

# Long-Time Behavior for the Full One-Dimensional Frémond Model for Shape Memory Alloys

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## Abstract

This note deals with a well-posed one-dimensional initial-boundary value problem related to the Frémond thermomechanical model of structural phase transitions in shape memory materials. The long-time behavior is investigated and it is proved that the  $\omega$ -limit set in a suitable topology only contains stationary states.

**Key words:** shape memory alloys, existence and uniqueness, long-time behavior.

**AMS (MOS) Subject Classification:** 35K55, 35B40.

## 1 Introduction

This paper is concerned with the following system of partial differential equations in terms of the unknown functions  $\vartheta$ ,  $u$ ,  $\chi_1$ , and  $\chi_2$ ,

$$\partial_t(c_0\vartheta - L\chi_1) + \partial_t((\alpha(\vartheta) - \vartheta\alpha'(\vartheta))\chi_2u_x) - h\vartheta_{xx} = \alpha(\vartheta)\chi_2u_{xt}, \quad (1.1)$$

$$\partial_x(u_x + \beta\alpha(\vartheta)\chi_2) = 0, \quad (1.2)$$

$$\mu\partial_t\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} - \lambda\partial_{xx}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + \begin{pmatrix} \ell(\vartheta - \vartheta^*) \\ \alpha(\vartheta)u_x \end{pmatrix} + \partial I_{\mathcal{K}}(\chi_1, \chi_2) \ni \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (1.3)$$

a.e. in  $Q_T := (0, 1) \times (0, T)$ , where  $T$  stands for some reference time, and  $c_0, L, h, \beta, \mu, \lambda, \ell$ , and  $\vartheta^*$  are positive parameters. Here,  $\partial I_{\mathcal{K}}$  denotes the subdifferential of the indicator function of a non-empty, bounded, convex and closed subset  $\mathcal{K}$  of  $\mathbb{R}^2$ , while  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function with suitable properties to be specified later.

The nonlinear system (1.1)-(1.3) has been proposed by M. Frémond [7, 9, 10, 11, 12] in order to describe the macroscopic thermomechanical evolution of a shape memory body. The latter is a metallic alloy, which exhibits this surprising behavior: it could be permanently deformed (avoiding fractures) and consequently be forced to recover its original shape just by thermal means. Indeed, in the microscopic scale, such phenomenon has been interpreted (see, e.g., [1]) as the effect of a *structural phase transition* (solid-solid) between two (or more) variants of the metallic lattice, due to thermomechanical treatments.

In this concern,  $\vartheta$  has to be regarded as the absolute temperature of the sample, while  $u$  stands for the one-dimensional longitudinal displacement and  $\chi_1, \chi_2$  are related to the pointwise proportions of the phases. Namely, the system (1.1)-(1.3) results from the coupling of the universal conservation laws for energy and momentum (the latter in a *quasi-stationary* form, since the term  $u_{tt}$  is omitted) with a variational inequality describing the evolution of the phases. Let us point out that  $\alpha(\vartheta)$  represents the *thermal expansion* of the system and  $\lambda$  is a diffusion parameter related to the phase dissipation. Moreover, we refer to [7] for the physical meaning of the constants  $c_0, L, h, \beta, \mu, \ell$ , and  $\vartheta^*$ .

The system (1.1)-(1.3) has to be supplemented with suitable boundary and initial conditions. Namely, we prescribe

$$h\vartheta_x(0, t) - k(\vartheta(0, t) - \vartheta_e) = 0 \quad \text{for a.e. } t \in (0, T), \quad (1.4)$$

$$h\vartheta_x(1, t) + k(\vartheta(1, t) - \vartheta_e) = 0 \quad \text{for a.e. } t \in (0, T), \quad (1.5)$$

$$u(0, t) = 0 \quad \text{for any } t \in (0, T), \quad (1.6)$$

$$(u_x + \beta\alpha(\vartheta)\chi_2)(1, t) - \beta g(t) = 0 \quad \text{for a.e. } t \in (0, T), \quad (1.7)$$

$$\chi_{i,x}(0, t) = \chi_{i,x}(1, t) = 0 \quad \text{for a.e. } t \in (0, T), \quad i = 1, 2, \quad (1.8)$$

$$\vartheta(x, 0) = \vartheta_0 \quad \text{for any } x \in [0, 1], \quad (1.9)$$

$$\chi_i(x, 0) = \chi_{i,0} \quad \text{for any } x \in [0, 1], \quad i = 1, 2, \quad (1.10)$$

where  $k$  is a positive heat exchange coefficient, while the constant  $\vartheta_e > 0$  and  $g : (0, T) \rightarrow \mathbb{R}$  account for the interactions with the medium surrounding the domain. In particular,  $\vartheta_e$  stands for an external temperature and  $g$  is a traction.

Although the well-posedness of problem (1.1)-(1.10) has never been investigated, various existence and uniqueness results for initial and boundary value problems for systems close to (1.1)-(1.3) have been obtained recently. First of all, let us quote the paper [8] where a well-posedness result for the system (1.1)-(1.3) with  $\lambda = 0$  is proved. Indeed, the referred paper deals with the physically meaningful case of *diffusion free* martensitic phase transitions. The analytic argument devised there may be easily adjusted for the case of positive  $\lambda$  so that we will widely refer to [8] for our well-posedness result. As regards the one-dimensional problem including the inertial term  $u_{tt}$ , we stress that an existence and uniqueness result has been jointly proved in the two

papers [4, 15]. Indeed, the paper [4] provides also a uniqueness result for the *quasi-stationary* three-dimensional problem, which turns out to admit a global solution according to the paper [6]. For the sake of completeness, we mention also [5] which provides a solution to the *non-stationary* three-dimensional problem under the linearization of the energy balance equation and [2] where non-Fourier memory-type heat flux laws are considered along with diffusive effects for the phase proportions ( $\lambda > 0$ ). Finally, let us quote [14], where the authors also deal with the case of a positive coefficient  $\lambda$  but they neglect one of the three nonlinearities which appear in the energy balance equation (1.1). However, for a detailed derivation of a general model including phase diffusion, we refer to the first part of the book [11] in which such effects are discussed along with other contributions due to internal microscopic velocities.

Up to now, it seems that no long-time analysis has been performed for the Frémond model. Nevertheless, the investigation and characterization of long-time behavior are especially important within applications. Indeed, shape memory materials are used in several fields (from aerospace sciences to bio-engineering) for the construction of small devices whose behavior for long times turns out to be crucial. The main novelty of this paper is that of investigating the stabilization of solutions to the full one-dimensional Frémond model. In particular, we aim to analyze the set of cluster points of trajectories as time goes to  $+\infty$ . Moreover, we will completely characterize such points as solutions to the related stationary problem.

## 2 Main Results

Before stating our main results, let us fix some notation. We set, for the sake of simplicity,

$$H := L^2(0, 1), \quad V := H^1(0, 1), \quad W := H^2(0, 1),$$

identify  $H$  with its dual space  $H'$ , and let  $(\cdot, \cdot)$  and  $\|\cdot\|$  denote the scalar product and the norm in  $H$ , respectively.

Next, let  $\mathcal{K}$  be any non-empty, bounded, convex, and closed subset of  $\mathbb{R}^2$ , and define the (convex and closed) subset of  $H \times H$

$$K := \{(\gamma_1, \gamma_2) \in H \times H : (\gamma_1, \gamma_2) \in \mathcal{K} \text{ a.e. in } (0, 1)\}. \quad (2.1)$$

Then, it is straightforward to find a positive constant  $c_{\mathcal{K}}$  such that

$$\left(|\gamma_1|^2 + |\gamma_2|^2\right)^{1/2} \leq c_{\mathcal{K}} \quad \text{a.e. in } (0, 1), \quad \forall (\gamma_1, \gamma_2) \in K. \quad (2.2)$$

We assume that the data fulfill

$$g \in H^1(0, +\infty), \quad g_t \in L^1(0, +\infty), \quad (2.3)$$

$$\vartheta_0 \in V, \quad (\chi_{1,0}, \chi_{2,0}) \in V \cap K, \quad (2.4)$$

and require  $\alpha$  to be a smooth function on  $(0, +\infty)$ , vanishing in the interval  $(\vartheta_c, +\infty)$ , where  $\vartheta_c > 0$  stands for the so called Curie temperature, such that

$$\alpha \in C^2(\mathbb{R}) \text{ and the set } \{\xi \in \mathbb{R} : \alpha'(\xi) \neq 0\} \text{ is contained in } [0, \vartheta_c]. \quad (2.5)$$

We shall also require that the quantity  $c_\alpha := \|\alpha''\|_{L^\infty(\mathbb{R})}$  is sufficiently small. Indeed, owing to (2.5), we easily check that

$$|\alpha'(\xi)|, |\xi\alpha''(\xi)| \leq \vartheta_c c_\alpha \quad \text{and} \quad |\alpha(\xi)|, |\xi\alpha'(\xi)| \leq \vartheta_c^2 c_\alpha \quad \forall \xi \in \mathbb{R}. \quad (2.6)$$

Moving from this consideration, the smallness assumption on  $c_\alpha$  we are looking for is

$$\beta\vartheta_c c_\alpha c_{\mathcal{K}} (\|g\|_{L^\infty(0,+\infty)} + \vartheta_c^2 c_\alpha c_{\mathcal{K}}) < c_0. \quad (2.7)$$

**Remark 2.1.** Note that a consequence of (2.7) is that (see also the following (2.12)) the factor of  $\vartheta_t$  in (1.1), namely

$$c_0 - \vartheta\alpha''(\vartheta)\chi_2 u_x,$$

representing the actual specific heat of the shape memory body, is strictly positive everywhere. In this sense, (2.7) has to be regarded as a non degeneracy condition for the energy balance equation in (1.1). Indeed, from (2.3) we deduce that the left hand side of (2.7) is bounded anyway and our assumption turns out to be just a compatibility condition among the data. Moreover, we point out that other properties of  $\alpha$  (which is nonnegative and decreasing as well, see [8]) are neglected in our analysis.

Thus, we are now in a position of stating a well-posedness result for problem (1.1)-(1.10), in which the pointwise inclusion (1.3) is equivalently rewritten as variational inequality.

**Theorem 2.2.** *Under the assumptions (2.3)-(2.5) and (2.7), there exists a unique quadruple  $(\vartheta, u, \chi_1, \chi_2)$  such that, for each  $T \in (0, +\infty)$ ,  $\vartheta \in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W)$ ,  $u \in H^1(0, T; V) \cap C^0([0, T]; W)$ ,  $\chi_1, \chi_2 \in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W)$  and relations (1.1)-(1.2), (1.4)-(1.10) and the following conditions hold*

$$(\chi_1(\cdot, t), \chi_2(\cdot, t)) \in K \quad \forall t \in [0, T], \quad (2.8)$$

$$\sum_{j=1}^2 (\mu\partial_t \chi_j - \lambda\partial_{xx} \chi_j, \chi_j - \gamma_j) + \ell(\vartheta - \vartheta^*, \chi_1 - \gamma_1) + (\alpha(\vartheta)u_x, \chi_2 - \gamma_2) \leq 0$$

a.e. in  $(0, T)$ ,  $\forall (\gamma_1, \gamma_2) \in K.$  (2.9)

We will not give a proof of the previous result. Indeed, let us observe that the argument devised in [8] for the problem without diffusion on the phase parameters (i.e.,  $\lambda \equiv 0$ ) applies to the case above with slight modifications. The main idea of [8] is to rewrite the problem in a completely equivalent form independent of  $u_x$ . This possibility is ensured by relation (1.2) which, in particular, gives

$$u_x = \beta(g - \alpha(\vartheta)\chi_2) \quad \text{a.e. in } Q_T, \quad (2.10)$$

whence, by (1.6)

$$u(x, t) = \beta \int_0^x (g - \alpha(\vartheta)\chi_2)(\xi, t) d\xi \quad \forall (x, t) \in \overline{Q_T}. \quad (2.11)$$

Let us recall that  $Q_T = (0, 1) \times (0, T)$  and set  $Q_\infty := (0, 1) \times (0, +\infty)$ . Note that the boundedness of  $\mathcal{K}$ , (2.3), (2.5), (2.10), and (2.11) ensure

$$\|u\|_{L^\infty(Q_\infty)} + \|u_x\|_{L^\infty(Q_\infty)} \leq C_1, \quad (2.12)$$

for some constant  $C_1$  depending on  $\|g\|_{L^\infty(0,+\infty)}$ ,  $\|\alpha\|_{L^\infty(\mathbb{R})}$ ,  $c_K$ , and  $\beta$  only. Moreover, for the sake of clarity we stress that (1.3) and (2.9) are completely equivalent and refer to [2, Appendix] for an independent study of variational inequalities of that form.

We are interested in studying the long-time behavior of the solution to (1.1)-(1.10) defined above. More precisely, if  $(\vartheta, u, \chi_1, \chi_2)$  is a solution to (1.1)-(1.10) in the sense of Theorem 2.2, we aim to study the cluster points as  $t \rightarrow +\infty$  of  $(\vartheta(t), u(t), \chi_1(t), \chi_2(t))$  in a suitable topology. In our setting it turns out that a natural topology for the study of the long-time behavior of  $(\vartheta, u, \chi_1, \chi_2)$  is that of  $H \times V \times H \times H$ . We thus define the  $\omega$ -limit set of  $(\vartheta, u, \chi_1, \chi_2)$  in  $H \times V \times H \times H$  by

$$\begin{aligned} \omega(\vartheta, u, \chi_1, \chi_2) := & \left\{ (\vartheta_\infty, u_\infty, \chi_{1,\infty}, \chi_{2,\infty}) \in H \times V \times H \times H \text{ such that there exists} \right. \\ & \text{a sequence of positive real numbers } \{t_n\} \text{ with } t_n \longrightarrow +\infty \text{ and} \\ & \left. (\vartheta(t_n), u(t_n), \chi_1(t_n), \chi_2(t_n)) \longrightarrow (\vartheta_\infty, u_\infty, \chi_{1,\infty}, \chi_{2,\infty}) \text{ in } H \times V \times H \times H \right\}. \end{aligned} \quad (2.13)$$

As for establishing a long-time behavior result, we need some further assumptions on the data. Namely, we impose  $c_\alpha$  to be small with respect to some given quantities. In particular, letting  $C_2$  be the largest constant, depending on  $h$  and  $k$ , defined by the Poincaré inequality

$$C_2 \|v\|_V^2 \leq h \int_0^1 |v_x|^2 + k \sum_{i=0}^1 v^2(i), \quad \forall v \in V,$$

and setting  $C_3 := L\mu/\ell$ , we ask for

$$\beta \|g\|_{L^\infty(0,+\infty)} \vartheta_c^2 c_\alpha + \beta \vartheta_c^3 c_\alpha^2 c_K (2 \max\{2\vartheta_c - \vartheta_e, \vartheta_e\} + \vartheta_c + 2L/\ell) < 2\sqrt{C_2 C_3}. \quad (2.14)$$

Let us point out that both (2.7) and the relation above are fulfilled upon choosing the quantity  $c_\alpha$  small enough. This possibility is actually ensured by physical considerations. Indeed,  $c_\alpha$  is known to be very small [7].

Owing to the definition of the  $\omega$ -limit set above, we may state the main result of our paper.

**Theorem 2.3.** *Assume that (2.3)-(2.5), (2.7), and (2.14) are fulfilled and let  $(\vartheta, u, \chi_1, \chi_2)$  be the solution to (1.1)-(1.10) in the sense of Theorem 2.2. Then  $\omega(\vartheta, u, \chi_1, \chi_2)$  is a non-empty, compact, and connected subset of  $H \times V \times H \times H$ . Moreover, if  $(\vartheta_\infty, u_\infty, \chi_{1,\infty}, \chi_{2,\infty})$  belongs to  $\omega(\vartheta, u, \chi_1, \chi_2)$ , one has*

$$\vartheta_\infty = \vartheta_e \quad \text{a.e. in } \Omega, \quad (2.15)$$

$$u_\infty(x) = -\beta \alpha(\vartheta_e) \int_0^x \chi_{2,\infty}(\xi) d\xi \quad \forall x \in (0, 1), \quad (2.16)$$

$$-\lambda \partial_{xx} \begin{pmatrix} \chi_{1,\infty} \\ \chi_{2,\infty} \end{pmatrix} + \begin{pmatrix} \ell(\vartheta_e - \vartheta^*) \\ -\beta(\alpha(\vartheta_e))^2 \chi_{2,\infty} \end{pmatrix} + \partial I_K(\chi_{1,\infty}, \chi_{2,\infty}) \ni \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (2.17)$$

$$\chi_{i,\infty,x}(0) = \chi_{i,\infty,x}(1) = 0, \quad i = 1, 2. \quad (2.18)$$

Let us stress that, though the above theorem identifies the possible cluster points of  $(\vartheta(t), u(t), \chi_1(t), \chi_2(t))$  as  $t \rightarrow +\infty$ , it does not indicate whether  $(\vartheta(t), u(t), \chi_1(t), \chi_2(t))$  has a limit or not as  $t \rightarrow +\infty$ . However, we note that, since  $\vartheta_\infty$  is uniquely determined (cf. (2.15)) it turns out that the entire trajectory of  $\vartheta(t)$  converges to  $\vartheta_e$  as  $t \rightarrow +\infty$ .

The proof of the above result is to be found in the forthcoming section.

### 3 Long-Time Behavior

This section brings to the proof of Theorem 2.3 by establishing some preparatory results. First, we have the following result:

**Lemma 3.1.** *Assume that (2.3)-(2.5), (2.7), and (2.14) are fulfilled and let  $(\vartheta, u, \chi_1, \chi_2)$  be the unique solution to (1.1)-(1.10) in the sense of Theorem 2.2. Then, there exists a positive constant  $C_4$ , depending on  $\|g\|_{H^1(0,+\infty)}$ ,  $\|g_t\|_{L^1(0,+\infty)}$ ,  $\|\vartheta_0\|_V$ ,  $\|\chi_{i,0}\|_V$  ( $i = 1, 2$ ),  $\beta$ ,  $\vartheta_c$ ,  $c_\alpha$ ,  $c_\kappa$ ,  $c_0$ ,  $L$ ,  $\ell$ ,  $\vartheta_e$ ,  $C_2$ , and  $\mu$ , such that, for each  $t \in (0, +\infty)$ ,*

$$\begin{aligned} & \|\vartheta(t) - \vartheta_e\|_V^2 + \int_0^t (\|\vartheta_t(s)\|^2 + \|\vartheta_{xx}(s)\|^2) ds + \|u_x(t)\|_V^2 \\ & + \int_0^t \|u_{xt}(s)\|^2 ds + \sum_{i=1}^2 \left( \|\chi_i(t)\|_V^2 + \int_0^t \|\chi_{i,t}(s)\|^2 ds \right) \leq C_4. \end{aligned} \quad (3.1)$$

*Proof.* We begin by multiplying (1.1) by  $\vartheta(t) - \vartheta_e$  and integrating in space and time, one readily obtains

$$\begin{aligned} & c_0 \int_0^t \int_0^1 \vartheta_t(\vartheta - \vartheta_e) + h \int_0^t \int_0^1 |(\vartheta - \vartheta_e)_x|^2 + k \int_0^t \sum_{i=0}^1 (\vartheta(i, s) - \vartheta_e)^2 ds \\ & \leq \int_0^t \int_0^1 L \chi_{1,t}(\vartheta - \vartheta_e) + \sum_{j=1}^3 I_j(t) \quad \forall t \in [0, T], \end{aligned} \quad (3.2)$$

where

$$\begin{aligned} I_1(t) & := \int_0^t \int_0^1 \vartheta \alpha''(\vartheta) \chi_2 u_x \vartheta_t (\vartheta - \vartheta_e), \\ I_2(t) & := \int_0^t \int_0^1 (\vartheta \alpha'(\vartheta) - \alpha(\vartheta)) u_x \chi_{2,t} (\vartheta - \vartheta_e), \\ I_3(t) & := \int_0^t \int_0^1 \vartheta \alpha'(\vartheta) \chi_2 u_{xt} (\vartheta - \vartheta_e). \end{aligned}$$

On the other hand, from (2.9) it is not difficult to deduce the following estimate

$$\begin{aligned} C_3 \int_0^t \int_0^1 \sum_{j=1}^2 |\chi_{j,t}|^2 + \frac{L\lambda}{2\ell} \int_0^1 \sum_{j=1}^2 |\chi_{j,x}(t)|^2 + L \int_0^t \int_0^1 (\vartheta - \vartheta^*) \chi_{1,t} \\ + \frac{L}{\ell} \int_0^t \int_0^1 \alpha(\vartheta) u_x \chi_{2,t} \leq \frac{L\lambda}{2\ell} \int_0^1 \sum_{j=1}^2 |\partial_x \chi_{j,0}|^2 \quad \forall t \in [0, T], \end{aligned} \quad (3.3)$$

by taking  $\gamma_j = \chi_j(t - \tau)$ ,  $j = 1, 2$ , in (2.9), multiplying by  $L/\ell\tau$ , integrating with respect to time, and then passing to the limit as  $\tau \rightarrow 0$  (we recall that  $C_3 = L\mu/\ell$ ).

Taking now the sum of the last two inequalities and using the Poincaré inequality we obtain

$$\begin{aligned} \frac{c_0}{2} \|(\vartheta - \vartheta_e)(t)\|^2 + C_2 \int_0^t \|\vartheta - \vartheta_e\|_V^2 + C_3 \int_0^t \int_0^1 \sum_{j=1}^2 |\chi_{j,t}|^2 \\ + \frac{L\lambda}{2\ell} \int_0^1 \sum_{j=1}^2 |\chi_{j,x}(t)|^2 \leq \frac{c_0}{2} \|\vartheta_0 - \vartheta_e\|^2 + \frac{L\lambda}{2\ell} \int_0^1 \sum_{j=1}^2 |\partial_x \chi_{j,0}|^2 \\ + L \int_0^1 (\vartheta^* - \vartheta_e)(\chi_1(t) - \chi_{1,0}) + \sum_{j=1}^4 I_j(t), \quad \forall t \in [0, T], \end{aligned} \quad (3.4)$$

where

$$I_4(t) := -\frac{L}{\ell} \int_0^t \int_0^1 \alpha(\vartheta) u_x \chi_{2,t}.$$

Henceforth, we will denote by  $C$  any constant depending on the data but independent of  $T$ . Of course,  $C$  may vary from line to line.

We exploit (2.10) and deduce that

$$\begin{aligned} I_1(t) &= \beta \int_0^t \int_0^1 \vartheta \alpha''(\vartheta) \chi_2 (g - \alpha(\vartheta) \chi_2) \vartheta_t (\vartheta - \vartheta_e), \\ I_3(t) &= \beta \int_0^t \int_0^1 \vartheta \alpha'(\vartheta) \chi_2 (g_t - \alpha'(\vartheta) \vartheta_t \chi_2 - \alpha(\vartheta) \chi_{2,t}) (\vartheta - \vartheta_e). \end{aligned}$$

Let us introduce, for the sake of clarity, two auxiliary functions  $\psi_1$  and  $\psi_2$  defined by

$$\psi_1(\vartheta) := \int_{\vartheta_e}^{\vartheta} \xi \alpha''(\xi) (\xi - \vartheta_e) d\xi, \quad \vartheta \in \mathbb{R}, \quad (3.5)$$

$$\psi_2(\vartheta) := \int_{\vartheta_e}^{\vartheta} \xi ((\alpha'(\xi))^2 + \alpha(\xi) \alpha''(\xi)) (\xi - \vartheta_e) d\xi, \quad \vartheta \in \mathbb{R}. \quad (3.6)$$

According to assumption (2.5), it turns out that the two functions above are uniformly bounded in terms of the data. Moreover, by integrating by parts one easily infers

$$\begin{aligned} \psi_1(\vartheta) &= \vartheta \alpha'(\vartheta) (\vartheta - \vartheta_e) - \int_{\vartheta_e}^{\vartheta} \alpha'(\xi) (2\xi - \vartheta_e) d\xi, \quad \vartheta \in \mathbb{R}, \\ \psi_2(\vartheta) &= \vartheta \alpha(\vartheta) \alpha'(\vartheta) (\vartheta - \vartheta_e) - \int_{\vartheta_e}^{\vartheta} \alpha(\xi) \alpha'(\xi) (2\xi - \vartheta_e) d\xi, \quad \vartheta \in \mathbb{R}. \end{aligned}$$

By making use of the above relations it is not difficult to handle jointly  $I_1(t)$  and  $I_3(t)$  as follows.

$$\begin{aligned}
 I_1(t) + I_3(t) &= \beta \int_0^t \int_0^1 \chi_2 g(\psi_1(\vartheta))_t - \beta \int_0^t \int_0^1 \chi_2^2(\psi_2(\vartheta))_t \\
 &+ \beta \int_0^t \int_0^1 \vartheta \alpha'(\vartheta) \chi_2 g_t(\vartheta - \vartheta_e) \\
 &- \beta \int_0^t \int_0^1 \vartheta \alpha(\vartheta) \alpha'(\vartheta) \chi_2 \chi_{2,t}(\vartheta - \vartheta_e) \\
 &= \beta \int_0^1 (\chi_2 g \psi_1(\vartheta))(t) - \beta \int_0^1 \chi_{2,0} g(0) \psi_1(\vartheta_0) \\
 &- \beta \int_0^t \int_0^1 \psi_1(\vartheta) (\chi_2 g_t + \chi_{2,t} g) - \beta \int_0^t \int_0^1 (\chi_2^2 \psi_2(\vartheta))(t) \\
 &+ \beta \int_0^1 \chi_{2,0}^2 \psi_2(\vartheta_0) + 2\beta \int_0^t \int_0^1 \psi_2(\vartheta) \chi_2 \chi_{2,t} + \beta \int_0^t \int_0^1 \vartheta \alpha'(\vartheta) \chi_2 g_t(\vartheta - \vartheta_e) \\
 &- \beta \int_0^t \int_0^1 \vartheta \alpha(\vartheta) \alpha'(\vartheta) \chi_2 \chi_{2,t}(\vartheta - \vartheta_e).
 \end{aligned}$$

Thus, according to (2.2)-(2.3) and (2.6), we easily obtain that

$$\begin{aligned}
 I_1(t) + I_3(t) &\leq C - \beta \int_0^t \int_0^1 \psi_1(\vartheta) \chi_{2,t} g \\
 &+ \beta \int_0^t \int_0^1 \chi_2 g_t \left( \int_{\vartheta_e}^{\vartheta} \alpha'(\xi) (2\xi - \vartheta_e) d\xi \right) \\
 &+ \beta \int_0^t \int_0^1 \chi_2 \chi_{2,t} \left( \vartheta \alpha(\vartheta) \alpha'(\vartheta) (\vartheta - \vartheta_e) - 2 \int_{\vartheta_e}^{\vartheta} \alpha(\xi) \alpha'(\xi) (2\xi - \vartheta_e) d\xi \right).
 \end{aligned}$$

Next, by (2.10) we have that

$$\begin{aligned}
 I_2(t) &= \beta \int_0^t \int_0^1 (\vartheta \alpha'(\vartheta) - \alpha(\vartheta)) (\vartheta - \vartheta_e) g \chi_{2,t} \\
 &- \beta \int_0^t \int_0^1 \chi_2 \chi_{2,t} \alpha(\vartheta) (\vartheta \alpha'(\vartheta) - \alpha(\vartheta)) (\vartheta - \vartheta_e).
 \end{aligned}$$

Combining the above two estimates and recalling the formulae which give  $\psi_1$ , we infer

$$\begin{aligned}
 (I_1 + I_2 + I_3)(t) &\leq C + \beta \int_0^t \int_0^1 (g \chi_2)_t \left( \int_{\vartheta_e}^{\vartheta} \alpha'(\xi) (2\xi - \vartheta_e) d\xi \right) \\
 &- \beta \int_0^t \int_0^1 g \chi_{2,t} \alpha(\vartheta) (\vartheta - \vartheta_e) \\
 &+ \int_0^t \int_0^1 \chi_2 \chi_{2,t} \left( \alpha^2(\vartheta) (\vartheta - \vartheta_e) - 2 \int_{\vartheta_e}^{\vartheta} (2\xi - \vartheta_e) \alpha(\xi) \alpha'(\xi) d\xi \right).
 \end{aligned}$$

We now fix  $\varepsilon > 0$  to be chosen later. Using again (2.2)-(2.3) and (2.6) we finally obtain

$$\begin{aligned}
 (I_1 + I_2 + I_3)(t) &\leq C + C \int_0^t (|g_t| + |g| \|\chi_{2,t}\|) \\
 &\quad + \beta \|g\|_{L^\infty(0,+\infty)} \vartheta_c^2 c_\alpha \int_0^t \|\chi_{2,t}\| \|\vartheta - \vartheta_e\| \\
 &\quad + \beta c_\mathcal{K} \vartheta_c^4 c_\alpha^2 \int_0^t \|\chi_{2,t}\| \|\vartheta - \vartheta_e\| \\
 &\quad + 2\beta c_\mathcal{K} \vartheta_c^3 c_\alpha^2 \max\{2\vartheta_c - \vartheta_e, \vartheta_e\} \int_0^t \|\chi_{2,t}\| \|\vartheta - \vartheta_e\| \\
 &\leq C + \varepsilon \int_0^t \|\chi_{2,t}\|^2 ds + \beta \|g\|_{L^\infty(0,+\infty)} \vartheta_c^2 c_\alpha \int_0^t \|\chi_{2,t}\| \|\vartheta - \vartheta_e\| \\
 &\quad + \beta c_\mathcal{K} \vartheta_c^3 c_\alpha^2 (\vartheta_c + 2 \max\{2\vartheta_c - \vartheta_e, \vartheta_e\}) \int_0^t \|\chi_{2,t}\| \|\vartheta - \vartheta_e\|. \tag{3.7}
 \end{aligned}$$

Next, we deal with  $I_4(t)$  by integrating by parts. With the help of (2.10) one has

$$\begin{aligned}
 I_4(t) &= -\frac{\beta L}{\ell} \int_0^t \int_0^1 \alpha(\vartheta) g \chi_{2,t} + \frac{\beta L}{\ell} \int_0^t \int_0^1 \alpha^2(\vartheta) \chi_2 \chi_{2,t} \\
 &= -\frac{\beta L}{\ell} \int_0^t \int_0^1 \alpha(\vartheta) g \chi_{2,t} + \frac{\beta L}{\ell} \int_0^t \int_0^1 \alpha^2(\vartheta_e) \chi_2 \chi_{2,t} \\
 &\quad + \frac{\beta L}{\ell} \int_0^t \int_0^1 (\alpha^2(\vartheta) - \alpha^2(\vartheta_e)) \chi_2 \chi_{2,t} \\
 &= -\frac{\beta L}{\ell} \int_0^t \int_0^1 \alpha(\vartheta) g \chi_{2,t} + \frac{\beta L}{2\ell} \int_0^1 (\alpha^2(\vartheta_e) \chi_2^2)(t) - \frac{\beta L}{2\ell} \int_0^1 \alpha^2(\vartheta_e) \chi_{2,0}^2 \\
 &\quad + \frac{2\beta L}{\ell} \int_0^t \int_0^1 \chi_2 \chi_{2,t} \left( \int_{\vartheta_e}^{\vartheta} \alpha(\xi) \alpha'(\xi) d\xi \right).
 \end{aligned}$$

Thus, owing to (2.2)-(2.3) and (2.6), we infer that

$$\begin{aligned}
 I_4(t) &\leq \frac{\beta L}{\ell} \vartheta_c^2 c_\alpha \int_0^t \int_0^1 |g| |\chi_{2,t}| + \frac{\beta L}{\ell} (\vartheta_c^4 c_\alpha^2 c_\mathcal{K}^2) + \frac{2\beta L}{\ell} \vartheta_c^3 c_\alpha^2 c_\mathcal{K} \int_0^t \|\chi_{2,t}\| \|\vartheta - \vartheta_e\| \\
 &\leq C + \varepsilon \int_0^t \|\chi_{2,t}\|^2 + \frac{2\beta L}{\ell} \vartheta_c^3 c_\alpha^2 c_\mathcal{K} \int_0^t \|\chi_{2,t}\| \|\vartheta - \vartheta_e\|. \tag{3.8}
 \end{aligned}$$

Whence, collecting (3.7)-(3.8) in (3.4), accounting for (2.14), and choosing  $\varepsilon$  small enough, we obtain, in particular, that

$$\|\vartheta(t) - \vartheta_e\|^2 + \int_0^t \|\vartheta(s)\|_V^2 ds + \sum_{j=1}^2 \left( \int_0^t \|\chi_{j,t}(s)\|^2 ds + \|\chi_j(t)\|_V^2 \right) \leq C. \tag{3.9}$$

Observe that also conditions (2.2) and (2.8) have been used. The latter bound turns out to be crucial in the following second estimate. First of all, we exploit (2.10) to get from (1.1) that

$$\begin{aligned} & \left( c_0 + \beta\chi_2^2\vartheta(\alpha'(\vartheta))^2 - \beta g\chi_2\vartheta\alpha''(\vartheta) + \beta\chi_2^2\vartheta\alpha(\vartheta)\alpha'(\vartheta) \right) \vartheta_t - h\vartheta_{xx} \\ & = L\chi_{1,t} + \psi_3g_t + \psi_4\chi_{2,t}, \quad \text{a.e. in } Q_\infty \end{aligned} \quad (3.10)$$

where the quantities  $\psi_3$  and  $\psi_4$  are defined by

$$\begin{aligned} \psi_3 & := \beta\vartheta\alpha'(\vartheta)\chi_2, \\ \psi_4 & := -\beta(\alpha(\vartheta) - \vartheta\alpha'(\vartheta))g + \beta\alpha(\vartheta)(\alpha(\vartheta) - 2\vartheta\alpha'(\vartheta))\chi_2. \end{aligned}$$

Let us stress that, owing to (2.2)-(2.3) and (2.5), both  $\psi_3$  and  $\psi_4$  are uniformly bounded in terms of the data. We multiply (3.10) by  $\vartheta_t$  and take the integral in space and time. Accounting for (2.7), one easily obtains a constant  $\delta > 0$  such that

$$0 < \delta \leq \left( c_0 + \beta\chi_2^2\vartheta(\alpha'(\vartheta))^2 - \beta g\chi_2\vartheta\alpha''(\vartheta) + \beta\chi_2^2\vartheta\alpha(\vartheta)\alpha'(\vartheta) \right) \quad \text{a.e. in } Q_\infty.$$

Thus, recalling also (2.4), we deduce

$$\begin{aligned} & \delta \int_0^t \int_0^1 |\vartheta_t|^2 + \frac{h}{2} \int_0^1 |\vartheta_x(t)|^2 + \frac{k}{2} \sum_{i=0}^1 (\vartheta(i, t) - \vartheta_e)^2 \\ & \leq C + \max\{L, \|\psi_3\|_{L^\infty(Q_\infty)}, \|\psi_4\|_{L^\infty(Q_\infty)}\} \int_0^t \int_0^1 (|\chi_{1,t}| + |\chi_{2,t}| + |g_t|) |\vartheta_t|. \end{aligned}$$

Owing to (2.3), (3.9), and the elementary Young inequality, it is now straightforward to control the right hand side above. Hence, the assertion of Lemma 3.1 follows from some comparison in (1.1) and (2.10).  $\square$

A first consequence of Lemma 3.1 is that the set  $\{(\vartheta(t), u(t), \chi_1(t), \chi_2(t)), t \geq 0\}$  is bounded in  $V \times W \times V \times V$  and thus is relatively compact in  $H \times V \times H \times H$ . Therefore  $\omega(\vartheta, u, \chi_1, \chi_2)$  is a non-empty and compact subset of  $H \times V \times H \times H$ . In addition, since

$$(\vartheta, u, \chi_1, \chi_2) \in C^0([0, +\infty), H \times V \times H \times H),$$

a classical argument from the theory of dynamical systems ensures that  $\omega(\vartheta, u, \chi_1, \chi_2)$  is connected in  $H \times V \times H \times H$  (see e.g. [13, p. 12]).

Consider now  $(\vartheta_\infty, u_\infty, \chi_{1,\infty}, \chi_{2,\infty}) \in \omega(\vartheta, u, \chi_1, \chi_2)$ . There is a sequence  $\{t_n\}$  of positive real numbers such that  $t_n \rightarrow +\infty$  and

$$(\vartheta(t_n), u(t_n), \chi_1(t_n), \chi_2(t_n)) \longrightarrow (\vartheta_\infty, u_\infty, \chi_{1,\infty}, \chi_{2,\infty}) \quad \text{in } H \times V \times H \times H. \quad (3.11)$$

For  $n$  and  $t \geq 0$ , we define

$$\begin{aligned} \vartheta_n(t) & := \vartheta(t_n + t), & u_n(t) & := u(t_n + t), \\ \chi_{1,n}(t) & := \chi_1(t_n + t), & \chi_{2,n}(t) & := \chi_2(t_n + t). \end{aligned} \quad (3.12)$$

As  $(\vartheta, u, \chi_1, \chi_2)$  is a solution to (1.1)-(1.10) in the sense of Theorem 2.2, we can introduce a pair of auxiliary functions  $(\xi_{1,n}, \xi_{2,n})$  such that the quadruple  $(\vartheta_n, u_n, \chi_{1,n}, \chi_{2,n})$  solves the system (1.1)-(1.2),

$$\mu \partial_t \begin{pmatrix} \chi_{1,n} \\ \chi_{2,n} \end{pmatrix} - \lambda \partial_{xx} \begin{pmatrix} \chi_{1,n} \\ \chi_{2,n} \end{pmatrix} + \begin{pmatrix} \ell(\vartheta_n - \vartheta^*) \\ \alpha(\vartheta_n) u_{n,x} \end{pmatrix} + \begin{pmatrix} \xi_{1,n} \\ \xi_{2,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{a.e. in } Q_T, \quad (3.13)$$

$$\begin{pmatrix} \xi_{1,n} \\ \xi_{2,n} \end{pmatrix} \in \partial I_{\mathcal{K}}(\chi_1, \chi_2) \quad \text{a.e. in } Q_T, \quad (3.14)$$

as well as the boundary conditions (1.4)-(1.8), for all  $T \in (0, +\infty)$ . Note that, however, in (1.7)  $g$  has to be replaced by  $g_n := g(\cdot + t_n)$ . We also point out the initial conditions

$$\vartheta_n(x, 0) = \vartheta(x, t_n) \quad \text{for any } x \in [0, 1], \quad (3.15)$$

$$\chi_{1,n}(x, 0) = \chi_1(x, t_n) \quad \text{for any } x \in [0, 1], \quad (3.16)$$

$$\chi_{2,n}(x, 0) = \chi_2(x, t_n) \quad \text{for any } x \in [0, 1]. \quad (3.17)$$

Owing to Lemma 3.1 it is not difficult to prove some estimates for the functions  $\vartheta_n, u_n, \chi_{1,n}$ , and  $\chi_{2,n}$  which are uniform with respect to  $n$ . The proof of the next result is omitted since it is straightforward (indeed, one may refer to [2, Appendix] for the related regularities  $L^2(0, T; W)$  of  $\chi_{i,n}$  and  $L^2(0, T; H)$  of  $\xi_{i,n}$ ,  $i = 1, 2$ ).

**Lemma 3.2.** *Let  $T > 0$ . Then, letting  $\xi_{1,n}$  and  $\xi_{2,n}$  be as in (3.13)-(3.14), there exists a positive constant  $C_6$  such that*

$$\begin{aligned} & \|\vartheta_n\|_{H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)} + \|u_n\|_{C^0([0,T];W) \cap H^1(0,T;V)} \\ & + \sum_{i=1}^2 \|\chi_{i,n}\|_{H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)} + \sum_{i=1}^2 \|\xi_{i,n}\|_{L^2(0,T;H)} \leq C_6. \end{aligned} \quad (3.18)$$

Here  $C_6$  depends on the same data as  $C_5$  and on  $T$ .

Another consequence of Lemma 3.1 is to allow the identification of the limit of  $\vartheta_n, u_n, \chi_{1,n}$ , and  $\chi_{2,n}$  as  $n \rightarrow +\infty$ . More precisely, we have the following

**Lemma 3.3.** *For every  $T > 0$  there holds*

$$\vartheta_{n,t}, u_{n,xt}, \chi_{i,n,t} \longrightarrow 0 \quad \text{in } L^2(0, T; H), \quad i = 1, 2, \quad (3.19)$$

$$\vartheta_n \longrightarrow \vartheta_\infty \quad \text{in } C^0([0, T]; H), \quad (3.20)$$

$$u_{n,x} \longrightarrow u_{\infty,x} \quad \text{in } C^0([0, T]; H), \quad (3.21)$$

$$\chi_{i,n} \longrightarrow \chi_{i,\infty} \quad \text{in } C^0([0, T]; H), \quad i = 1, 2, \quad (3.22)$$

and

$$u_{\infty,x} = -\beta \alpha(\vartheta_\infty) \chi_{2,\infty} \quad \text{in } (0, 1). \quad (3.23)$$

*Proof.* Accounting for (3.1) it is straightforward to check that

$$\int_0^T \|\vartheta_{n,t}\|^2 = \int_{t_n}^{t_n+T} \|\vartheta_t\|^2 \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \quad (3.24)$$

This remark applies to  $u_{n,t}$ ,  $\chi_{1,n,t}$ , and  $\chi_{2,n,t}$  as well. Hence, the convergences in (3.19) follow. As for (3.20), by the Hölder inequality we easily deduce

$$\begin{aligned} \|\vartheta_n(t) - \vartheta_\infty\| &\leq \|\vartheta_n(t) - \vartheta_n(0)\| + \|\vartheta(t_n) - \vartheta_\infty\| \\ &\leq T^{1/2} \|\vartheta_{n,t}\|_{L^2(0,T;H)} + \|\vartheta(t_n) - \vartheta_\infty\| \end{aligned}$$

Owing to (3.11) and (3.24), the right hand side of the latter inequality goes to zero as  $n \rightarrow +\infty$ . Hence, (3.20) is proved. A similar argument ensures that (3.21)-(3.22) hold true. Finally, due to (2.10), it is straightforward to check that

$$u_{n,x} = \beta(g_n - \alpha(\vartheta_n)\chi_{2,n}).$$

Thanks to (2.3), (2.5), and (3.20)-(3.22) we may let  $n \rightarrow +\infty$  in the above identity and obtain (3.23). The proof of Lemma 3.3 is complete.  $\square$

After these preliminaries, we may prove Theorem 2.3 by passing to the limit as  $n \rightarrow +\infty$  in equations (1.1)-(1.2) for the quadruple  $(\vartheta_n, u_n, \chi_{1,n}, \chi_{2,n})$  and in relations (3.13)-(3.14). Thanks to Lemmata 3.2 and 3.3 and well-known compactness results we find a subsequence (not relabeled) of  $\vartheta_n, u_n, \chi_{1,n}$ , and  $\chi_{2,n}$  and a pair  $(\xi_{1,\infty}, \xi_{2,\infty})$  such that the following hold

$$\vartheta_n \longrightarrow \vartheta_\infty \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad (3.25)$$

$$u_n \longrightarrow u_\infty \quad \text{weakly star in } L^\infty(0, T; W) \cap H^1(0, T; V), \quad (3.26)$$

$$\chi_{1,n} \longrightarrow \chi_{1,\infty} \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad (3.27)$$

$$\chi_{2,n} \longrightarrow \chi_{2,\infty} \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad (3.28)$$

$$\xi_{1,n} \longrightarrow \xi_{1,\infty} \quad \text{weakly in } L^2(0, T; H), \quad (3.29)$$

$$\xi_{2,n} \longrightarrow \xi_{2,\infty} \quad \text{weakly in } L^2(0, T; H). \quad (3.30)$$

These convergences, along with (3.19)-(3.22), are sufficient to pass to the limit in equation (1.1) and conditions (1.4)-(1.5), properly rewritten for the quadruple  $(\vartheta_n, u_n, \chi_{1,n}, \chi_{2,n})$ . Namely, note that the following convergences hold

$$\begin{aligned} -\vartheta_n \alpha''(\vartheta_n) \vartheta_{n,t} \chi_{2,n} u_{n,x} &\longrightarrow 0 \quad \text{strongly in } L^2(0, T; H), \\ (\alpha(\vartheta_n) - \vartheta \alpha'(\vartheta_n)) \chi_{2,n,t} u_{n,x} &\longrightarrow 0 \quad \text{strongly in } L^2(0, T; H), \\ -\vartheta \alpha'(\vartheta_n) \chi_{2,n} u_{n,xt} &\longrightarrow 0 \quad \text{strongly in } L^2(0, T; H). \end{aligned}$$

As far as relations (3.13)-(3.14) are concerned, we observe that

$$\begin{aligned} \begin{pmatrix} \ell(\vartheta_n - \vartheta^*) \\ \alpha(\vartheta_n) u_{n,x} \end{pmatrix} &\longrightarrow \begin{pmatrix} \ell(\vartheta_\infty - \vartheta^*) \\ \alpha(\vartheta_\infty) u_{\infty,x} \end{pmatrix} \quad \text{strongly in } (C^0([0, T], H))^2, \\ \mu \partial_t \begin{pmatrix} \chi_{1,n} \\ \chi_{2,n} \end{pmatrix} &\longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{strongly in } (L^2(0, T; H))^2, \\ \lambda \partial_{xx} \begin{pmatrix} \chi_{1,n} \\ \chi_{2,n} \end{pmatrix} &\longrightarrow \lambda \partial_{xx} \begin{pmatrix} \chi_{1,\infty} \\ \chi_{2,\infty} \end{pmatrix} \quad \text{weakly in } (L^2(0, T; H))^2. \end{aligned}$$

Thus, we just need to identify the limit of  $(\xi_{1,n}, \xi_{2,n})$ . Indeed, from (3.27) and (3.29)-(3.30) we easily deduce that

$$\int_0^t \int_0^1 \begin{pmatrix} \xi_{1,n} \\ \xi_{2,n} \end{pmatrix} \cdot \begin{pmatrix} \chi_{1,n} \\ \chi_{2,n} \end{pmatrix} \longrightarrow \int_0^t \int_0^1 \begin{pmatrix} \xi_{1,\infty} \\ \xi_{2,\infty} \end{pmatrix} \cdot \begin{pmatrix} \chi_{1,\infty} \\ \chi_{2,\infty} \end{pmatrix} \quad \forall t \in [0, T].$$

The classical theory of maximal monotone operators (see, e.g., [3, Prop. 2.5, p. 27]) then entails

$$\begin{pmatrix} \xi_{1,\infty} \\ \xi_{2,\infty} \end{pmatrix} \in \partial_{\mathcal{K}}(\chi_{1,\infty}, \chi_{2,\infty})$$

and the proof of Theorem 2.3 is complete.

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