A}_1+\mathcal{E})(u)'+(\tilde{\mathcal{B}}u)\big)'+(\mathcal{A}_2+\mathcal{E})(u)\ni\tilde{f}, \tag{5.13}$$

where both the operators $\mathcal{A}_i : V \longrightarrow V^*$, $i=1,2$, satisfy assumption (H1) and the operator $\tilde{\mathcal{B}} : L^2(0,T;V) \longrightarrow L^2(0,T;V^*)$ fulfills (H2) (possibly being of the type $\tilde{\mathcal{B}} = \mathcal{B}_1+\mathcal{B}_2$ as above). Moreover, we ask $\mathcal{A}_2$ to be symmetric, $\tilde{f}\in L^2(0,T;V^*)$ and the initial conditions

$$(\mathcal{A}_1+\mathcal{E})(u)(0)\ni v^0+\mathcal{E}u^0, \tag{5.14}$$

$$\big((\mathcal{A}_1+\mathcal{E})(u)'+(\tilde{\mathcal{B}}u)\big)(0)\ni z^0, \tag{5.15}$$

to be fulfilled for suitable $[u^0,v^0]\in\mathcal{A}_1$, $z^0\in V^*+R(\mathcal{A}_2)$.

According to Theorem 2.1, the equation (5.13) along with conditions (5.14)-(5.15) turns out to admit at least one solution $u\in H^1(0,T;V)$ with $(\mathcal{A}_1+\mathcal{E})(u)\in H^1(0,T;V^*)$ and $\big((\mathcal{A}_1+\mathcal{E})(u)'+(\tilde{\mathcal{B}}u)\big)\in H^1(0,T;V^*)$. Indeed, equation (5.13) is equivalent to the first order system

$$\begin{pmatrix}\mathcal{E}+\mathcal{A}_1 & \\ & \mathcal{E}+\mathcal{A}_2\end{pmatrix}\begin{pmatrix}u\\ z\end{pmatrix}'+\begin{pmatrix}\tilde{\mathcal{B}} & -(\mathcal{E}+\mathcal{A}_2)\\ \mathcal{E}+\mathcal{A}_2 & \end{pmatrix}\begin{pmatrix}u\\ z\end{pmatrix}\ni\begin{pmatrix}0\\ \tilde{f}\end{pmatrix}, \tag{5.16}$$

and the initial conditions (5.14)-(5.15) are rewritten as

$$\begin{pmatrix}\mathcal{E}+\mathcal{A}_1 & \\ & \mathcal{E}+\mathcal{A}_2\end{pmatrix}\begin{pmatrix}u\\ z\end{pmatrix}(0)\ni\begin{pmatrix}\mathcal{E}u^0+v^0\\ z^0\end{pmatrix}.$$

We shall apply Theorem 2.1 in the space $V=V\times V$ with the choices

$$\begin{aligned}
\big([u_1,v_1],[u_2,v_2]\big)_{V\times V} &:= (u_1,u_2)_V+(v_1,v_2)_V,\\
\mathcal{R}+\mathcal{A} &:= \begin{pmatrix}\mathcal{E}+\mathcal{A}_1 & \\ & \mathcal{E}+\mathcal{A}_2\end{pmatrix},\\
\mathcal{B} &:= \begin{pmatrix}\tilde{\mathcal{B}} & -(\mathcal{E}+\mathcal{A}_2)\\ \mathcal{E}+\mathcal{A}_2 & \end{pmatrix},\\
f &:= \begin{pmatrix}0\\ \tilde{f}\end{pmatrix},
\end{aligned}$$

with obvious notations. It may be easily seen that both hypothesis (H1) and (H2) are fulfilled in the present situation. Indeed, if $\mathcal{A}_j = \partial(i \circ \varphi_j)$ for $j = 1, 2$, we have that $\mathcal{A} = \partial(i \circ \varphi) : V \times V \longrightarrow V^* \times V^*$ where

$$\varphi(u, z) := \varphi_1(u) + \varphi_2(z) \quad \forall [u, z] \in W \times W$$

Hence, our second order problem admits at least a weak solution.

For a concrete example for the above discussed abstract second order equation we may consider the following integrodifferential equation

$$\big((\alpha_0(u) - \Delta u)_t + \mathcal{K}_1 u\big)_t + \beta_0(u) - \Delta u \ni f,$$

properly complemented with suitable boundary conditions. Here, both α_0 and β_0 are maximal monotone sublinear graphs in $\mathbb{R} \times \mathbb{R}$, possibly being multivalued. The equation above may arise in connection with some nonlinear and nonlocal viscoelasticity problems. In particular, the term $-\Delta u_{tt}$ is expected to account for some contribution of the *microscopic scale* to the *macroscopic* one in elasticity phenomena. The reader is referred to [37] for a contribution in this direction.

4. Second order equations II.

We shall now present a further example of second order differential equation which may be handled owing to our results. Indeed, we consider the following

$$(\tilde{\mathcal{A}} + \mathcal{E})(u')' + (\tilde{\mathcal{B}} u') + \mathcal{E} u \ni f \tag{5.17}$$

where $\tilde{\mathcal{B}} = \mathcal{B}_1 + \mathcal{B}_2$, $\tilde{f} \in L^2(0, T; V^*)$ along with the initial conditions

$$u(0) = u^0, \tag{5.18}$$

$$(\tilde{\mathcal{A}} + \mathcal{E})(u')(0) \ni y^0 + \mathcal{E} z^0, \tag{5.19}$$

for suitable $u^0, z^0 \in V$, $y^0 \in R(\tilde{\mathcal{A}})$. The latter equation (5.17) looks rather different from those discussed in the previous examples since the operators are applied both on the solution u and on its time derivative u'. Nevertheless, the problem (5.17)-(5.19) may be easily transformed into the equivalent first order system

$$\begin{pmatrix} \mathcal{E} & \\ & \mathcal{E} + \tilde{\mathcal{A}} \end{pmatrix} \begin{pmatrix} u \\ z \end{pmatrix}' + \begin{pmatrix} & -\mathcal{E} \\ \mathcal{E} & \tilde{\mathcal{B}} \end{pmatrix} \begin{pmatrix} u \\ z \end{pmatrix} \ni \begin{pmatrix} 0 \\ \tilde{f} \end{pmatrix}, \tag{5.20}$$

and the initial conditions (5.14)-(5.15) are rewritten as

$$\begin{pmatrix} \mathcal{E} & \\ & \mathcal{E} + \tilde{\mathcal{A}} \end{pmatrix} \begin{pmatrix} u \\ z \end{pmatrix}(0) \ni \begin{pmatrix} \mathcal{E} u^0 \\ \mathcal{E} z^0 + y^0 \end{pmatrix}.$$

We aim now to apply the existence result of Theorem 2.1 with the choices $\mathcal{V} = V \times V$,

$$\begin{aligned}
\Big([u_1, v_1], [u_2, v_2]\Big)_{V\times V} &:= (u_1, u_2)_V + (v_1, v_2)_V, \\
\mathcal{R} + \mathcal{A} &= \begin{pmatrix} \mathcal{E} & \\ & \mathcal{E} + \tilde{\mathcal{A}} \end{pmatrix}, \\
\mathcal{B} &= \begin{pmatrix} & -\mathcal{E} \\ \mathcal{E} & \tilde{\mathcal{B}} \end{pmatrix}, \\
f &= \begin{pmatrix} 0 \\ \tilde{f} \end{pmatrix},
\end{aligned}$$

with obvious notations. As before, it may be plainly checked that $\mathcal{A}$ and $\mathcal{B}$ satisfy both (H1) and (H2) (in the $V \times V - V^* \times V^*$ setting) whenever $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ satisfy both (H1) and (H2) (in the $V - V^*$ setting, respectively). In particular, if $\tilde{\mathcal{A}} = \partial(i \circ \tilde{\varphi}) : V \longrightarrow V^*$, then $\mathcal{A} = \partial(i \circ \varphi) : V \times V \longrightarrow V^* \times V^*$ with

$$\varphi(u, z) = \tilde{\varphi}(z) \quad \forall [u, z] \in W \times W.$$

As far as we are interested in an example of an integropartial differential equation which may be formulated as (5.17) we consider the following second order inclusion.

$$\big(\alpha_0(u_t) - \Delta u_t\big)_t + \mathcal{K}_1(u_t) - \Delta u \ni f.$$

Suitable boundary conditions are imposed and $\alpha_0 \subset \mathbb{R} \times \mathbb{R}$ is a maximal monotone graph fulfilling (5.2). Once again, the above equation may rise in the study of some viscoelastic problems accounting for memory effects (see [37]).

References

[1] S. Aizicovici, P. Colli, and M. Grasselli. On a class of degenerate nonlinear Volterra equations. *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, 132:135–152, 1998.

[2] S. Aizicovici, P. Colli, and M. Grasselli. Doubly nonlinear evolution equations with memory. *Funkcial. Ekval.*, to appear, 2001.

[3] S. Aizicovici and Q. Yan. Convergence theorems for abstract doubly nonlinear differential equations. *Panamer. Math. J.*, 7(1):1–17, 1997.

[4] H. Attouch. *Variational Convergence for Functions and Operators.* Pitman, London, 1984.

[5] V. Barbu. *Nonlinear semigroups and differential equations in Banach spaces.* Noordhoff International Publishing, 1976.

[6] V. Barbu. Existence for a nonlinear Volterra equation in Hilbert spaces. *SIAM J. Math. Anal.*, 10(3):552–569, 1979.

[7] V. Barbu, P. Colli, G. Gilardi, and M. Grasselli. Existence, uniqueness, and longtime behavior for a nonlinear Volterra integrodifferential equation. *Differential Integral Equations*, to appear.

[8] V. Barbu and A. Favini. Existence for implicit differential equations in Banach spaces. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. (9) Mat. Appl.*, 3:203–215, 1992.

[9] V. Barbu and A. Favini. Existence for an implicit nonlinear differential equation. *Nonlinear Anal.*, 32(1):33–40, 1998.

[10] F. Bernis. Existence results for doubly nonlinear higher order parabolic equations on unbounded domains. *Math. Ann.*, 279:373–394, 1988.

[11] H. Brezis. Monotonicity methods in Hilbert spaces and some application to nonlinear partial differential equations. In *Contrib. to nonlin. functional analysis. Proc. Sympos. Univ. Wisconsin, Madison*, pages 101–156. Academic Press, New York, 1971.

[12] H. Brezis. *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert.* Number 5 in North Holland Math. Studies. North-Holland, Amsterdam, 1973.

[13] R. W. Carroll and R. E. Showalter. Singular and degenerate Cauchy problems. Number 127 in Mathematics in Science and Engineering. Academic Press, New York, 1976.

[14] P. Colli, G. Gilardi, and M. Grasselli. Global smooth solution to the standard phase-field model with memory. *Adv. Differential Equations*, 2(3):453–486, 1997.

[15] P. Colli, G. Gilardi, and M. Grasselli. Weak solutions to hyperbolic Stefan problems with memory. *NoDEA Nonlinear Differential Equations Appl.*, 4(1):123–132, 1997.

[16] P. Colli, G. Gilardi, and M. Grasselli. Well-posedness of the weak formulation for the phase-field model with memory. *Adv. Differential Equations*, 2(3):487–508, 1997.

[17] P. Colli and M. Grasselli. Phase transition problems in materials with memory. *J. Integral Equations Appl.*, 5:1–22, 1993.

[18] P. Colli and M. Grasselli. Degenerate nonlinear Volterra integrodifferential equations. In C. Corduneanu and I.W. Sandberg, editors, *Volterra equations and applications*, volume 10 of *Stability Control Theory Methods Appl.*, pages 187–195. Gordon and Breach, Amsterdam, 2000.

[19] P. Colli and A. Visintin. On a class of doubly nonlinear evolution equations. *Comm. Partial Differential Equations*, 15(5):737–756, 1990.

[20] M. G. Crandall and A. Pazy. Semi-groups of nonlinear contractions and dissipative sets. *J. Funct. Anal.*, 3:376–418, 1969.

[21] E. Di Benedetto and R. E. Showalter. Implicit degenerate evolution equations and applications. *SIAM J. Math. Anal.*, 12(5):731–751, 1981.

[22] G. Gentili and C. Giorgi. Thermodynamic properties and stability for the heat flux equation with linear memory. *Quart. Appl. Math.*, 51(2):343–362, 1993.

[23] O. Grange and F. Mignot. Sur la résolution d'une équation et d'une inéquation paraboliques non linéaires. *J. Funct. Anal.*, 11:77–92, 1972.

[24] G. Gripenberg, S. -O. Londen, and O. Staffans. *Volterra Integral and Functional Equations*. Cambridge University Press, 1990.

[25] M. E. Gurtin and A.C. Pipkin. A general theory of heat conduction with finite wave speeds. *Arch. Rational Mech. Anal.*, 31:113–126, 1968.

[26] V. -M. Hokkanen. On nonlinear Volterra equations in Hilbert spaces. *Differential and Integral Equations*, 5(3):647–669, 1992.

[27] J.W. Jerome. *Approximation of nonlinear evolution systems*. Number 164 in Math. Sci. Engrg. Academic Press, Orlando, 1983.

[28] D. D. Joseph and L. Preziosi. Heat waves. *Rev. Modern Phys.*, 61(1):41–73, 1989.

[29] D. D. Joseph and L. Preziosi. Addendum to the paper "Heat waves". *Rev. Modern Phys.*, 62(2):375–391, 1990.

[30] J.-L. Lions. *Sur quelques question d'analyse de mechanique et de contrôle optimal*. Les Presses de l'Université de Montréal, Montréal, 1976.

[31] J.-L. Lions and E. Magenes. *Non-homogeneus boundary value problems and applications*, volume 1. Springer-Verlag, New York - Heidelberg, 1972.

[32] T. Senba. On some nonlinear evolution equation. *Funkcial. Ekval.*, 29:243–257, 1986.

[33] R. E. Showalter. *Monotone operators in Banach space and nonlinear differential equations*, volume 49 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1997.

[34] R. E. Showalter and N. Walkington. A hyperbolic Stefan problem. *Quart. Appl. Math.*, 45(4):769–781, 1987.

[35] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[36] U. Stefanelli. Well-posedness and time discretization of a nonlinear Volterra integrodifferential equation. *J. Integral Equations Appl.*, to appear.

[37] F. Theil and V. I. Levitas. A study of a Hamiltonian model for martensitic phase transformations including microkinetic energy. *Math. Mech. Solids*, 5(3):337–368, 2000.