Global existence for a class of generalized systems for irreversible phase changes

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Abstract

A nonlinear evolution system is investigated. It can describe a wide class of phase transition phenomena, including irreversible phase changes. The nonlinearities are of various kind and two maximal monotone graphs appear in the phase relaxation equation. An existence result is established for the related Cauchy-Neumann problem by using regularization, truncation, and monotonicity techniques.

Key words: phase transitions, microscopic movements, subdifferential operators, nonlinear evolution systems, existence results, maximum principle.

AMS (MOS) subject classification: 35K55, 80A22, 35B50, 34C11.

1 Introduction and derivation of the model

In the last two decades, some models describing phase transition phenomena have been proposed and gave rise to a class of interesting mathematical problems for systems of partial differential equations. We are referring, in particular, to phase relaxation and the more recent phase field models (cf., e.g., [10, 13, 18, 19]), which turn out to be generalizations and/or regularizations of the classical Stefan problem and advanced Stefan-like models. Many mathematical results have been already proved for a variety of related initial-boundary value problems. Let us especially quote the two monographs [9] and [20] and, just to mention some specific work, the papers [11, 12, 14] where suitable phase transition dynamics are discussed also in absence of diffusive effects for the order parameter.

Here, we want to address the analytical study of a new phase relaxation system which is derived from a recent model by Frémond (see [5, 6]). This model, that is going to be described below, is based on the consideration that microscopic movements give rise to macroscopic effects. Moreover, the model is able to deal with irreversible phase changes. It is worthy noting that irreversibility is not a mere theoretical feature: even materials of the daily life, such as eggs or glue, do not re-melt after solidification. In performing the derivation of the full model, we have the chance to show the general expression of the laws governing the thermal evolution of the material and to give an overview about some related papers.

Let us consider a two-phase substance contained in a domain $\Omega \subset \mathbb{R}^3$ and let T be a given final time. We want to describe the heat diffusion inside the body. For this purpose, we choose the volume fraction of one of the phases as state quantity and denote it by $\chi = \chi(x,t)$, for $x \in \Omega$ and $t \in]0,T[$. Thus, the order parameter χ satisfies the relation $0 \le \chi \le 1$, and, assuming that no voids appear in the mixture, the volume fraction of the other phase is simply given by $1-\chi$. Of course, the absolute temperature $\theta = \theta(x,t)$ is the other state variable for the thermodynamical system, and it has to be non-negative.

Although the material can be macroscopically regarded as a rigid body, the phase transition is a consequence of microscopic movements as well. Then, we decide to take into account the power of these movements in the energy balance equation

$$\partial_t e + \operatorname{div} \mathbf{q} = B\,\partial_t \chi + \mathbf{H} \cdot \nabla \partial_t \chi, \tag{1.1}$$

where e denotes the internal energy and $\mathbf{q} = -k\nabla\theta$ is the heat flux vector (i.e., the Fourier law is assumed for a constant thermal conductivity k). Indeed, B and $\mathbf{H}$ are respectively a scalar quantity and a vector resulting from the microscopic interior forces and obeying the following relation

$$-\operatorname{div} \mathbf{H} + B = A, \tag{1.2}$$

which is consequence of the virtual power principle and where the right hand side A collects the amount of external forces.

We are now introducing the expressions of the free energy Ψ and of a pseudo-potential of dissipation Φ in order to state the constitutive laws. Towards this aim, we specify I_K, the indicator function of a convex set $K \subseteq \mathbb{R}$, as follows

$$I_K(r) = \begin{cases} 0 & \text{if } r \in K, \\ +\infty & \text{if } r \notin K. \end{cases} \tag{1.3}$$

In the case when $K \neq \varnothing$, it is well known that I_K is a l.s.c. (lower semicontinuous) proper convex function and that its subdifferential ∂I_K, defined by

$$p \in \partial I_K(r) \quad \Longleftrightarrow \quad I_K(q) \geq I_K(r) + p(q-r) \quad \text{for all } q \in \mathbb{R} \tag{1.4}$$

is a maximal monotone operator in $\mathbb{R}$, i.e. a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ (see, e.g., [7]). Therefore, we can choose the free energy Ψ as

$$\Psi(\chi, \nabla\chi, \theta) = -c_s \theta \log\theta - \frac{L}{\theta_c}(\theta - \theta_c)\chi + I_{[0,1]}(\chi) + \frac{\nu}{2}|\nabla\chi|^2 , \tag{1.5}$$

where $L > 0$ stands for the latent heat at the critical transition temperature $\theta_c > 0$, $c_s > 0$ represents the specific heat, the parameter $\nu \geq 0$ is the factor of the interfacial energy term, and $I_{[0,1]}$ plays the role of the constraint for the phase proportion χ. Analogously, we set

$$\Phi(\partial_t\chi, \nabla\partial_t\chi) = \frac{\mu}{2}(\partial_t\chi)^2 + I_{[0,+\infty[}(\partial_t\chi) + \frac{\delta}{2}|\nabla\partial_t\chi|^2 \tag{1.6}$$

for the pseudo-potential of dissipation, where $\mu > 0$ and $\delta \geq 0$ are two coefficients related to the evolution of the interface. Note that the above position accounts for the irreversibility of the phase change because of the presence of the term $I_{[0,+\infty[}(\partial_t\chi)$. On the other hand, more general situations including *reversible* phase transitions can be obtained by replacing in (1.6) the term $I_{[0,+\infty[}(\partial_t\chi)$ by $\phi(\partial_t\chi)$ for a general convex l.s.c. function $\phi : \mathbb{R} \to [0, +\infty]$.

Now, in order to satisfy the basic laws of Thermodynamics, it turns out that admissible and straigthforward choices for the quantities B and $\mathbf{H}$ are provided by the following constitutive laws

$$B = \frac{\partial\Psi}{\partial\chi} + \frac{\partial\Phi}{\partial(\partial_t\chi)} , \quad \mathbf{H} = \frac{\partial\Psi}{\partial(\nabla\chi)} + \frac{\partial\Phi}{\partial(\nabla\partial_t\chi)} , \tag{1.7}$$

while the internal energy e is related to the free energy Ψ and to the entropy $s = -\partial\Psi/\partial\theta$ by the rather classical relationship

$$e = \Psi + s\theta = \Psi - \theta\frac{\partial\Psi}{\partial\theta}. \tag{1.8}$$

Since the two potentials Ψ and Φ contain non-differentiable terms, we use subgradients and point out that

$$\partial I_{[0,1]}(\chi) = \begin{cases}]-\infty, 0] & \text{if } \chi = 0, \\ 0 & \text{if } 0 < \chi < 1, \\ [0, +\infty[& \text{if } \chi = 1, \end{cases} \tag{1.9}$$

$$\partial I_{[0,+\infty[}(\partial_t \chi) = \begin{cases}]-\infty, 0] & \text{if } \partial_t \chi = 0, \\ 0 & \text{if } \partial_t \chi > 0. \end{cases} \tag{1.10}$$

Let us set $\beta = \partial I_{[0,1]}$ and denote by $\alpha = \partial \phi$ a general maximal monotone graph which, in particular, can coincide with $\partial I_{[0,+\infty[}$.

Then, the balance and constitutive laws, coupled with definitions (1.5) and (1.6), yield the following *full* system

$$c_s \partial_t \theta + \frac{L}{\theta_c} \theta \partial_t \chi - k \Delta \theta = \mu (\partial_t \chi)^2 + \xi \partial_t \chi + \delta |\nabla \partial_t \chi|^2, \tag{1.11}$$

$$\mu \partial_t \chi + \xi - \delta \Delta \partial_t \chi - \nu \Delta \chi + \eta = \frac{L}{\theta_c} (\theta - \theta_c) + A, \tag{1.12}$$

where

$$\eta \in \beta(\chi) \quad \text{and} \quad \xi \in \alpha(\partial_t \chi) \tag{1.13}$$

almost everywhere in the space-time domain.

Now, one can readily see that the system (1.11–1.13) is highly nonlinear and looks difficult to handle. As far as we know, no existence result has beeen shown yet for any initial-boundary value problem relying on (1.11–1.13). However, let us mention some work concerning certain simplified versions of the system. Before going on, we just remark that the term A is a datum and then it is not so important for the analytical study, so that we can restrict ourselves to the particular but meaningful case $A = 0$.

First, we notice that, in the pure reversible case (i.e., $\alpha \equiv 0$), a Cauchy-Neumann problem related to (1.11–1.13) is studied in [6] where anyway the term $\delta |\nabla \partial_t \chi|^2$ is missing in (1.11). A local in time existence result is obtained through a regularization procedure combined with a fixed point argument, by exploiting the dissipation effects of the term $-\delta \Delta \partial_t \chi$ of (1.12). Moreover, it seems that global existence can be proved in the one-dimensional setting and this will be the subject of the paper [17]. Second, a further investigation deals with the choice $\alpha = \partial I_{[0,+\infty[}$ in the paper [16], where system (1.11–1.13) is studied in the physically relevant case $\delta = 0$, though the whole right hand side of (1.11) is omitted. Using regularization and monotonicity arguments, the existence of a solution is obtained in the general multidimensional framework.

The aim of this paper is to face a class of PDE problems obtained by keeping a general α in (1.13) (then the irreversibility may be taken into account) and trying to handle the nonlinearities in (1.11). Unfortunately, we have to neglect the term $\xi \partial_t \chi$ (vanishing indeed in the reference model with $\alpha = \partial I_{[0,+\infty[}$). Moreover, as in [5, 16] we consider the case $\delta = \nu = 0$ like in the phase relaxation models examined in [3, 13, 19]. Indeed, in the subsequent analysis we need to use sharply the monotonicity properties of α and the particular structure of the graph β, that we are forced to assume equal to $\partial I_{[0,1]}$ (this is the most significant choice from the point of view of thermodynamics, anyway). Finally, we complement the so-modified system (1.11–1.13) with no-flux boundary conditions and prescribed initial conditions. Getting rid of most of the physical parameters which do not affect our analysis, we actually address the following Cauchy-Neumann problem

$$\partial_t\theta + \theta\,\partial_t\chi - \Delta\theta = (\partial_t\chi)^2 \quad \text{in } Q, \tag{1.14}$$

$$\partial_t\chi + \alpha(\partial_t\chi) + \beta(\chi) \ni \theta - \theta_c \quad \text{in } Q, \tag{1.15}$$

$$\theta(\cdot,0) = \theta_0, \qquad \chi(\cdot,0) = \chi_0 \quad \text{in } \Omega, \tag{1.16}$$

$$\partial_{\mathbf{n}}\theta = 0 \quad \text{on } \Sigma, \tag{1.17}$$

where $Q = \Omega\times]0,T[$, $\Sigma = \Gamma\times]0,T[$, $\Gamma := \partial\Omega$, and $\partial_{\mathbf{n}}$ denotes the outward normal derivative to Γ.

One novel feature of the system (1.14–1.17) is the nonlinear part involving $\partial_t\chi$ in (1.14), that causes difficulties in the existence proof and, in our opinion, prevents from getting uniqueness. In fact, we stress that a more usual expression of the energy balance equation in standard phase transition problems (see, e.g., [10]) is the following *linear* relation

$$\partial_t\theta + \partial_t\chi - \Delta\theta = 0\,. \tag{1.18}$$

Observe that in [5] an initial-boundary value problem was investigated for the system related to (1.18) and to the inclusion

$$\mu\partial_t\chi + \alpha(\partial_t\chi) - \nu\Delta\chi + \beta(\chi) \ni \theta - \theta_c, \tag{1.19}$$

which is a reduction of (1.12–1.13), covering also the limiting case $\mu = 0$. We also quote the papers [3] and [4] which couple equation (1.18) with an inclusion like (1.15). The authors obtained some existence and uniqueness results, via regularization and monotonicity techniques. It must be noted that general maximal monotone graphs α and β can be handled in [5, 3], but the main application remains $\alpha = \partial I_{[0,+\infty[}$ and $\beta = \partial I_{[0,1]}$.

The plan of the paper is as follows. In the next section we introduce some notation and state our main (existence) result, whose proof is started by considering the Yosida regularization β_ε ($\varepsilon > 0$) of the maximal monotone graph β and by introducing a suitable procedure of truncation. Thus, we obtain a family of approximating problem $(P_{*,\varepsilon})$ and, in Section 3, we show the existence of a local in time solution to $(P_{*,\varepsilon})$. Such a solution is global, thanks to the *a priori* estimates (independent of the regularization parameter ε) derived in Section 4. These uniform estimates are derived through the sharp use of the maximum principle and a comparison argument; moreover, they allow to establish that the solution of truncated problem also solves the ε-regularized problem. In Section 5, we carry out the passage to the limit as ε tends to 0. With the help of well-known compactness tools and a non-standard monotonicity technique, we can even prove the strong convergence for the sequence of the time derivatives of the χ's, which allows us to completely characterize the limit elements η and ξ as required by (1.13).

2 Main result

First of all, let us recall that Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\Gamma = \partial\Omega$ and put $Q_t = \Omega\times]0,t[$, for $t \in]0,T]$. We also set $H := L^2(\Omega)$, $V := H^1(\Omega)$, in order that, identifying H with H' as usual, we get $V \subset H \subset V'$, whence (V, H, V') forms a *Hilbert triplet.* We also denote by $(\cdot,\cdot)$ the scalar product of H, by $|\cdot|$ the norm of H and by $\|\cdot\|$ that of V. Looking back at equations (1.14–1.17), we assume the following hypotheses

$$0 < \theta_c < \theta^* \quad \text{assigned constants,} \tag{2.1}$$

$$\theta_0, \chi_0 \in L^\infty(\Omega), \quad \text{with } 0 \leq \theta_0 \leq \theta^*, \quad 0 \leq \chi_0 \leq 1 \text{ a.e. in } \Omega, \tag{2.2}$$

$$\alpha \subset \mathbb{R}\times\mathbb{R} \quad \text{maximal monotone graph such that } 0 \in \alpha(0), \tag{2.3}$$

$$\beta \subset \mathbb{R}\times\mathbb{R} \quad \text{maximal monotone graph given by } \beta = \partial I_{[0,1]}. \tag{2.4}$$

It is well-known that under these conditions there exists a convex, l.s.c. and proper function $\phi : \mathbb{R} \to [0,+\infty]$ such that $\alpha = \partial\phi$, with $0 = \min\phi = \phi(0)$. Also, for the sake of simplicity, we set $\psi := I_{[0,1]}$ in order that $\beta = \partial\psi$.

We can now state the main result of this paper.

Theorem 2.1. *There exist real valued functions (θ,χ,ξ,η) on Q with regularity*

$$\theta \in H^1(0,T;V') \cap L^2(0,T;V) \cap L^\infty(Q), \tag{2.5}$$

$$\chi \in L^\infty(Q), \qquad \chi_t \in L^\infty(Q), \tag{2.6}$$

$$\xi, \eta \in L^\infty(Q), \tag{2.7}$$

and such that the following equations hold at least almost everywhere

$$\partial_t\theta + \theta\,\partial_t\chi - \Delta\theta = (\partial_t\chi)^2 \quad \text{in } Q, \tag{2.8}$$

$$\partial_t\chi + \xi + \eta = \theta - \theta_c \quad \text{in } Q, \tag{2.9}$$

$$\xi \in \alpha(\partial_t\chi) \quad \text{in } Q, \tag{2.10}$$

$$\eta \in \beta(\chi) \quad \text{in } Q, \tag{2.11}$$

$$\theta(\cdot,0) = \theta_0, \qquad \chi(\cdot,0) = \chi_0 \quad \text{in } \Omega, \tag{2.12}$$

$$\partial_{\mathbf{n}}\theta = 0 \quad \text{on } \Sigma. \tag{2.13}$$

Moreover, we have that

$$0 \leq \theta \leq \theta^* \quad \text{for a.e. } (x,t) \in Q, \tag{2.14}$$

$$-\theta_c \leq \eta \leq \theta^* - \theta_c \quad \text{for a.e. } (x,t) \in Q, \tag{2.15}$$

$$|\partial_t\chi| \leq \theta^* \quad \text{for a.e. } (x,t) \in Q. \ \blacksquare \tag{2.16}$$

Remark 2.2. First of all, we emphasize that (2.5) yields, by interpolation, $\theta \in C^0([0,T];H)$, so that the initial conditions in (2.12) make sense. Next, any pair

(θ, χ) solving (2.5–2.16) is such that the solution component θ is even more regular than (2.5). For instance, for all $\delta > 0$ there holds

$$\theta \in W^{1,p}(\delta, T; L^p(\Omega)) \cap L^p(\delta, T; W^{2,p}(\Omega)) \quad \text{for all } p \in [1, \infty[, \tag{2.17}$$

since $(\partial_t \chi)^2 - \theta\, \partial_t \chi \in L^\infty(Q)$ in (2.8). Indeed, observe that the function $v(t) := t\theta(t)$ solves the initial-boundary value problem

$$\partial_t v - \Delta v = \theta + t\theta\, \partial_t \chi + t\, (\partial_t \chi)^2 \quad \text{a.e. in } Q, \tag{2.18}$$

$$\partial_{\mathbf{n}} v = 0 \quad \text{a.e. on } \Sigma, \tag{2.19}$$

$$v(0) = 0 \quad \text{a.e. in } \Omega \tag{2.20}$$

and then we can apply, for instance, [8, Thm. X.12, p. 220]. Moreover, let us point out that if the initial datum θ_0 is smoother, say $\theta_0 \in V$, then we can recover additional global regularity; in particular, the validity of (2.17) extends to $\delta = 0$ in the case $p = 2$.

The proof of Theorem 2.1 will be carried out all over the remainder of the paper. As a first step, we introduce an approximate statement through a regularization technique joint with a cutoff of the unknown θ in (2.8–2.9). We first give a related notation: for any number $\gamma \in \mathbb{R}$, we shall indicate in the sequel as $\tau(\gamma)$ the *Stampacchia truncation* of γ at level θ^*, i.e., $\tau(\gamma) := \max\{-\theta^*, \min\{\gamma, \theta^*\}\}$. Notice that the function $\gamma \mapsto \tau(\gamma)$ is contractive:

$$|\tau(\gamma_2) - \tau(\gamma_1)| \le |\gamma_2 - \gamma_1| \quad \text{for all } \gamma_1, \gamma_2 \in \mathbb{R}. \tag{2.21}$$

Hence, we point out that we are addressing a different problem, where θ is replaced by $\tau(\theta)$ in (2.8–2.9). However, note that if we are able to find solutions (θ, χ) of the latter problem with the property (2.14), then these pairs (θ, χ) actually solve (2.8–2.13) too. Moreover, if we set $J(r) := (\mathrm{Id} + \alpha)^{-1}(r)$, for $r \in \mathbb{R}$ (with Id staying for the identity function), it is well-known that J, as a monotone graph in $\mathbb{R} \times \mathbb{R}$, is a single-valued contraction; hence, relations (2.9–2.10) are equivalent to the following

$$\partial_t \chi = J(\theta - \theta_c - \eta) \quad \text{a.e. in } Q, \tag{2.22}$$

which simplifies the form of the original problem and suggests that the more delicate nonlinearity, requiring a careful approximation, is that of β.

Consequently, for any $\varepsilon \in]0, 1[$, we can introduce the Yosida-regularization β_ε of the graph β (we refer to the text [7] for its main properties), whose precise mathematical expression is given by

$$\beta_\varepsilon(r) = \begin{cases} \varepsilon^{-1} r & \text{if } r < 0, \\ 0 & \text{if } 0 \le r \le 1, \\ \varepsilon^{-1}(r - 1) & \text{if } r > 1. \end{cases} \tag{2.23}$$

In addition, indicate by ψ_ε the primitive of β_ε verifying $\psi_\varepsilon(0) = 0$. Let us point out that ε is intended to go to 0 in the limit.

We are now ready to state our regularization of (2.8–2.13). We formulate it as a local in time problem.

Problem $(\mathbf{P}_\varepsilon^\tau)$. For any $\varepsilon \in]0,1[$, we look for a small time $T_0 \leq T$, possibly depending on ε, such that the system

$$\partial_t\theta + \tau(\theta)\,\partial_t\chi - \Delta\theta = (\partial_t\chi)^2 \tag{2.24}$$

$$\partial_t\chi = J(\tau(\theta) - \theta_c - \beta_\varepsilon(\chi)) \tag{2.25}$$

(coupled with the Cauchy and boundary conditions (2.12–2.13)) has a (suitably regular) solution (θ,χ), defined at least over the time interval $[0,T_0]$. ■

3 Resolution of the approximate problem

In this section we handle problem $(\mathrm{P}_\varepsilon^\tau)$ through the classical Banach fixed point theorem in the space $\times(T_0) := \{u \in C^0([0,T_0];H) : u(0) = \theta_0\}$ equipped with the usual norm, where $T_0 \leq T$ is a positive time to be chosen later. More precisely, for a prescribed value $\overline{\theta} \in \times(T_0)$, we denote by $\theta = \mathcal{M}_\varepsilon(\overline{\theta})$ the component of the unique pair (θ,χ) solving the system

$$\partial_t\theta + \tau(\overline{\theta})\,\partial_t\chi - \Delta\theta = (\partial_t\chi)^2 \quad \text{in } Q, \tag{3.1}$$

$$\partial_t\chi = J(\tau(\overline{\theta}) - \theta_c - \beta_\varepsilon(\chi)) \quad \text{in } Q, \tag{3.2}$$

together with the usual Cauchy and boundary conditions (2.12–2.13).

Let us point out a particular feature of equation (3.2): when $X = L^p(\Omega)$ for some $p \in [1,\infty]$ (and this notation will be kept also in the sequel), if $\overline{\theta}$ and ε are fixed, the operator

$$v \mapsto J(\tau(\overline{\theta}) - \theta_c - \beta_\varepsilon(v)) \tag{3.3}$$

is Lipschitz continuous from X to X with Lipschitz constant $1/\varepsilon$.

Consequently, for any assigned $\overline{\theta} \in \times(T_0)$, owing to the boundedness of χ_0 and $\tau(\overline{\theta})$, we can use a Cauchy-Lipschitz-Picard type argument (see, e.g., [8, Thm. VII.3, p. 104]) for ODE's in the space $L^\infty(\Omega)$ to deduce that (3.2) has a unique solution χ such that $\chi \in C^1([0,T_0];H)$ and $\chi, \partial_t\chi \in L^\infty(Q_{T_0})$ (note that the regularity $\chi \in C^1([0,T_0];L^\infty(\Omega))$ is not guaranteed, since $\tau(\overline{\theta})$ is just continuous in time with respect to the $L^2(\Omega)$-norm).

Substituting now the value obtained for χ into (3.1), and using standard results on parabolic equations, we easily derive that the solution θ to (3.1), (2.12–2.13) exists, is unique, and has the desired regularity $C^0([0,T_0];H)$ (of course, much more is true); thus, the operator $\mathcal{M}_\varepsilon$ is well-defined and maps $\times(T_0)$ into itself for every choice of T_0.

In order to prove that $\mathcal{M}_\varepsilon$ is a strict contraction mapping, we now derive some a priori estimates for the solution of (3.1–3.2), (2.12–2.13). As before, X will stay for $L^p(\Omega)$, no restriction being assumed on the exponent $p \in [1,\infty]$. So, choose a couple of proposed values $\overline{\theta}_1, \overline{\theta}_2 \in \times(T_0)$ and name by (θ_1, χ_1) and (θ_2, χ_2) the corresponding solutions to (3.1–3.2), (2.12–2.13). Set also $\overline{\theta} := \overline{\theta}_2 - \overline{\theta}_1$, $\theta := \theta_2 - \theta_1$, and $\chi := \chi_2 - \chi_1$.

The following estimates work for any arbitrarily fixed $T_0 \in [0, T]$, whereas the correct choice of T_0 for the fixed point argument will be performed at the end. Furthermore, the constant C may vary from line to line, and will be assumed to depend only on θ^*, θ_c and in particular not on t, T_0.

First estimate. Write (3.2) first for $\overline{\theta}_2$ and then for $\overline{\theta}_1$; take the difference and integrate it over $]0, t[$, where $t \le T_0$. Considering the X-norm of both hands sides of the resulting relation, owing also to (3.3), we deduce

$$\|\chi(t)\|_X \le \int_0^t \left(\|\tau(\overline{\theta}_2)(s) - \tau(\overline{\theta}_1)(s)\|_X + \varepsilon^{-1}\|\chi(s)\|_X\right) ds, \tag{3.4}$$

whence, on account of (2.21) and of Gronwall's inequality, we infer

$$\|\chi(t)\|_X \le \|\overline{\theta}\|_{L^1(0,T_0;X)} e^{t/\varepsilon} \quad \text{for every } t \in [0, T_0]. \tag{3.5}$$

Second estimate. Take again the difference of equations (3.2), and consider the X-norm of the result. In view of the Lipschitz continuity of J, β_ε, and τ, from (3.5) we easily get

$$\begin{aligned} \|\partial_t \chi(t)\|_X &\le \|\overline{\theta}(t)\|_X + \varepsilon^{-1}\|\chi(t)\|_X \\ &\le \|\overline{\theta}(t)\|_X + \varepsilon^{-1}\|\overline{\theta}\|_{L^1(0,T_0;X)} e^{t/\varepsilon} \quad \text{for all } t \in [0, T_0]. \end{aligned} \tag{3.6}$$

Third estimate. Writing equation (3.1) first for $\overline{\theta}_2, \theta_2, \chi_2$, then for $\overline{\theta}_1, \theta_1, \chi_1$, and taking the difference, we obtain

$$\partial_t \theta - \Delta\theta = \partial_t\chi_2(\partial_t\chi_2 - \tau(\overline{\theta}_2)) - \partial_t\chi_1(\partial_t\chi_1 - \tau(\overline{\theta}_1)). \tag{3.7}$$

Hence, multiplying by θ and integrating over Q_t, with $t \le T_0$, we infer

$$\frac{1}{2}|\theta(t)|^2 + \|\nabla\theta\|^2_{L^2(Q_t)} = I_1(t) + I_2(t), \tag{3.8}$$

where

$$\begin{aligned} I_1(t) &:= \int_0^t \int_\Omega \partial_t\chi(\partial_t\chi_2 - \tau(\overline{\theta}_2))\theta \, dx\, ds \\ &\le \|\partial_t\chi_2 - \tau(\overline{\theta}_2)\|_{L^\infty(Q)} \int_0^t |\theta(s)||\partial_t\chi(s)|\, ds \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} I_2(t) &:= \int_0^t \int_\Omega \partial_t\chi_1(\partial_t\chi - (\tau(\overline{\theta}_2) - \tau(\overline{\theta}_1)))\theta \, dx\, ds \\ &\le \|\partial_t\chi_1\|_{L^\infty(Q)} \int_0^t |\theta(s)|(|\partial_t\chi(s)| + |\overline{\theta}(s)|)\, ds. \end{aligned} \tag{3.10}$$

Now we observe that the L^∞-norms in the above computations are bounded. Indeed, to verify this it suffices to repeat the procedure which led to the first two estimates, with the only difference that now it is necessary to reason on a single solution χ_1 or χ_2. In this way, choosing $X = L^\infty(\Omega)$, we precisely derive (compare with (3.6)) that

$$\|\partial_t\chi_1\|_{L^\infty(Q_{T_0})}, \|\partial_t\chi_2\|_{L^\infty(Q_{T_0})} \leq (\theta^* + \theta_c)\left(1 + T_0\varepsilon^{-1}e^{T_0/\varepsilon}\right), \tag{3.11}$$

since the truncation procedure immediately yields

$$\|\tau(\theta_1)\|_{L^\infty(Q_{T_0})} \leq \theta^*, \qquad \|\tau(\theta_2)\|_{L^\infty(Q_{T_0})} \leq \theta^*. \tag{3.12}$$

Collecting now (3.8–3.12), it is easy to infer

$$|\theta(t)|^2 \leq C(1 + T_0\varepsilon^{-1}e^{T_0/\varepsilon}) \int_0^t |\theta(s)|(|\partial_t\chi(s)| + |\overline{\theta}(s)|)\, ds \quad \text{for all } t \in [0, T_0], \tag{3.13}$$

and, exploiting a well-known variant of Gronwall's inequality,

$$\begin{aligned} |\theta(t)| &\leq C(1 + T_0\varepsilon^{-1}e^{T_0/\varepsilon}) \int_0^t (|\partial_t\chi(s)| + |\overline{\theta}(s)|)\, ds \\ &\leq C(1 + T_0\varepsilon^{-1}e^{T_0/\varepsilon})\, t\, (\|\partial_t\chi\|_{C^0([0,t];H)} + \|\overline{\theta}\|_{C^0([0,t];H)}) \quad \text{for all } t \in [0, T_0]. \end{aligned} \tag{3.14}$$

Resolution of Problem ($\mathbf{P}_\varepsilon^\tau$). Choosing $X = H$ in (3.6) and using it in (3.14), we deduce that

$$\|\theta\|_{C^0([0,T_0];H)} \leq CT_0(1 + T_0\varepsilon^{-1}e^{T_0/\varepsilon})^2 \|\overline{\theta}\|_{C^0([0,T_0];H)}. \tag{3.15}$$

Thus, it is possible to fix T_0 sufficiently small (depending on $\theta_c, \theta^*, \varepsilon$), in order that $\mathcal{M}_\varepsilon$ is a strict contraction of $\times(T_0)$. This proves that Problem ($\mathrm{P}_\varepsilon^\tau$) admits a unique local solution (θ, χ). Also, from the above argument, it follows that

$$\theta \in C^0([0, T_0]; H) \cap L^2(0, T_0; V), \tag{3.16}$$

$$\chi \in H^1(0, T_0; H), \qquad \text{with } \chi, \partial_t\chi \in L^\infty(Q_{T_0}) \tag{3.17}$$

(of course, much more could be said on regularity).

4 Boundedness of solutions

In this section, we derive some boundedness estimates, uniform with respect to ε, for the solutions to Problem ($\mathrm{P}_\varepsilon^\tau$); they are based on a careful use of the maximum principle joint with a monotonicity argument. We point out that, if we were able to solve directly the original problem (with the graph β instead of β_ε), the proofs of these results would become much simpler. However, it does not seem possible to get the local solution without performing some approximation as we did in the previous section.

Moreover, we remark that, since the a priori estimates we are going to construct are uniform in time, they will guarantee that the local solution to $(\mathrm{P}_\varepsilon^\tau)$ will in fact be a global one. For this reason, let us carry out the next computations with T in place of T_0. For the sake of brevity, henceforth the solutions to $(\mathrm{P}_\varepsilon^\tau)$ will be simply denoted by (θ, χ), with no emphasis on the dependence on ε; finally, for any function f, we shall indicate by f^+ (f^-) the positive (negative, respectively) part of f.

Positivity of $\boldsymbol{\theta}$. Multiply equation (2.24) by $-\theta^-$ and integrate the result over $]0,t[$, with $t \le T$. Standard integrations by parts and the positivity of θ_0 lead to

$$\frac{1}{2}|\theta^-(t)|^2 + \int_0^t |\nabla\theta^-(s)|^2\,ds = -\int_0^t\int_\Omega \partial_t\chi^2\,\theta^-\,dx\,ds + \int_0^t\int_\Omega \theta^-\,\tau(\theta)\,\partial_t\chi\,dx\,ds. \tag{4.1}$$

Now, the first term on the right hand side is clearly negative. Regarding the second one, we notice that $|\theta^-\tau(\theta)| \le |\theta^-|^2$, whence

$$\int_0^t\int_\Omega \theta^-\,\tau(\theta)\,\partial_t\chi\,dx\,ds \le \int_0^t \|\partial_t\chi(s)\|_{L^\infty(\Omega)}|\theta^-(s)|^2\,ds, \tag{4.2}$$

so that the simple boundedness of $\partial_t\chi$ (note that no uniformity is required with respect to ε) and the Gronwall lemma allow us to conclude that $\theta^- = 0$ a.e. in Q_t.

We now establish a global boundedness property for the solutions to $(\mathrm{P}_\varepsilon^\tau)$, which will allow us to remove the truncation operator from equations (2.24–2.25). With this aim, we first state a monotonicity property for the solutions to an ODE which is strictly related to (2.25).

Lemma 4.1. *Let $f_1, f_2 \in L^1(0,T)$ such that $f_1(t) \ge f_2(t)$ for a.e. $t \in]0,T[$. For $u_0 \in \mathbb{R}$ and $i = 1, 2$, let $u_i \in W^{1,1}(0,T)$ be the solution to the Cauchy problem*

$$u_i' = J(f_i - \theta_c - \beta_\varepsilon(u_i)) \qquad a.e.\ in\ ]0,T[, \tag{4.3}$$

$$u_i(0) = u_0. \tag{4.4}$$

Then, we have that $u_1(t) \ge u_2(t)$ for every $t \in [0,T]$.

Proof. Subtract (4.3) with $i = 1$ from the same relation written for $i = 2$. Multiplying the result by $(u_2 - u_1)^+$, and integrating over $]0,t[$, with $t \le T$, we easily derive

$$\begin{aligned}\frac{1}{2}|u(t)^+|^2 &= \int_0^t \big(J(f_2 - \theta_c - \beta_\varepsilon(u_2))(s) - J(f_2 - \theta_c - \beta_\varepsilon(u_1))(s)\big)u^+(s)\,ds \\ &\quad + \int_0^t \big(J(f_2 - \theta_c - \beta_\varepsilon(u_1))(s) - J(f_1 - \theta_c - \beta_\varepsilon(u_1))(s)\big)u^+(s)\,ds \\ &=: I_3(t) + I_4(t),\end{aligned} \tag{4.5}$$

where we have set $u := u_2 - u_1$. We now show that both $I_3(t)$ and $I_4(t)$ are non-positive. Concerning the first integral, we see that the function $r \mapsto J(f_2(s) - \theta_c - \beta_\varepsilon(r))$ is non-increasing for a.e. $s \in [0,T]$, whence it turns out that $I_3(t) \le 0$ (indeed, it suffices to argue on the set of the s such that $u^+(s) > 0$). As regards $I_4(t)$, we simply observe that $f_1(s) \ge f_2(s)$ and $u^+(s) \ge 0$ for a.e. $s \in [0,t]$, thus one concludes thanks to the monotonicity of J. ■

We now discuss the behaviour of a subsolution and a supersolution to equation (2.25).

Lemma 4.2. *Define $\overline{\chi}$ and $\underline{\chi}$ as the solutions of the following Cauchy problems:*

$$\begin{cases} \partial_t \overline{\chi} = J(\theta^* - \theta_c - \beta_\varepsilon(\overline{\chi})) & \text{a.e. in } Q \\ \overline{\chi}(0) = \chi_0 \end{cases} \tag{4.6}$$

$$\begin{cases} \partial_t \underline{\chi} = J(-\theta_c - \beta_\varepsilon(\underline{\chi})) & \text{a.e. in } Q \\ \underline{\chi}(0) = \chi_0 \end{cases} \tag{4.7}$$

Then, for a.e. $x \in \Omega$, we have:

$$\overline{\chi} - 1 < \varepsilon(\theta^* - \theta_c) \quad \text{and} \quad \underline{\chi} > -\varepsilon\theta_c \quad \text{for every } t \in [0,T]. \tag{4.8}$$

Proof. It is rather straightforward, once one observes that, for a.e. $x \in \Omega$, the above systems (4.6–4.7) are Cauchy problems for autonomous ODE's over the time interval $[0,T]$. Consider, for instance, the problem (4.6): by freezing $x \in \Omega$ and letting $t = 0$ into the differential equation, in view of (2.2) we infer that $\partial_t \overline{\chi}(x,0) = J(\theta^* - \theta_c)$. Owing to (2.1), to the monotonicity of J, and to $J(0) = 0$, two things may happen: either $J(\theta^* - \theta_c) = 0$, so that $\overline{\chi}(x,t) \equiv \chi_0(x)$ for all $t \in [0,T]$ by the uniqueness property for the ODE (consequence of the elementary theory), or $J(\theta^* - \theta_c) > 0$. Anyway, also in this second case, for the same reason as before, for no $t \in [0,T]$ it could be $J(\theta^* - \theta_c - \beta_\varepsilon(\chi(x,t))) = 0$ and, *in particular*, we have that $\theta^* - \theta_c - \beta_\varepsilon(\overline{\chi}(x,t)) > 0$ for all $t \in [0,T]$. The first part of the thesis follows now from (2.23), while the proof of the second inequality is very similar. ■

Corollary 4.3. *The solution (θ, χ) to Problem $(\mathrm{P}_\varepsilon^\tau)$ fulfils the following bounds*

$$-\varepsilon\theta_c \le \chi \le 1 + \varepsilon(\theta^* - \theta_c) \quad \text{for a.e. } (x,t) \in Q, \tag{4.9}$$

$$-\theta_c \le \beta_\varepsilon(\chi) \le \theta^* - \theta_c \quad \text{for a.e. } (x,t) \in Q. \tag{4.10}$$

Proof. To prove (4.9), fix $x \in \Omega$ and apply Lemma 4.1, first with the choices $f_1 = \tau(\theta)$ and $f_2 = 0$ (recall that we have already proved that $\theta \ge 0$, so that $\tau(\theta) \ge 0$ as well), then with the choices $f_1 = \theta^*$ and $f_2 = \tau(\theta)$. Therefore, (4.9) becomes an easy consequence of Lemma 4.2. On the other hand, (4.10) follows immediately from (4.9) and (2.23). ■

Boundedness of θ. Test equation (2.24) by $(\theta - \theta^*)^+$. Integrating as usual over

$]0,t[$, thanks to (2.2) we derive

$$\frac{1}{2}|(\theta-\theta^*)^+(t)|^2 + \|\nabla(\theta-\theta^*)^+\|^2_{L^2(0,t;H)} = \int_0^t\int_\Omega \partial_t\chi(\partial_t\chi-\tau(\theta))(\theta-\theta^*)^+\,dx\,ds. \tag{4.11}$$

Observe that the multiple integral on the right hand side can be restricted to the set $Q_t^*; = \{(x,t)\in Q_t : \theta(x,t)>\theta^*\}$ (in which $\tau(\theta)=\theta^*$), and let us substitute therein the value of $\partial_t\chi$ by drawing it from equation (2.25). Thus, we obtain

$$\begin{aligned}\int_0^t\int_\Omega &\partial_t\chi(\partial_t\chi-\tau(\theta))(\theta-\theta^*)^+\,dx\,ds\\ &= \iint_{Q_t^*} J(\theta^*-\theta_c-\beta_\varepsilon(\chi))(J(\theta^*-\theta_c-\beta_\varepsilon(\chi))-\theta^*)(\theta-\theta^*)^+\,dx\,ds.\end{aligned} \tag{4.12}$$

On account of the right inequality in (4.10), we now see that the first factor of the integrand on the right hand side is surely non-negative. Concerning the second factor, by monotonicity and contractivity of J and by $J(0)=0$ we deduce that

$$J(\theta^*-\theta_c-\beta_\varepsilon(\chi))-\theta^* \le -\theta_c-\beta_\varepsilon(\chi)\le 0, \tag{4.13}$$

where the last inequality follows from the first of (4.10). This entails that the left hand side of (4.11) is (less or) equal to 0, as desired. At this point, we can observe that (2.14) is completely proved for the solution to $(\mathrm{P}_\varepsilon^\tau)$.

Corollary 4.4. *If the pair (θ,χ) solves problem (P_ε^τ), then there holds*

$$|\partial_t\chi|\le\theta^* \quad \text{for a.e. } (x,t)\in Q. \tag{4.14}$$

Proof. It follows directly from equation (2.25) by simple computations exploiting the properties of J, estimate (4.10), and the bound $0\le\theta\le\theta^*$ a.e. in Q. ∎

As a final consequence of the above procedure, we can notice that every solution of system (2.24–2.25) is in fact a solution of the system obtained by suppressing the truncation operator τ from it.

Remark 4.5. Other boundary conditions could be taken for the absolute temperature θ in place of (1.17) without affecting the previous (and further, of course) analysis. Indeed, letting (cf. (2.5) and (2.2))

$$g\in H^1(0,T;V')\cap L^2(0,T;V)\cap L^\infty(Q), \quad \text{with } 0\le g\le\theta^* \quad \text{a.e. in } Q$$

and denoting by g_Γ the trace of g on the lateral boundary, one can substitute the homogeneous Neumann boundary condition (2.13) either with the Dirichlet boundary condition

$$\theta = g_\Gamma \quad \text{on } \Sigma \tag{4.21}$$

or with the third-type boundary conditions

$$\partial_{\mathbf{n}}\theta + \gamma(\theta - g_\Gamma) = 0 \quad \text{on } \Sigma, \tag{4.22}$$

where γ is a suitable proportionality coefficient. While the meaning of (4.21) is rather clear, the latter condition (4.22) states that the heat flux on the boundary is proportional to the difference between the internal and external values of the absolute temperature on the boundary. Now, it seems to us that both variants are admissible and Theorem 2.1 still holds in both cases. Details are not so straightforward to check, but we leave them to the reader since the treatment of boundary conditions is not the main aim of this paper. We just note that, provided a standard modification of the variational formulations, in both situations we can repeat the crucial estimates yielding the positivity and the boundedness of θ, by suitably adapting those estimates to the new frameworks.

5 Passage to the limit

We begin by collecting all the convergences in ε that we can derive from the procedure of the previous section: all of them hold at least for a subsequence $\varepsilon_n \to 0$ as $n \to \infty$. However, let us avoid the use of a double index and argue directly on ε. In the sequel, we denote by $(\theta_\varepsilon, \chi_\varepsilon)$ the solution to Problem $(\mathrm{P}_\varepsilon^\tau)$ (and we introduce also the auxiliary functions ξ_ε and η_ε) for $0 < \varepsilon < 1$, reserving the notations θ, χ, ξ, η to the limits as $\varepsilon \to 0$. Hence, dropping also the truncation operator τ from (2.24–2.25), we are now concerned with a solution $(\theta_\varepsilon, \chi_\varepsilon)$ of the system

$$\partial_t\theta_\varepsilon + \theta_\varepsilon\,\partial_t\chi_\varepsilon - \Delta\theta_\varepsilon = (\partial_t\chi_\varepsilon)^2 \quad \text{a.e. in } Q, \tag{5.1}$$

$$\partial_t\chi_\varepsilon = J(\theta_\varepsilon - \theta_c - \beta_\varepsilon(\chi_\varepsilon)) \quad \text{a.e. in } Q \tag{5.2}$$

complemented by conditions (2.12–2.13). First of all, owing to the uniform boundedness of θ_ε and to Corollaries 4.3, 4.4, we easily obtain

$$\chi_\varepsilon \to \chi \quad \text{weakly star in } L^\infty(Q), \tag{5.3}$$

$$\partial_t\chi_\varepsilon \to \partial_t\chi \quad \text{weakly star in } L^\infty(Q), \tag{5.4}$$

$$\eta_\varepsilon := \beta_\varepsilon(\chi_\varepsilon) \to \eta \quad \text{weakly star in } L^\infty(Q), \tag{5.5}$$

$$\theta_\varepsilon \to \theta \quad \text{weakly star in } L^\infty(Q). \tag{5.6}$$

In particular, note that $\theta, \eta, \partial_t\chi$ fulfil (2.14–2.16). Observing now that equation (5.2) can be rewritten in the equivalent form

$$\xi_\varepsilon := \theta_\varepsilon - \theta_c - \partial_t\chi_\varepsilon - \beta_\varepsilon(\chi_\varepsilon) \in \alpha(\partial_t\chi_\varepsilon) \quad \text{a.e. in } Q, \tag{5.7}$$

by a comparison of terms we easily deduce that

$$\xi_\varepsilon \to \xi := \theta - \theta_c - \partial_t\chi - \eta \quad \text{weakly star in } L^\infty(Q). \tag{5.8}$$

We remark that, for the moment, we are not able to give an interpretation of the elements ξ, η in relations (5.5–5.8) in terms of the operators α, β; we can only say that ξ, η are some $L^\infty(Q)$–functions. Of course, our aim is to prove that the quadruple $(\theta, \chi, \xi, \eta)$ solves problem (2.8–2.13).

Now, rewriting equation (5.1) in the form

$$\partial_t \theta_\varepsilon - \Delta \theta_\varepsilon = (\partial_t \chi_\varepsilon)^2 - \theta_\varepsilon \, \partial_t \chi_\varepsilon \quad \text{a.e. in } Q, \tag{5.9}$$

we observe that the right hand sides are uniformly bounded in $L^\infty(Q)$ with respect to ε. Moreover, for any ε, θ_ε satisfies also the Cauchy condition $\theta_\varepsilon(0) = \theta_0$, where it is $\theta_0 \in L^\infty(\Omega)$. Then, it is not difficult to recover the boundedness of θ_ε and the further convergence

$$\theta_\varepsilon \to \theta \quad \text{weakly in } H^1(0,T;V') \cap L^2(0,T;V). \tag{5.10}$$

Hence, applying the Aubin compactness theorem [15, Thm. 5.1, p. 58], we also infer

$$\theta_\varepsilon \to \theta \quad \text{strongly in } L^2(0,T;H). \tag{5.11}$$

Moreover, on account of (5.3–5.4), we easily deduce that, for any $p \in [1, \infty[$, it is

$$\chi_\varepsilon \to \chi \quad \text{weakly in } W^{1,p}(0,T;L^p(\Omega)) \tag{5.12}$$

(we remark that relations (5.3–5.4) are not sufficient to have a convergence of χ_ε in $W^{1,\infty}(0,T;L^\infty(\Omega))$; indeed, this would require further differentiability properties).

Our task is now to pass to the limit in the nonlinear terms of system (5.1–5.2). ¿From (5.4) and (5.11) it follows that

$$\partial_t \chi_\varepsilon \, \theta_\varepsilon \to \partial_t \chi \theta \quad \text{weakly in } L^2(0,T;H), \tag{5.13}$$

while the other nonlinear term of (5.1) is tougher, since at present we have no strong convergence for $\partial_t \chi_\varepsilon$. At this point, we can only deduce by boundedness that, for some $w \in L^\infty(Q)$,

$$(\partial_t \chi_\varepsilon)^2 \to w \quad \text{weakly star in } L^\infty(Q). \tag{5.14}$$

Anyway, this is enough to write the limit of equations (5.1–5.2) and we can actually say that the functions $\theta, \chi, \xi, \eta, w$ satisfy the relations

$$\partial_t \theta + \theta \, \partial_t \chi - \Delta \theta = w, \tag{5.15}$$

$$\partial_t \chi + \xi + \eta = \theta - \theta_c, \tag{5.16}$$

a.e. in Q. Moreover, from (5.10) and (5.12), we deduce that the limit functions θ, χ, satisfy the Cauchy conditions (2.12) and the boundary condition (2.13) (cf. also Remark 2.2). Then, what still remains to do is only to interpret the nonlinear terms w, ξ, η, which indeed deserve a more careful treatment.

Strong convergence for χ_ε. To derive it, we use a Cauchy estimate for the sequence χ_ε, which is due to Blanchard, Damlamian and Ghidouche [3, Lemma 3.3]. We report their argument for the sake of clarity.

Name $(\theta_\delta, \chi_\delta)$ another solution to system (5.1–5.2) at the level $\delta > 0$. Writing (5.2) first for ε, then for δ, taking the difference, and multiplying it by $\chi_\varepsilon - \chi_\delta$, through an integration over Q_t and the addition and simultaneous subtraction of one term on the right hand side, we obtain:

$$\begin{aligned}
&\frac{1}{2}|(\chi_\varepsilon - \chi_\delta)(t)|^2 \\
&\qquad = \int_0^t \int_\Omega \big(J(\theta_\varepsilon - \theta_c - \beta_\varepsilon(\chi_\varepsilon)) - J(\theta_\varepsilon - \theta_c - \beta_\delta(\chi_\delta))\big)(\chi_\varepsilon - \chi_\delta)\,dx\,ds \\
&+ \int_0^t \int_\Omega \big(J(\theta_\varepsilon - \theta_c - \beta_\delta(\chi_\delta)) - J(\theta_\delta - \theta_c - \beta_\delta(\chi_\delta))\big)(\chi_\varepsilon - \chi_\delta)\,dx\,ds \\
&\qquad =: I_5(t) + I_6(t).
\end{aligned} \tag{5.17}$$

Now, on account of the contractivity of J, for the last integral it is easy to infer

$$I_6(t) \le \int_0^t |\chi_\varepsilon - \chi_\delta||\theta_\varepsilon - \theta_\delta|\,ds. \tag{5.18}$$

On the contrary, some work has to be done on $I_5(t)$. Denoting now by R_ε, R_δ the resolvent operators of β at the steps ε and δ, note that they have the very simple expression

$$R_\varepsilon(r) = R_\delta(r) = \max\{0, \min\{r, 1\}\} \quad \text{for all } r \in \mathbb{R}. \tag{5.19}$$

Moreover, a standard relationship (see, e.g., [7, p. 28]) postulates that

$$\chi_\varepsilon - \chi_\delta = R_\varepsilon(\chi_\varepsilon) - R_\delta(\chi_\delta) + \varepsilon\beta_\varepsilon(\chi_\varepsilon) - \delta\beta_\delta(\chi_\delta). \tag{5.20}$$

Substituting into $I_5(t)$, we easily deduce

$$\begin{aligned}
I_5(t) = &\int_0^t \int_\Omega \frac{J(\theta_\varepsilon - \theta_c - \beta_\varepsilon(\chi_\varepsilon)) - J(\theta_\varepsilon - \theta_c - \beta_\delta(\chi_\delta))}{\beta_\varepsilon(\chi_\varepsilon) - \beta_\delta(\chi_\delta)} \\
&\qquad\qquad \times (\beta_\varepsilon(\chi_\varepsilon) - \beta_\delta(\chi_\delta))(R_\varepsilon(\chi_\varepsilon) - R_\delta(\chi_\delta))\,dx\,ds \\
+ &\int_0^t \int_\Omega \big(J(\theta_\varepsilon - \theta_c - \beta_\varepsilon(\chi_\varepsilon)) - J(\theta_\varepsilon - \theta_c - \beta_\delta(\chi_\delta))\big) \\
&\qquad\qquad \times (\varepsilon\beta_\varepsilon(\chi_\varepsilon) - \delta\beta_\delta(\chi_\delta))\,dx\,ds =: I_7(t) + I_8(t),
\end{aligned} \tag{5.21}$$

where the integrand of $I_7(t)$ is meant to be 0 whenever $\beta_\varepsilon(\chi_\varepsilon) = \beta_\delta(\chi_\delta)$. Now, owing to the monotonicity of J and β and to the relations $\beta_\varepsilon(\chi_\varepsilon) \in \beta(R_\varepsilon(\chi_\varepsilon))$, $\beta_\delta(\chi_\delta) \in \beta(R_\delta(\chi_\delta))$ (cf. [7, p. 28]), we easily derive that $I_7(t) \le 0$. Finally, it is straightforward to verify that $I_8(t) \le Ct(\varepsilon + \delta)$, the C depending only on the $L^\infty(Q)$-bound for $\beta_\varepsilon(\chi_\varepsilon)$, $0 < \varepsilon < 1$, besides the measure $|\Omega|$.

Collecting all the information of (5.17–5.21), we see that it is possible to apply the Gronwall inequality in the form of [7, Lemme A.5, p. 157], which yields

$$|(\chi_\varepsilon - \chi_\delta)(t)| \leq \big(2C(\varepsilon+\delta)\big)^{1/2} + \|\theta_\varepsilon - \theta_\delta\|_{L^1(0,T;H)} \quad \text{for any } t \leq T, \tag{5.22}$$

so that, on account of (5.11), we derive

$$\chi_\varepsilon \to \chi \quad \text{strongly in } C^0([0,T];H). \tag{5.23}$$

Finally, thanks to (5.5) and to the usual monotonicity argument of [2, Prop. 1.1, p. 42], here applied in the space $L^2(Q)$, we can deduce relation (2.11), i.e. the desired interpretation of the term η in (5.16).

At this point, we still have to identify the limits ξ and w of the nonlinear terms ξ_ε and $(\partial_t\chi_\varepsilon)^2$. To this end, due to the monotone structure of the nonlinearities, some semicontinuity argument is required. We follow a probably nonstandard way, by first reporting a particular convergence result for maximal monotone operators together with its proof. Indeed, this general result may be useful in other situations and, in our opinion, permits to simplify the subsequent semicontinuity – comparison procedure.

Lemma 5.1. *Let $\mathcal{H}$ be a Hilbert space, A a maximal monotone operator of $\mathcal{H}$. Suppose also that $[x_n, y_n] \in A$ for any $n \in \mathbb{N}$. Finally, assume that*

$$x_n \to x, \qquad y_n \to y \qquad \textit{weakly in } \mathcal{H}, \quad \textit{for some } x, y \in \mathcal{H}. \tag{5.24}$$

Then, denoting by $(\cdot,\cdot)$ the scalar product of $\mathcal{H}$, we have

$$(y, x) \leq \liminf_{n\to\infty} (y_n, x_n). \tag{5.25}$$

Proof. We first point out that from the hypotheses of the lemma it does not follow that $[x, y] \in A$ (this property is instead implied by the *upper semicontinuity* inequality opposite to (5.25)). Choose an arbitrary element $[\xi, \eta] \in A$. By monotonicity, one has $(y_n - \eta, x_n - \xi) \geq 0$ for all $n \in \mathbb{N}$. Thus, passing to the lim inf and using the weak convergences given by (5.24), it follows that

$$\liminf_{n\to\infty} (y_n, x_n) \geq (\eta, x) - (\eta, \xi) + (y, \xi) \qquad \text{for any } [\xi, \eta] \in A. \tag{5.26}$$

Next, we rewrite the above relation for a particular choice of ξ, η. Indeed, taking $\lambda > 0$ and denoting by $J_\lambda = (\mathrm{Id} + \lambda A)^{-1}$ the resolvent and by $A_\lambda = \lambda^{-1}(\mathrm{Id} - J_\lambda)$ the Yosida-regularization of the operator A, we can choose $\xi = J_\lambda x$, $\eta = A_\lambda x$ (recalling again that $A_\lambda x \in A(J_\lambda x)$) and deduce

$$\begin{aligned} \liminf_{n\to\infty} (y_n, x_n) &\geq (A_\lambda x, x) - (A_\lambda x, J_\lambda x) + (y, J_\lambda x) \\ &= \lambda \|A_\lambda x\|^2_{\mathcal{H}} + (y, J_\lambda x) \geq (y, J_\lambda x) \qquad \text{for every } \lambda > 0. \end{aligned} \tag{5.27}$$

We now pass to the limit with respect to $\lambda \to 0$ in the above relation. Denoting by $\overline{D(A)}$ the closure of the domain $D(A)$ in the *strong* topology of $\mathcal{H}$, which is a convex set on account of [7, Thm. 2.2, p. 27], we have that (thanks to the same theorem)

$$J_\lambda x \to P_{\overline{D(A)}}x \quad \text{strongly in } \mathcal{H}, \tag{5.28}$$

where we have denoted by $P_{\overline{D(A)}}x$ the projection of x onto $\overline{D(A)}$.

Now, as $x_n \in D(A)$ for any $n \in \mathbb{N}$, by the first convergence in (5.24) we see that x belongs to the weak closure of $\overline{D(A)}$, which coincides with $\overline{D(A)}$, because $\overline{D(A)}$ is convex. This means that $P_{\overline{D(A)}}x = x$, so that (5.27–5.28) enable us to conclude. ∎

Characterization of $\boldsymbol{\xi}$. We multiply equation (5.2) by $\partial_t \chi_\varepsilon$ and integrate over the whole set Q. Referring to the remarks before (5.8) for the meaning of ξ_ε, we easily infer

$$\begin{aligned}\|\partial_t \chi_\varepsilon\|^2_{L^2(Q)} &= \int_0^T \int_\Omega \theta_\varepsilon \, \partial_t \chi_\varepsilon \, dx \, ds - \int_0^T \int_\Omega \theta_c \, \partial_t \chi_\varepsilon \, dx \, ds \\ &\quad - \int_0^T \int_\Omega \beta_\varepsilon(\chi_\varepsilon) \, \partial_t \chi_\varepsilon \, dx \, ds - \int_0^T \int_\Omega \xi_\varepsilon \, \partial_t \chi_\varepsilon \, dx \, ds =: \sum_{j=9}^{12} I_j(\varepsilon).\end{aligned} \tag{5.29}$$

We now want to prove a strong convergence for $\partial_t \chi_\varepsilon$; in particular, we are trying to deduce

$$\limsup_{\varepsilon \to 0} \|\partial_t \chi_\varepsilon\|^2_{L^2(Q)} \le \|\partial_t \chi\|^2_{L^2(Q)}, \tag{5.30}$$

which is enough since $\partial_t \chi_\varepsilon \to \partial_t \chi$ weakly in $L^2(Q)$ (cf. (5.4) and [2, Prop. 1.4, p. 14]). With this purpose, take the lim sup of (5.29): on account of (5.4) and (5.11), it easy to see that

$$\lim_{\varepsilon \to 0} I_9(\varepsilon) = \int_0^T \int_\Omega \theta \, \partial_t \chi \, dx \, ds, \tag{5.31}$$

$$\lim_{\varepsilon \to 0} I_{10}(\varepsilon) = -\int_0^T \int_\Omega \theta_c \, \partial_t \chi \, dx \, ds. \tag{5.32}$$

Regarding $I_{11}(\varepsilon)$, by explicit integration in time, with the help of (2.2) we have that

$$\limsup_{\varepsilon \to 0} I_{11}(\varepsilon) = -\liminf_{\varepsilon \to 0} \int_\Omega \psi_\varepsilon(\chi_\varepsilon(T)) \, dx \le -\int_\Omega \psi(\chi(T)) \, dx \tag{5.33}$$

and the last inequality holds since the functional induced by ψ_ε on $L^2(\Omega)$, namely

$$\Psi_\varepsilon(v) := \int_\Omega \psi_\varepsilon(v) \, dx \quad \text{for } v \in L^2(\Omega), \tag{5.34}$$

converges to

$$\Psi(v) := \begin{cases} 0 & \text{if } v \in L^2(\Omega) \text{ and } 0 \leq v \leq 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise} \end{cases} \tag{5.35}$$

in the sense of Mosco [1, Prop. 3.56, p. 354] and, furthermore, by virtue of (5.12) we have

$$\chi_\varepsilon(T) \to \chi(T) \quad \text{weakly in } L^2(\Omega) \tag{5.36}$$

(actually, $\chi_\varepsilon(T) \to \chi(T)$ strongly in H because of (5.23)). We finally discuss $I_{12}(\varepsilon)$: recalling Lemma 5.1 (with the choices $\mathcal{H} = L^2(Q)$, $A = \alpha$, $x_n = \chi_\varepsilon$, $y_n = \xi_\varepsilon$), one easily sees that

$$\limsup_{\varepsilon\to 0} I_{12}(\varepsilon) = -\liminf_{\varepsilon \to 0} \int_0^T \int_\Omega \xi_\varepsilon \, \partial_t \chi_\varepsilon \, dx\, ds \leq - \int_0^T \int_\Omega \xi \, \partial_t \chi \, dx\, ds. \tag{5.37}$$

Collecting now (5.31–5.37), we infer from (5.29)

$$\begin{aligned} \limsup_{\varepsilon \to 0} \|\partial_t \chi_\varepsilon\|^2_{L^2(Q)} &\leq \int_0^T \int_\Omega \theta \, \partial_t \chi \, dx\, ds \\ &- \int_0^T \int_\Omega \theta_c \, \partial_t \chi \, dx\, ds - \int_\Omega \psi(\chi(T))\, dx - \int_0^T \int_\Omega \xi \, \partial_t \chi \, dx\, ds. \end{aligned} \tag{5.38}$$

If one now multiplies the limit equation (5.16) by $\partial_t \chi$ and integrates over Q, (2.11), (5.4), and [7, Lemme 3.3, p. 73] can be exploited for the integration in time of the term with $\eta \, \partial_t \chi$. After the computations, a final comparison with the right hand side of (5.38) enables to derive (5.30), which yields the desired convergence

$$\partial_t \chi_\varepsilon \to \partial_t \chi \quad \text{strongly in } L^2(0, T; H) \tag{5.39}$$

and implies that (cf. (5.14)) $w = (\partial_t \chi)^2$. Therefore, (2.8) follows from (5.15) and, recalling (5.8), the (already used) monotonicity argument of [2, Prop. 1.1, p. 42] gives (2.10). This concludes the proof of Theorem 2.1.

6 Final remarks on uniqueness

Thanks to the limit procedure of the previous section, we have been able to prove the existence of a solution (θ, χ) to problem (2.8–2.13), with the further properties (2.14–2.16). Moreover, we have seen that such a solution can be found as limit of a subsequence of solutions to the approximate Problems $(\mathrm{P}_\varepsilon^\tau)$. Concerning uniqueness, we have to note that it seems rather difficult to recover it, even if one assumes the additional boundedness condition (2.14) (under this condition, the approximating problems with β_ε replacing β possess only one solution). On the other hand, the uniqueness argument of [3] could exploit the monotonicity properties related to equation (2.9) and control the variation of θ by the variation of χ in their equation

$$\partial_t(\theta + \chi) - \Delta\theta = 0 \quad \text{in } Q \tag{6.1}$$

replacing our (2.8). Now, the reader understands that the highly nonlinear part of (2.8) allows us to estimate the difference of two θ's only in terms of the difference of $\partial_t \mathcal{X}$'s (and not of $\mathcal{X}$'s!).

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