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Magnetic Shape Memory Alloys: three-dimensional modeling and analysis*

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We present a three-dimensional phenomenological model for the magneto-mechanical behavior of magnetic shape memory materials such as Ni_2MnGa . Moving from micro-magnetic considerations, we advance some thermodynamically consistent constitutive relations describing the magnetically-induced cubic-to-tetragonal martensitic transformation in a single crystal. We present an existence analysis for both the constitutive relation problem and the three-dimensional quasi-static evolution problem. Finally, we discuss the reduction of this model to some simpler one by means of a rigorous Γ -convergence analysis.

Keywords: Magnetic Shape Memory Alloys; Rate-independence; Existence; Energetic solutions.

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1. Introduction

The description of the thermomechanical behavior of Shape Memory Alloys (SMAs) has recently attracted a great deal of attention. In particular, this interest is triggered by a variety of innovative applications to biomedical, aeronautical, structural, and earthquake technologies^{13,14} just to mention a few hot topics. The motivation

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for this impressive success relies on the amazing material behavior of SMAs: severely deformed specimens regain the original shape after a thermal cycle (*shape memory* effect) and, at some higher temperature regimes, large strains are completely recovered by unloading (*superelastic* effect), see 18. These phenomena arise as the macroscopic effect of a solid-solid phase change in the crystallographic structure of the material from a highly symmetric variant called *austenite* (mostly cubic, predominant at high temperatures and low stresses) and less regular variants called *martensites* (energetically preferred at low temperatures).

The efficient and accurate modeling of the thermomechanical behavior of SMAs is a crucial task and a whole menagerie of models has been advanced by addressing different alloys (NiTi, CuAlNi, Ni₂MnGa, among many others) at different scales (atomistic, microscopic with micro-structures, mesoscopic with volume fractions, macroscopic) and emphasizing different principles (minimization of stored energy vs. maximization of dissipation, phenomenology vs. rational crystallography and Thermodynamics) and different structures (single crystals vs. polycrystalline aggregates, possibly including intragranular interaction)⁴⁵. Consequently, the Engineering literature on SMA modeling is vast. By limiting ourselves to macroscopic phenomenological models we can refer, without claim of completeness, to 10, 16, 17, 19, 20, 29, 30, 42, 41, 43, 44, 47, 48.

In the last decade a new class of materials called magnetic (or ferromagnetic) shape memory alloys (MSMAs) has been intensively investigated. These are SMAs presenting an impressive magnetostrictive effect: under suitable temperature and loading regimes, comparably large strains (up to 5%) can be achieved by applying moderate external magnetic fields. This happens as the effect of the ferromagnetic nature of martensites in MSMAs. In particular, the martensitic phase of the material presents the classical ferromagnetic texture of magnetic domains. This mesostructure can of course be reoriented and changed as an effect of the application of an external magnetic field by magnetic domain wall motion and magnetization vector rotation. Moreover, an extra phenomenon appears as magnetic-field driven martensitic variant reorientation can be observed. We shall assume that each martensitic variant is *magnetically uniaxial*, namely, that it presents just one so-called *easy axis*. This is specifically the case of tetragonal martensites as in the Heusler alloy Ni₂MnGa (see Figure 1). By applying a magnetic field one favours one variant against the others, depending on the respective easy axis orientation.

The focus of this paper is to advance a three-dimensional phenomenological model for MSMAs and provide the corresponding mathematical analysis. Working within the theory of irreversible thermodynamics, we propose a description of the constitutive response of MSMAs as an effect of changes in the internal magnetic

Fig. 1. Schematic crystal structure of the austenite and the tetragonal martensite variants with their associated easy axes in Ni₂MnGa.

field \mathbf{H} , the total stress $\boldsymbol{\sigma}$, and the absolute temperature T . Moreover, the linearized (small) strain $\boldsymbol{\varepsilon}$ is additively decomposed into $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\text{el}} + \boldsymbol{z}$ where $\boldsymbol{\varepsilon}_{\text{el}}$ is the elastic part of the strain whereas \boldsymbol{z} is the inelastic (or transformation) part. This last tensor \boldsymbol{z} is an internal variable of the model and shall be regarded as the descriptor of the martensitic structure of the material. A second internal variable is the local (signed) proportion α of magnetic domains oriented in the direction of the easy axis. Additional modeling discussions together with numerical experiments are reported in 1, 2.

Our main modeling choice is that of directly connecting magnetic and mechanical variables by prescribing the magnetization \mathbf{M} of the material to be given in terms of \boldsymbol{z} and α as

$$\mathbf{M} = \alpha \mathcal{A} \boldsymbol{z}.$$

Here, \mathcal{A} is a suitable affine operator whose exact form is completely characterized moving from micromagnetic considerations (see Subsection 2.4). In particular, we assume that the magnetic anisotropy of the material is sufficiently strong so that the magnetization stays rigidly attached to the easy axes of the martensitic variants and no magnetization rotation occurs. This assumption is in large agreement with experiments on Ni_2MnGa ^{49,40}. The specific form of \mathcal{A} will turn out to be compatible with material symmetries and yields the natural constraint $|\mathbf{M}| \leq m_{\text{sat}}$ where the latter stands for the saturation magnetization.

Before moving on we shall comment that the Engineering literature on MSMA is already quite vast and the reader shall at least be referred, with no claim of completeness, to 12, 22, 31, 40, 49, see also the review in 24. As for phenomenological models of internal-variable-type, we mention the results by HIRSINGER & LEXCELLENT ²¹ and KIEFER & LAGODAS ²⁶. These two models, albeit basically informed by our same principles, differ from ours as they are essentially restricted to two dimensions (or two martensitic variants). Both these models assume the *scalar* local proportion of one martensitic variant with respect to the other as an internal variable. Moreover, the shape of the Gibbs energy of these models is different from ours and comparably more complex. In particular, anisotropy is directly built in by means of the choice of specific anisotropic energy contributions. On the contrary, our model is *three-dimensional* in nature and develops an anisotropic behavior as an effect of the (affine) structure of \mathcal{A} . One has to mention that, the model by KIEFER & LAGODAS ²⁶ allows for rotations of local magnetizations with respect to the corresponding (signed) easy axes. As the macroscopic effect of magnetization rotations is observed to be small compared with martensitic reorientation ⁴⁹ we have neglected this phenomenon here. However, we plan to address the possible extension of our model to magnetization rotation in a forthcoming paper.

The paper is organized as follows. Section 2 is devoted to the thermodynamically consistent derivation of the model and the presentation of corresponding constitutive and evolution laws. Section 3 is then concerned with the existence analysis for the model. In particular, we prove existence of *energetic solutions* for the model

4 *A.-L. Bessoud and U. Stefanelli*

³², both the constitutive relation and the three-dimensional quasi-static evolution level. Eventually, Section 4 proves that the present model reduces to the original and well-validated Souza-Auricchio model for non-magnetic SMAs as the effect of some specific parameter asymptotics.

2. The model

2.1. Tensors

We will use bold uppercase letters for vectors $\mathbf{A} = (A_i) \in \mathbb{R}^3$, bold lowercase letters for 2-tensors $\mathbf{a} = (a_{ij}) \in \mathbb{R}^{3 \times 3}$, and bold letters $\mathbb{A} = (A_{ijk}) \in \mathbb{R}^{3 \times 3 \times 3}$ and $\mathbb{A} = (A_{ijkl}) \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$ for both 3- and 4-tensors. Given, $\mathbf{H} \in \mathbb{R}^3$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{3 \times 3}$, and $\mathbb{A} \in \mathbb{R}^{3 \times 3 \times 3}$ we classically define $\mathbf{H} \cdot \mathbb{A} \in \mathbb{R}^{3 \times 3}$, $\mathbf{a} : \mathbf{b} \in \mathbb{R}$, and $\mathbb{A} : \mathbf{a} \in \mathbb{R}^3$ as (summation convention)

$$(\mathbf{H} \cdot \mathbb{A})_{ij} = H_k A_{kij}, \quad \mathbf{a} : \mathbf{b} = a_{ij} b_{ij}, \quad (\mathbb{A} : \mathbf{a})_i = A_{ijk} a_{jk}.$$

Moreover, let us denote by $\mathbb{R}_{\text{sym}}^{3 \times 3}$ the subspace of symmetric 2-tensors endowed with the natural scalar product $\mathbf{a} : \mathbf{b} := \text{tr}(\mathbf{a}\mathbf{b})$ where $\text{tr}(\mathbf{a}) := a_{ii}$ and the corresponding norm $|\mathbf{a}|^2 := \mathbf{a} : \mathbf{a}$. The space $\mathbb{R}_{\text{sym}}^{3 \times 3}$ is orthogonally decomposed into $\mathbb{R}_{\text{sym}}^{3 \times 3} = \mathbb{R}_{\text{dev}}^{3 \times 3} \oplus \mathbb{R} \mathbf{1}_2$, where $\mathbb{R} \mathbf{1}_2$ is the subspace spanned by the identity 2-tensor $\mathbf{1}_2$ and $\mathbb{R}_{\text{dev}}^{3 \times 3}$ is the subspace of deviatoric (i.e. traceless) symmetric tensors. In particular, for all $\mathbf{a} \in \mathbb{R}_{\text{sym}}^{3 \times 3}$, by letting $\text{dev } \mathbf{a} := \mathbf{a} - (\text{tr } \mathbf{a}) \mathbf{1}_2 / 3$, we have that $\mathbf{a} = \text{dev } \mathbf{a} + (\text{tr } \mathbf{a}) \mathbf{1}_2 / 3$.

2.2. Constitutive equations

Assume to be given a material specimen occupying the reference configuration $\Omega \subset \mathbb{R}^3$. Within the small-strain regime, we additively decompose the linearized deformation $\boldsymbol{\varepsilon}(\mathbf{U}) = (\boldsymbol{\varepsilon}(\mathbf{U}))_{ij} = (U_{i,j} + U_{j,i})/2 \in \mathbb{R}_{\text{sym}}^{3 \times 3}$ (where $\mathbf{U} = (U_1, U_2, U_3) \in \mathbb{R}^3$ is the displacement from the reference configuration) into the elastic part $\boldsymbol{\varepsilon}_{\text{el}} \in \mathbb{R}_{\text{sym}}^{3 \times 3}$ and the inelastic (or transformation) part $\mathbf{z} \in \mathbb{R}_{\text{dev}}^{3 \times 3}$ as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\text{el}} + \mathbf{z}. \quad (2.1)$$

As martensitic transformations are classically observed to be volume preserving, the inelastic strain \mathbf{z} is assumed here to be (symmetric and) deviatoric.

The super-elastic effect in SMAs is the result of a structural phase transition between different configurations of the material lattice, namely the *parent phase* (austenite and twinned martensite) and its shared counterpart termed *product phase* (detwinned or single-variant martensite). In this regard, the internal variable \mathbf{z} is assumed to be descriptive of the mechanical (tensorial) effect of the detwinning observed in the material.

We focus here on the description of an alloy presenting a cubic-to-tetragonal martensitic behavior (this is particularly the case of Ni_2MnGa). Hence, we take into account three martensitic variants only. We assume that the magnetic anisotropy of the material is so strong that magnetizations are rigidly attached to the easy axes

in the martensitic phase. This assumption is largely acceptable in the case of high-anisotropy materials like Ni_2MnGa where variant nucleation and rearrangement are energetically favoured energy-reduction mechanisms with respect to magnetization rotation^{49,40}. We shall introduce the (local) magnetization vector $\mathbf{M} \in \mathbb{R}^3$ as the result of the linearly combined effect of the magnetization of each of the three martensitic variants. As each variant is assumed to present only one easy axis (recall that martensites are tetragonal), we are indeed assuming that

$$\mathbf{M} = \alpha(\xi_1 \mathbf{A}_1 + \xi_2 \mathbf{A}_2 + \xi_3 \mathbf{A}_3). \quad (2.2)$$

Here, \mathbf{A}_i is the (directed) easy axis of the i -th variant. Hence, each pure variant is assumed to be magnetically oriented either as \mathbf{A}_i or $-\mathbf{A}_i$. The scalar $\xi_i \in [0, 1]$ is the local proportion of variant i whereas the scalar internal variable $\alpha \in [-1, 1]$ represents the (signed) proportion of each variant which is magnetically oriented in direction \mathbf{A}_i . In particular, α can be interpreted as a local indicator of magnetic domains orientation. In particular, by letting $\mathbf{z}_i \in \mathbb{R}_{\text{dev}}^{3 \times 3}$ be the inelastic strain related to the i -th martensitic variant, we shall impose that

$$\mathbf{A}_i := \mathbf{A}_0 + \mathbb{A}:\mathbf{z}_i$$

where

$$\mathbf{A}_0 := \frac{1}{3} m_{\text{sat}} (1, 1, 1)^\top,$$

\mathbb{A} is some fixed 3-tensor, and $m_{\text{sat}} > 0$ denotes the saturation magnetization of martensites (constant for all variants). Taking into account (2.2), we introduce the affine operator $\mathcal{A} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^3$ such that

$$\mathbf{M} = \alpha \mathcal{A} \mathbf{z} := \alpha (\mathbf{A}_0 + \mathbb{A}:\mathbf{z}). \quad (2.3)$$

In particular, for $\mathbf{z} = \mathbf{0}$ (equal proportion of the three martensitic variants) we assume that the magnetization of the material is provided by the *mean easy axis* vector \mathbf{A}_0 , namely the easy axis related to the so-called *equivariant state* $\mathbf{z} = \mathbf{0}$ ⁴⁰. A concrete choice for \mathbb{A} (and hence for \mathcal{A}) is discussed in Subsection 2.4 below. This choice in particular preserves the usual constraint $|\mathbf{M}| \leq m_{\text{sat}}$.

The constitutive relations for the model are derived from the specification of the Gibbs free energy density of the material (which we assume to be of a constant and normalized density) as a function of the stress $\boldsymbol{\sigma}$, the internal magnetic field \mathbf{H} , the absolute temperature T , and the internal variables \mathbf{z} and α as

$$\begin{aligned} G(\boldsymbol{\sigma}, \mathbf{H}, T, \mathbf{z}, \alpha) := & -\frac{1}{2} \boldsymbol{\sigma}:\mathbb{C}^{-1}:\boldsymbol{\sigma} - \boldsymbol{\sigma}:\mathbf{z} + \beta(T)|\mathbf{z}| + \frac{h}{2} |\mathbf{z}|^2 + I(\mathbf{z}) \\ & + \frac{1}{2\delta} \alpha^2 + I_{[-1,1]}(\alpha) - \mu_0 \mathbf{H} \cdot \alpha \mathcal{A} \mathbf{z}. \end{aligned} \quad (2.4)$$

The first line in (2.4) is exactly the Gibbs energy of the *Souza-Auricchio model* for non-magnetic SMAs originally advanced by SOUZA, MAMIYA, & ZOUAIN⁴⁶ and then combined with finite elements by AURICCHIO & PETRINI^{4,5,6} and considered in 3, 7, 8, 9, 27, 28, 33, 34, 35. In particular \mathbb{C} is the isotropic elasticity 4-tensor,

6 *A.-L. Bessoud and U. Stefanelli*

$T \mapsto \beta(T) \geq 0$ is a given function of the temperature, $h > 0$ is an isotropic hardening modulus, and I is the *indicator function* of the simplex $Z := \{\mathbb{R}_{\text{dev}}^{3 \times 3} \ni \mathbf{z} = \xi_1 \mathbf{z}_1 + \xi_2 \mathbf{z}_2 + \xi_3 \mathbf{z}_3 : \xi_i \geq 0, \xi_1 + \xi_2 + \xi_3 = 1\}$ (namely $I(\mathbf{z}) = 0$ if $\mathbf{z} \in Z$ and $I(\mathbf{z}) = \infty$ otherwise), where \mathbf{z}_i , $i = 1, 2, 3$, correspond to the transformation strain of the pure martensitic variants. Namely,

$$\mathbf{z}_1 = \frac{\varepsilon_L}{\sqrt{6}} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{z}_2 = \frac{\varepsilon_L}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{z}_3 = \frac{\varepsilon_L}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

where $\varepsilon_L > 0$ is the maximal strain modulus obtainable by alignment of martensitic variants.

In the following, we shall use the short hand

$$F_{\text{SA}}(T, \mathbf{z}) := \beta(T)|\mathbf{z}| + \frac{h}{2}|\mathbf{z}|^2 + I(\mathbf{z}),$$

in order to indicate the specific choice of the original *Souza-Auricchio* model for the mechanical hardening part of the Gibbs energy.

The second line in the expression of the Gibbs energy (2.4) describes the magnetic behavior of the material. By (2.3), the term $-\mu_0 \mathbf{H} \cdot \alpha \mathcal{A} \mathbf{z}$ is nothing but the classical *Zeeman energy* term $-\mu_0 \mathbf{H} \cdot \mathbf{M}$. Note that \mathbf{H} stands here for the *internal* magnetic field. Namely, \mathbf{H} is the magnetic field which is actually experienced by the material when subjected to some (externally) applied field. In particular, \mathbf{H} corresponds to the sum of the applied external field and the corresponding induced demagnetization field. The indicator function $I_{[-1,1]}$ is constraining the scalar α to take values in $[-1, 1]$ and $1/\delta$ is a user-defined (dimensionalized in MPa) hardening parameter.

As temperature effects are not of interest here, we fix right from the beginning some suitable (comparably low) temperature T^* such that field-induced reorientation of martensitic variants may take place. Correspondingly, β^* stands for the constant value $\beta(T^*)$.

As the inelastic strain \mathbf{z} belongs to $\mathbb{R}_{\text{dev}}^{3 \times 3}$, we can equivalently rewrite the Gibbs energy as

$$G(\boldsymbol{\sigma}, \mathbf{H}, \mathbf{z}, \alpha) = -\frac{1}{2} \boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma} - \boldsymbol{\sigma} : \mathbf{z} + F_{\text{SA}}(\mathbf{z}) + \frac{1}{2\delta} \alpha^2 + I_{[-1,1]}(\alpha) - \mu_0 \mathbf{H} \cdot \alpha \mathcal{A}^* \mathbf{z} \quad (2.5)$$

where \mathcal{A}^* is defined by

$$\mathcal{A}^* \mathbf{z} := \mathbf{A}_0 + \mathbb{A}^* : \mathbf{z} \quad \text{with} \quad A_{ijk}^* := \frac{1}{2}(A_{ijk} + A_{ikj}) - \frac{1}{3} A_{i\ell\ell} \delta_{jk}. \quad (2.6)$$

Namely \mathbb{A}^* is the symmetrized deviatoric part of \mathbb{A} (with respect to the last two indices only).

Given the Gibbs energy, we classically derive the following constitutive equations

$$\boldsymbol{\varepsilon} = -\frac{\partial G}{\partial \boldsymbol{\sigma}} = \mathbb{C}^{-1} \cdot \boldsymbol{\sigma} + \mathbf{z}, \quad (2.7)$$

$$\mathbf{M} = -\frac{1}{\mu_0} \frac{\partial G}{\partial \mathbf{H}} = \alpha \mathcal{A}^* \mathbf{z}, \quad (2.8)$$

$$\mathbf{g} \in -\frac{\partial G}{\partial \mathbf{z}} = \boldsymbol{\sigma} - \beta^* \partial |\mathbf{z}| - h \mathbf{z} - \partial I(\mathbf{z}) + \mu_0 \alpha \mathbf{H} \cdot \mathcal{A}^*, \quad (2.9)$$

$$\gamma \in -\frac{\partial G}{\partial \alpha} = -\frac{1}{\delta} \alpha - \partial I_{[-1,1]}(\alpha) + \mu_0 \mathbf{H} \cdot \mathcal{A}^* \mathbf{z}. \quad (2.10)$$

In the latter, \mathbf{g} and γ stand for the so-called *thermodynamic forces* associated with the internal variables \mathbf{z} and α , respectively. Moreover, the symbol ∂ denotes the subdifferential in the sense of Convex Analysis. In particular,

$$\mathbf{b} \in \partial |\mathbf{z}| \iff \mathbf{b} : (\mathbf{a} - \mathbf{z}) \leq |\mathbf{a}| - |\mathbf{z}| \quad \forall \mathbf{a} \in \mathbb{R}_{\text{sym}}^{3 \times 3} \iff \mathbf{b} = \frac{\mathbf{z}}{|\mathbf{z}|} \text{ if } \mathbf{z} \neq \mathbf{0} \text{ and } |\mathbf{b}| \leq 1 \text{ else,}$$

$$\mathbf{b} \in \partial I(\mathbf{z}) \iff \mathbf{z} \in Z \text{ and } \mathbf{b} : (\mathbf{a} - \mathbf{z}) \leq 0 \quad \forall \mathbf{a} \in Z,$$

$$b \in \partial I_{[-1,1]}(\alpha) \iff \alpha \in [-1, 1] \text{ and } b(a - \alpha) \leq 0 \quad \forall a \in [-1, 1].$$

Note incidentally that $\mathbf{H} \cdot \mathcal{A}^* \in \mathbb{R}_{\text{dev}}^{3 \times 3}$, $\mathbf{g} \in \mathbb{R}_{\text{sym}}^{3 \times 3}$. Moreover, as $\partial I(\mathbf{z}) \subset \mathbb{R}_{\text{dev}}^{3 \times 3}$, we have that $\mathbf{g} - \boldsymbol{\sigma} \in \mathbb{R}_{\text{dev}}^{3 \times 3}$ and (2.7) and (2.9) can be possibly decoupled into the corresponding volumetric and deviatoric parts.

2.3. Evolution laws

We shall now address the dissipative character of the model. Our basic assumption is that the inelastic strain \mathbf{z} dissipates energy (exactly as in the original non-magnetic Souza-Auricchio model) whereas the variable α is non-dissipative. This is of course disputable as the dissipation in α is, for instance, the basic dissipative mechanism in ferromagnetic materials. Still, experiments show that, at small strains, the dissipation in α seems negligible with respect to that in \mathbf{z} ^{11,25}.

Moving from these considerations, the internal variable α can be obtained as a function of \mathbf{H} and \mathbf{z} for the thermodynamic force γ fulfills

$$\gamma = 0.$$

In particular, we have that

$$\alpha = \Pi_{[-1,1]}(\delta \mu_0 \mathbf{H} \cdot \mathcal{A}^* \mathbf{z}) := \max \{ -1, \min \{ 1, \delta \mu_0 \mathbf{H} \cdot \mathcal{A}^* \mathbf{z} \} \} \quad (2.11)$$

where $\Pi_{[-1,1]}$ denotes the usual projection operator onto $[-1, 1]$. In the following, we shall systematically use relation (2.11) in order to eliminate the variable α from the equations.

We now introduce the evolution law for \mathbf{z} in the usual complementary form by letting the yield function $F : \mathbb{R}_{\text{sym}}^{3 \times 3} \rightarrow \mathbb{R}$ be defined as

$$F(\mathbf{g}) := |\text{dev } \mathbf{g}| - R$$

$R > 0$ being the activation radius. Hence, we require \mathbf{z} to satisfy the flow rule and the complementary conditions

$$\dot{\mathbf{z}} = \dot{\zeta} \frac{\partial F}{\partial \mathbf{g}}(\mathbf{g}), \quad \dot{\zeta} \geq 0, \quad F \leq 0, \quad \dot{\zeta} F = 0. \quad (2.12)$$

Eventually, the dissipation function associated with \mathbf{z} is given by

$$D(\dot{\mathbf{z}}) = \sup \{ \mathbf{g} : \dot{\mathbf{z}} \quad : \quad F(\mathbf{g}) \leq 0 \} = \begin{cases} R|\dot{\mathbf{z}}| & \text{if } \dot{\mathbf{z}} \in \mathbb{R}_{\text{dev}}^{3 \times 3} \\ \infty & \text{else} \end{cases}$$

and, by taking position (2.11) into account, we can rewrite the constitutive equation (2.9) and the flow rule (2.12) in $\mathbb{R}_{\text{dev}}^{3 \times 3}$ as

$$R\partial|\dot{\mathbf{z}}| + \beta^* \partial|\mathbf{z}| + h\mathbf{z} + \partial I(\mathbf{z}) \ni \boldsymbol{\sigma} + \mu_0 \Pi_{[-1,1]}(\delta\mu_0 \mathbf{H} \cdot \mathcal{A}^* \mathbf{z}) \mathbf{H} \cdot \mathbb{A}^*. \quad (2.13)$$

2.4. The operator \mathcal{A}

This subsection presents a possible choice for the tensor \mathbb{A} , hence for operator \mathcal{A} . As $\mathbf{z} \in \mathbb{R}_{\text{dev}}^{3 \times 3}$, we shall indeed be concerned with the tensor $\mathbb{A}^* = (A_{ijk}^*)$ only. We propose to let

$$A_{iii}^* = -\frac{2}{3} \sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L}, \quad A_{ijj}^* = \frac{1}{3} \sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L}, \quad A_{ijk}^* = 0, \quad (2.14)$$

$\forall i, j, k = 1, 2, 3, \quad i \neq j \neq k$. The remainder of this Subsection is devoted to the motivation of the latter choice in view of micromagnetic considerations and to the description of its remarkable consequences.

2.4.1. The diagonal terms A_{ijj}^* .

The components A_{ijj}^* are uniquely determined via a micromagnetic ansatz. In particular, we ask the operator \mathcal{A} to map the inelastic strain \mathbf{z}_i related to the pure i -th martensitic variant into the corresponding (directed) easy axis \mathbf{A}_i . Namely, we impose ⁵⁰

$$\mathcal{A}\mathbf{z}_i = \mathbf{A}_i \quad i = 1, 2, 3,$$

that is

$$\mathcal{A}\mathbf{z}_1 = m_{\text{sat}}(1, 0, 0)^\top, \quad \mathcal{A}\mathbf{z}_2 = m_{\text{sat}}(0, 1, 0)^\top, \quad \mathcal{A}\mathbf{z}_3 = m_{\text{sat}}(0, 0, 1)^\top. \quad (2.15)$$

By solving the latter system, the components A_{iii}^* and A_{ijj}^* are completely characterized as in (2.14). Indeed, we have the following.

Lemma 2.1. *The operator \mathcal{A} fulfills (2.15) if and only if*

$$A_{iii}^* = -\frac{2}{3} \sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L}, \quad A_{ijj}^* = \frac{1}{3} \sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L} \quad \forall i, j = 1, 2, 3, \quad i \neq j.$$

Proof. Relations (2.15) are equivalent to the linear system

$$\begin{aligned} 2A_{111} - A_{122} - A_{133} &= -A_{211} + 2A_{222} - A_{233} = -A_{311} - A_{322} + 2A_{333} = -2\sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L} \\ 2A_{211} - A_{222} - A_{233} &= -A_{311} + 2A_{322} - A_{333} = -A_{111} - A_{122} + 2A_{133} = \sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L} \\ 2A_{311} - A_{322} - A_{333} &= -A_{111} + 2A_{122} - A_{133} = -A_{211} - A_{222} + 2A_{233} = \sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L} \end{aligned}$$

By possibly exchanging indices, we shall assume that $a = A_{iii}$ and $b = A_{ijj}$ for every $i, j = 1, 2, 3$, $i \neq j$. From the above system, we have

$$b = a + \sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L},$$

and, by taking the deviatoric part, one obtains

$$\begin{aligned} A_{iii}^* &= a - \frac{1}{3}(3a + 2\sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L}) = -\frac{2}{3}\sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L}, \\ A_{ijj}^* &= b - \frac{1}{3}(3a + 2\sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L}) = \frac{1}{3}\sqrt{\frac{2}{3}} \frac{m_{\text{sat}}}{\varepsilon_L}, \quad i \neq j. \quad \square \end{aligned}$$

Note incidentally that the diagonal terms A_{ijj} of the original tensor \mathbb{A} are determined by (2.15) up to an additive constant only.

2.4.2. The off-diagonal terms A_{ijk}^*

As for the off-diagonal terms of \mathbb{A}^* we shall require $A_{ijk}^* = 0$. Apart from simplicity, this position entails the possibility of reproducing at the macroscopic level the micromagnetic superposition law (2.2).

Let us preliminarily observe that all $\mathbf{z} \in \mathbb{R}_{\text{dev}}^{3 \times 3}$ can be decomposed as

$$\mathbf{z} = \xi_1 \mathbf{z}_1 + \xi_2 \mathbf{z}_2 + \xi_3 \mathbf{z}_3 + \eta_1 \mathbf{g}_1 + \eta_2 \mathbf{g}_2 + \eta_3 \mathbf{g}_3, \quad (2.16)$$

where

$$\mathbf{g}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{g}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{g}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

for some suitable choice of the coefficients ξ_i, η_i . We have the following.

Lemma 2.2. *We have that $\mathbb{A}:\mathbf{g}_i = \mathbf{0}$, $i = 1, 2, 3$, if and only if $A_{ijk}^* = 0$ for all $i, j, k = 1, 2, 3$, $j \neq k$.*

Proof. Letting $\mathbb{A}:\mathbf{g}_i = \mathbf{0}$, $i = 1, 2, 3$, is equivalent to impose the conditions

$$\begin{aligned} A_{112} + A_{121} &= A_{113} + A_{131} = A_{123} + A_{132} = 0 \\ A_{212} + A_{221} &= A_{213} + A_{231} = A_{223} + A_{232} = 0 \\ A_{312} + A_{321} &= A_{313} + A_{331} = A_{323} + A_{332} = 0 \end{aligned} \quad (2.17)$$

10 *A.-L. Bessoud and U. Stefanelli*

which are in turn equivalent to $A_{ijk}^* = 0$ for all $i, j, k = 1, 2, 3$, $j \neq k$. \square

Hence, as $\mathbf{z} \in Z$, we have

$$\begin{aligned} \mathbf{M} &= \alpha \mathcal{A} \mathbf{z} = \alpha \left(\mathbf{A}_0 + \xi_1 \mathbb{A}^* : \mathbf{z}_1 + \xi_2 \mathbb{A}^* : \mathbf{z}_2 + \xi_3 \mathbb{A}^* : \mathbf{z}_3 \right) \\ &= \alpha (\xi_1 \mathbf{A}_1 + \xi_2 \mathbf{A}_2 + \xi_3 \mathbf{A}_3) \end{aligned}$$

which is exactly (2.2).

2.4.3. The constraint $|\mathbf{M}| \leq m_{\text{sat}}$

The choice (2.14) entails in the validity of the classical constraint $|\mathbf{M}| \leq m_{\text{sat}}$. Indeed, one has that, for any $\mathbf{z} \in \mathbb{R}_{\text{dev}}^{3 \times 3}$, the vector $\mathbb{A}^* : \mathbf{z}$ is orthogonal to \mathbf{A}_0 so that

$$|\mathbf{M}| = |\alpha| \left(|\mathbf{A}_0|^2 + |\mathbb{A}^* : \mathbf{z}|^2 \right)^{1/2} \leq \left(\frac{1}{3} m_{\text{sat}}^2 + \varepsilon_L^2 \frac{2}{3} \frac{m_{\text{sat}}^2}{\varepsilon_L^2} \right)^{1/2} = m_{\text{sat}}.$$

Note moreover that equality holds for $\mathbf{z} = \mathbf{z}_i$, $i = 1, 2, 3$, and $\alpha = \pm 1$. Furthermore, for all $\mathbf{z} \in \mathbb{R}_{\text{dev}}^{3 \times 3}$, we have that $\mathcal{A} \mathbf{z} \neq \mathbf{0}$ so that the zero global magnetization state $\mathbf{M} = \mathbf{0}$ will be achieved by letting $\alpha = 0$.

2.4.4. Material symmetries

We shall discuss here the invariance of the Zeeman energy term under suitable material symmetry transformations (note that the mechanical part of the Gibbs energy is objective). Let

$$P_i \subset \text{O}(3) := \{ \mathbf{r} \in \mathbb{R}^{3 \times 3} : \mathbf{r} \mathbf{r}^\top = \mathbf{r}^\top \mathbf{r} = \mathbf{1}_2 \}$$

be the finite point group related to the undeformed crystal lattice of the i -th martensitic variant. This is clearly the dihedral group of order 8. More precisely, by letting $\mathbf{r}(\mathbf{V}, \theta)$ denote the rotation of angle θ and axis \mathbf{V} we have, for instance,

$$\begin{aligned} P_1 &= \left\{ \text{id}, \mathbf{r}((1, 0, 0), \pi/2), \mathbf{r}((1, 0, 0), -\pi/2), \mathbf{r}((1, 0, 0), \pi), \right. \\ &\quad \left. \mathbf{r}((0, 1, 0), \pi), \mathbf{r}((0, 0, 1), \pi), \mathbf{r}((0, 1, 1), \pi), \mathbf{r}((0, 1, -1), \pi) \right\}. \end{aligned}$$

An immediate observation is that $\mathbf{r} \mathbf{z}_i \mathbf{r}^\top = \mathbf{z}_i$ for all $\mathbf{r} \in P_i$. Hence, we have that

$$\mathcal{A}(\mathbf{r} \mathbf{z}_i \mathbf{r}^\top) = \mathcal{A} \mathbf{z}_i \quad \forall \mathbf{r} \in P_i.$$

Namely, the i -th variant easy axis is invariant by P_i material symmetry transformations. By carefully considering the possible sign change in α as an effect of the transformation, the corresponding Zeeman energy can be checked to be invariant as well, namely

$$\mathbf{H} \cdot \alpha \mathcal{A} \mathbf{z}_i = \mathbf{r} \mathbf{H} \cdot \alpha \mathcal{A}(\mathbf{r} \mathbf{z}_i \mathbf{r}^\top) \quad \text{for all } \mathbf{r} \in P_i.$$

On the other hand, by transforming the i -th variant \mathbf{z}_i via $\mathbf{r} \in P_j$ we have that

$$\mathbf{r}\mathbf{z}_i\mathbf{r}^\top = \begin{cases} \mathbf{z}_i & \text{if } \mathbf{r} \in P_i \\ \mathbf{z}_i \text{ or } \mathbf{z}_k & \text{if } \mathbf{r} \in P_j \end{cases}$$

where $i \neq j \neq k$. By working out the possibilities one discovers that indeed the Zeeman energy related to a pure martensitic variant is invariant for all $\mathbf{r} \in P := P_1 \cup P_2 \cup P_3$ as well. Namely

$$\mathbf{H} \cdot \alpha \mathcal{A} \mathbf{z}_i = \mathbf{r} \mathbf{H} \cdot \alpha \mathcal{A} (\mathbf{r} \mathbf{z}_i \mathbf{r}^\top) \quad \text{for all } \mathbf{r} \in P$$

where P is finite point group related to the undeformed cubic crystal lattice.

2.4.5. Blocking stress

Experimental evidence suggests that the ferromagnetic behavior of the material is highly stress-dependent^{23,39}. The reference experiment for this fact consists in applying to $\mathbf{z} = \mathbf{0}$ a sufficiently strong compressive stress in direction $(1, 0, 0)^\top$, for instance, and check that, above some fixed *blocking stress* level, by letting $\mathbf{H} = (0, H_2, H_3)$ no variant reorientation occurs.

This effect is predicted by our model. Indeed, as soon as the compressive stress is such that martensite is fully oriented into variant 1, namely $\mathbf{z} = \mathbf{z}_1$, the corresponding domain proportion is

$$\alpha = \Pi_{[-1,1]}(\delta\mu_0 \mathbf{H} \cdot \mathcal{A}^* \mathbf{z}_1) = \Pi_{[-1,1]} \left(\delta\mu_0 \mathbf{H} \cdot \frac{m_{\text{sat}}}{2} (1, 0, 0)^\top \right) = 0.$$

Hence, the magneto-mechanical coupling term $\mu_0 \alpha \mathbf{H} \cdot \mathcal{A}^*$ vanishes and no magnetically-induced variant reorientation occurs.

However, note that, independently of the stress level, in case $|\mathbf{z}| < \varepsilon_L$ the model predicts reorientation by $\mathbf{H} = (0, H_2, H_3)$. Finally, even in the saturated case $\mathbf{z} = \mathbf{z}_1$, one can induce reorientation by letting $H_1 \neq 0$. Under the paradigm of high magnetic anisotropy, these predictions agree with experiments. Indeed, it appears that applied magnetic fields move only variant interfaces that are already present whereas, starting from a pure variant state, nucleation of new martensitic variants is energetically less favorable than magnetization vectors rotation.

3. Existence of energetic solutions

This section addresses the issue of the existence of suitably weak solutions to the model. In order to shorten notations, let us normalize $\delta\mu_0 = 1$ and introduce the function $F_M : \mathbb{R} \rightarrow \mathbb{R}$ such that $\partial_{\mathbf{z}} F_M(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}) := \Pi_{[-1,1]}(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}) \mathbf{H} \cdot \mathcal{A}^*$. Namely,

$$F_M(r) = \min \left\{ \frac{r^2}{2}, |r| - \frac{1}{2} \right\} \quad \text{for all } r \in \mathbb{R}.$$

In particular, we have that $F'_M = \Pi_{[-1,1]}$.

12 *A.-L. Bessoud and U. Stefanelli*

In Subsection 3.1, we deal with the constitutive relation problem: given \mathbf{z}_0 and $t \mapsto \boldsymbol{\sigma}(t)$, to find $t \mapsto \mathbf{z}(t)$ such that (see (2.13))

$$\partial D(\dot{\mathbf{z}}) + \partial F_{\text{SA}}(\mathbf{z}) - \partial_{\mathbf{z}} F_{\text{M}}(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}) \ni \boldsymbol{\sigma} \quad \mathbf{z}(0) = \mathbf{z}_0.$$

Subsection 3.2 is then devoted to the existence of suitably weak solution for the quasi-static evolution problem obtained by coupling the constitutive relation and flow rule

$$\begin{pmatrix} \mathbf{0} \\ \partial D(\dot{\mathbf{z}}) \end{pmatrix} + \begin{pmatrix} \partial_{\boldsymbol{\sigma}} G(\boldsymbol{\sigma}, \mathbf{H}, \mathbf{z}) \\ \partial_{\mathbf{z}} G(\boldsymbol{\sigma}, \mathbf{H}, \mathbf{z}) \end{pmatrix} \ni \begin{pmatrix} -\varepsilon(\mathbf{U}) \\ \mathbf{0} \end{pmatrix} \quad (3.1)$$

with the equilibrium equation

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{F} = \mathbf{0} \text{ in } \Omega, \quad (3.2)$$

for some given body force \mathbf{F} and some prescribed boundary conditions. Note that we do not comply (3.1)-(3.2) with Maxwell's system. In particular, the magnetic field \mathbf{H} is intended to be given in all of the following.

We shall focus on a suitable weak solution notion, namely, the *energetic formulation*, which has been introduced by MIELKE & THEIL³⁸ and is naturally arising in connection with time discretization. In this direction, let us preliminarily introduce here some notation. We identify time partitions of $[0, T]$ with vectors of positive time steps $\boldsymbol{\tau}$. More precisely, given $\boldsymbol{\tau} = (\tau_1, \dots, \tau_{N_{\boldsymbol{\tau}}})$, the partition is

$$P_{\boldsymbol{\tau}} = \{0 = t_{\boldsymbol{\tau}}^0 < t_{\boldsymbol{\tau}}^1 < \dots < t_{\boldsymbol{\tau}}^{N_{\boldsymbol{\tau}}} = T\}$$

where

$$t_{\boldsymbol{\tau}}^0 := 0, \quad t_{\boldsymbol{\tau}}^i := t_{\boldsymbol{\tau}}^{i-1} + \tau_i, \quad i = 1, \dots, N_{\boldsymbol{\tau}}.$$

Given any vector $(w^0, \dots, w^{N_{\boldsymbol{\tau}}})$, we define $w_{\boldsymbol{\tau}}$ on $[0, T]$ as

$$w_{\boldsymbol{\tau}}(t) := w^i \quad \text{for } t \in [t_{\boldsymbol{\tau}}^i, t_{\boldsymbol{\tau}}^{i+1}), \quad i = 0, \dots, N_{\boldsymbol{\tau}} - 1,$$

and for every function w defined on $[0, T]$, we set

$$\bar{w}_{\boldsymbol{\tau}}(t) = w(t_{\boldsymbol{\tau}}^i) \quad \text{for } t \in (t_{\boldsymbol{\tau}}^{i-1}, t_{\boldsymbol{\tau}}^i], \quad i = 1, \dots, N_{\boldsymbol{\tau}}.$$

At last, we define the *mean operator* $M_{\boldsymbol{\tau}} : L^1(0, T, E) \longrightarrow L^1(0, T, E)$ (E Banach space) as

$$M_{\boldsymbol{\tau}}(\varphi)(t) := \frac{1}{\tau_i} \int_{t_{\boldsymbol{\tau}}^{i-1}}^{t_{\boldsymbol{\tau}}^i} \varphi(s) ds, \quad \text{for } t_{\boldsymbol{\tau}}^{i-1} < t \leq t_{\boldsymbol{\tau}}^i, \quad \forall \varphi \in L^1(0, T; E).$$

3.1. Analysis of the constitutive relation

This subsection is concerned with the constitutive relation problem. Namely, given $t \mapsto \boldsymbol{\sigma}(t)$, $t \mapsto \mathbf{H}(t)$, and the initial datum \mathbf{z}_0 , we are interested in finding $t \mapsto \mathbf{z}(t) \in Z$ solving

$$\mathbf{0} \in \partial D(\dot{\mathbf{z}}) + \partial F_{\text{SA}}(\mathbf{z}) - \boldsymbol{\sigma} - \partial_{\mathbf{z}} F_{\text{M}}(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}), \quad \mathbf{z}(0) = \mathbf{z}_0. \quad (3.3)$$

This differential problem may be rephrased in *energetic form* ³⁸ by asking for the initial condition $\mathbf{z}(0) = \mathbf{z}_0$ and the following relations, to be checked for every t in $[0, T]$,

$$\begin{aligned} F_{\text{SA}}(\mathbf{z}(t)) - F_{\text{M}}(\mathbf{H}(t) \cdot \mathcal{A}^* \mathbf{z}(t)) - \boldsymbol{\sigma}(t) : \mathbf{z}(t) \\ \leq F_{\text{SA}}(\hat{\mathbf{z}}) - F_{\text{M}}(\mathbf{H}(t) \cdot \mathcal{A}^* \hat{\mathbf{z}}) - \boldsymbol{\sigma}(t) : \hat{\mathbf{z}} + D(\mathbf{z}(t) - \hat{\mathbf{z}}), \quad \forall \hat{\mathbf{z}} \in Z \end{aligned} \quad (\text{S})$$

$$\begin{aligned} F_{\text{SA}}(\mathbf{z}(t)) - F_{\text{M}}(\mathbf{H}(t) \cdot \mathcal{A}^* \mathbf{z}(t)) - \boldsymbol{\sigma}(t) : \mathbf{z}(t) + \text{Diss}_D(\mathbf{z}, [0, t]) \\ = F_{\text{SA}}(\mathbf{z}_0) - F_{\text{M}}(\mathbf{H}(0) \cdot \mathcal{A}^* \mathbf{z}_0) - \boldsymbol{\sigma}(0) : \mathbf{z}_0 \\ + \int_0^t \left(-\dot{\boldsymbol{\sigma}}(s) : \mathbf{z}(s) - F'_{\text{M}}(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}) \dot{\mathbf{H}}(s) \cdot \mathcal{A}^* \mathbf{z}(s) \right) ds \end{aligned} \quad (\text{E})$$

where

$$\text{Diss}_D(\mathbf{z}, [0, T]) := \sup \left\{ \sum_{i=1}^{N_\tau} D(\mathbf{z}(t_\tau^i) - \mathbf{z}(t_\tau^{i-1})) : \{0 = t_\tau^0 < t_\tau^1 < \dots < t_\tau^{N_\tau} = T\} \right\}$$

is the total dissipation of the system on $[0, T]$ (recall that $D(\mathbf{a}) := R|\mathbf{a}|$ for all $\mathbf{a} \in \mathbb{R}_{\text{sym}}^{3 \times 3}$), the supremum being taken over all the partitions of $[0, T]$. We define the set of *stable states* at time $t \in [0, T]$ as

$$\begin{aligned} S(t) := \{ \mathbf{z} \in Z : F_{\text{SA}}(\mathbf{z}) - F_{\text{M}}(\mathbf{H}(t) \cdot \mathcal{A}^* \mathbf{z}) - \boldsymbol{\sigma}(t) : \mathbf{z} \\ \leq F_{\text{SA}}(\hat{\mathbf{z}}) - F_{\text{M}}(\mathbf{H}(t) \cdot \mathcal{A}^* \hat{\mathbf{z}}) - \boldsymbol{\sigma}(t) : \hat{\mathbf{z}} + D(\mathbf{z} - \hat{\mathbf{z}}) \quad \forall \hat{\mathbf{z}} \in Z \}. \end{aligned}$$

Being given a suitable initial condition $\mathbf{z}(0) = \mathbf{z}_0$, a solution of the system (S)-(E) is called an *energetic solution* of the constitutive relation (3.3). The problem of finding energetic solutions to (3.3) has already been considered for the Souza-Auricchio model without magnetic effect in 3 along with continuous dependence and approximation issues. Here we extend the latter theory to magnetic SMAs. In particular, an existence result reads as follows.

Theorem 3.1 (Existence for the constitutive equation). *Assume to be given $\boldsymbol{\sigma} \in W^{1,1}(0, T; \mathbb{R}_{\text{sym}}^{3 \times 3})$, $\mathbf{H} \in W^{1,1}(0, T; \mathbb{R}^3)$, and $\mathbf{z}_0 \in S(0)$. Then, there exists an energetic solution \mathbf{z} to system (S)-(E).*

The latter existence result is proved by means of the by-now classical existence proof for rate-independent systems ³² (also known as *6 steps proof*). Note however that the specific form of the magnetic part of the energy does not allow for a direct application of the general results. Indeed, available results generally apply to the situation where, for all fixed states, the *power* of the system is either uniformly bounded in time or integrable and linear with respect to the states. Here instead the *power* $t \mapsto \partial_t \left(F_{\text{SA}}(\mathbf{z}) - F_{\text{M}}(\mathbf{H}(t) \cdot \mathcal{A}^* \mathbf{z}) - \boldsymbol{\sigma}(t) : \mathbf{z} \right)$ is absolutely continuous yet non-linear. This forces us to cope with some small modification of the original argument from ³². We hence report in the following a full proof for the sake of completeness.

Let $P_\tau := \{0 = t_\tau^0 < t_\tau^1 < \dots < t_\tau^{N_\tau} = T\}$ be a partition of $[0, T]$ with diameter $|\tau| := \max_{i=1, \dots, N_\tau} (t_\tau^i - t_\tau^{i-1})$ and let $W : [0, T] \times \mathbb{R}_{\text{dev}}^{3 \times 3} \rightarrow \mathbb{R} \cup \{\infty\}$ be defined by

$$W(t, \mathbf{z}) = F_{\text{SA}}(\mathbf{z}) - F_{\text{M}}(\mathbf{H}(t) \cdot \mathcal{A}^* \mathbf{z}) - \boldsymbol{\sigma}(t) : \mathbf{z}.$$

Then, we are concerned with the following *incremental problem*:

$$\begin{cases} \text{set } \mathbf{z}^0 = \mathbf{z}_0 \text{ and, for all } k = 1, \dots, N_\tau, \text{ find } \mathbf{z}^k \text{ such that} \\ \mathbf{z}^k \in \arg \min \{W(t_\tau^k, \hat{\mathbf{z}}) + D(\mathbf{z}^{k-1} - \hat{\mathbf{z}}) \mid \hat{\mathbf{z}} \in Z\}. \end{cases} \quad (\text{IP})$$

It is easily seen that problem (IP) possesses a solution. Indeed, F_{SA} is a convex and coercive function. Moreover the maps $\mathbf{z} \mapsto \boldsymbol{\sigma} : \mathbf{z}$ and $\mathbf{z} \mapsto F_{\text{M}}(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z})$ are continuous so that the function $\mathbf{z} \mapsto F_{\text{SA}}(\mathbf{z}) - F_{\text{M}}(\mathbf{H}(t_\tau^k) \cdot \mathcal{A}^* \mathbf{z}) - \boldsymbol{\sigma}(t_\tau^k) : \mathbf{z} + D(\mathbf{z}^{k-1} - \mathbf{z})$ is coercive and lower semi-continuous for all $k = 1, \dots, N_\tau$.

We start from the following lemma, whose proof can be taken from [Thm 3.2, 32].

Lemma 3.1 (Stability and estimates at the discrete level). *Let \mathbf{z}^k be a solution of (IP). Then, for all $k = 1, \dots, N_\tau$,*

- i) $\mathbf{z}^k \in S(t_\tau^k)$;
- ii) the following estimate holds:

$$\begin{aligned} W(t_\tau^k, \mathbf{z}^k) + D(\mathbf{z}^{k-1} - \mathbf{z}^k) &\leq W(t_\tau^{k-1}, \mathbf{z}^{k-1}) \\ &+ \int_{t_\tau^{k-1}}^{t_\tau^k} \left(-\dot{\boldsymbol{\sigma}}(s) : \mathbf{z}^{k-1} - F'_M(\mathbf{H}(s) \cdot \mathcal{A}^* \mathbf{z}^{k-1}) \dot{\mathbf{H}}(s) \cdot \mathcal{A}^* \mathbf{z}^{k-1} \right) ds. \end{aligned}$$

As a straightforward consequence of Lemma 3.1, we state the following.

Corollary 3.1. *Let \mathbf{z}_τ be the piecewise constant interpolants of $(\mathbf{z}^k)_{k=1, \dots, N_\tau}$. Then,*

- a) $\forall t \in P_\tau, \mathbf{z}_\tau(t) \in S(t)$;
- b) $\forall s, t \in P_\tau$, with $s < t$, we have

$$\begin{aligned} W(t, \mathbf{z}_\tau(t)) + \text{Diss}_D(\mathbf{z}_\tau, [s, t]) &\leq W(s, \mathbf{z}_\tau(s)) \\ &- \int_s^t \left(\dot{\boldsymbol{\sigma}}(r) : \mathbf{z}_\tau(r) + F'_M(\mathbf{H}(r) \cdot \mathcal{A}^* \mathbf{z}_\tau(r)) \dot{\mathbf{H}}(r) \cdot \mathcal{A}^* \mathbf{z}_\tau(r) \right) dr. \end{aligned}$$

Proof. [Proof of Theorem 3.1] Let $P_{\tau_m} := \{0 = t_{\tau_m}^0 < t_{\tau_m}^1 < \dots < t_{\tau_m}^{N_{\tau_m}} = T\}$, $m \in \mathbb{N}$, be a sequence of partitions of $[0, T]$ with diameters $|\tau_m| \rightarrow 0$ when $m \rightarrow \infty$. For all $m \in \mathbb{N}$, the incremental problem

$$\begin{cases} \text{Fix } \mathbf{z}_m^0 = \mathbf{z}_0. \text{ For all } k = 1, \dots, N_{\tau_m}, \text{ find } \mathbf{z}_m^k \text{ such that} \\ \mathbf{z}_m^k \in \arg \min \{W(t_{\tau_m}^k, \mathbf{z}) + D(\mathbf{z}_m^{k-1} - \mathbf{z}) \mid \mathbf{z} \in Z\} \end{cases}$$

possesses a solution and we denote by \mathbf{z}_{τ_m} the corresponding right-continuous piecewise constant interpolant.

From the coercivity of W and assertion b) of Corollary 3.1, we obtain

$$\text{Diss}_D(\mathbf{z}_{\tau_m}, [0, T]) \leq C \quad \forall m \in \mathbb{N}$$

where, from now on, C denotes any positive constant depending just on data and possibly varying from line to line. In particular, C is independent on τ_m . Thus, \mathbf{z}_{τ_m} (is uniformly bounded and) has uniform bounded time-variation. From Helly's selection principle we deduce that there exist $\mathbf{z} : [0, T] \rightarrow Z$, $\Delta_\infty : [0, T] \rightarrow [0, C]$ and a not relabeled subsequence such that

$$\begin{aligned} \mathbf{z}_{\tau_m}(t) &\rightarrow \mathbf{z}(t) \quad \forall t \in [0, T]; \\ \text{Diss}_D(\mathbf{z}_{\tau_m}, [0, t]) &\rightarrow \Delta_\infty(t) \quad \forall t \in [0, T]. \end{aligned}$$

Let us prove that $\mathbf{z}(t)$ is a *stable trajectory*, namely $\mathbf{z}(t) \in S(t)$ for all $t \in [0, T]$. To this end we begin by proving that the set $S_{[0, T]} := \cup_{t \in [0, T]} S(t)$ is closed. Let $((t_\ell, \mathbf{z}_\ell))_{\ell \in \mathbb{N}}$ be a sequence in $S_{[0, T]}$ converging to some $(\tilde{t}, \tilde{\mathbf{z}})$. Then we have, for all $\tilde{\mathbf{z}} \in \mathbb{R}_{\text{sym}}^{3 \times 3}$ and $t \in [0, T]$,

$$\begin{aligned} &F_{\text{SA}}(\tilde{\mathbf{z}}) - F_{\text{M}}(\mathbf{H}(\tilde{t}) \cdot \mathcal{A}^* \tilde{\mathbf{z}}) - \sigma(\tilde{t}) : \tilde{\mathbf{z}} \\ &\leq \liminf_{\ell \rightarrow \infty} \left(F_{\text{SA}}(\mathbf{z}_\ell) - F_{\text{M}}(\mathbf{H}(t_\ell) \cdot \mathcal{A}^* \mathbf{z}_\ell) - \sigma(t_\ell) : \mathbf{z}_\ell \right) \\ &\leq \liminf_{\ell \rightarrow \infty} \left(F_{\text{SA}}(\tilde{\mathbf{z}}) - F_{\text{M}}(\mathbf{H}(t_\ell) \cdot \mathcal{A}^* \tilde{\mathbf{z}}) - \sigma(t_\ell) : \tilde{\mathbf{z}} + D(\mathbf{z}_\ell - \tilde{\mathbf{z}}) \right) \\ &= F_{\text{SA}}(\tilde{\mathbf{z}}) - F_{\text{M}}(\mathbf{H}(\tilde{t}) \cdot \mathcal{A}^* \tilde{\mathbf{z}}) - \sigma(\tilde{t}) : \tilde{\mathbf{z}} + D(\tilde{\mathbf{z}} - \tilde{\mathbf{z}}). \end{aligned}$$

Thus, $(\tilde{t}, \tilde{\mathbf{z}}) \in S_{[0, T]}$. Now let $t \in [0, T]$ be fixed and consider $\tau_{\tau_m}(t) := \max\{t_{\tau_m}^k, k = 1, \dots, N_{\tau_m} : t_{\tau_m}^k \leq t\}$. Then $\mathbf{z}_{\tau_m}(\tau_{\tau_m}(t)) = \mathbf{z}_{\tau_m}(t) \in S(\tau_{\tau_m}(t))$ so that $(\tau_{\tau_m}(t), \mathbf{z}_{\tau_m}(\tau_{\tau_m}(t))) \in S_{[0, T]}$. Moreover, we readily check that $\tau_{\tau_m}(t) \rightarrow t$ and $\mathbf{z}_{\tau_m}(t) \rightarrow \mathbf{z}(t)$. Thus the stability of the limit trajectory follows from the closure of $S_{[0, T]}$. This proves that \mathbf{z} fulfills (S).

We shall now check that \mathbf{z} satisfies (E). Let $t \in [0, T]$ be fixed. From Corollary 3.1 .b) we know that

$$\begin{aligned} &W(\tau_{\tau_m}(t), \mathbf{z}_{\tau_m}(t)) + \text{Diss}_D(\mathbf{z}_{\tau_m}, [0, \tau_{\tau_m}(t)]) \\ &\leq W(0, \mathbf{z}_0) - \int_0^{\tau_{\tau_m}(t)} \left(\dot{\sigma}(s) : \mathbf{z}_{\tau_m}(s) + F'_{\text{M}}(\mathbf{H}(s) \cdot \mathcal{A}^* \mathbf{z}_{\tau_m}(s)) \dot{\mathbf{H}}(s) \cdot \mathcal{A}^* \mathbf{z}_{\tau_m}(s) \right) ds. \end{aligned}$$

Passing to the limit we get

$$\begin{aligned} &W(t, \mathbf{z}(t)) + \text{Diss}_D(\mathbf{z}, [0, t]) \leq W(t, \mathbf{z}(t)) + \Delta_\infty(t) \\ &\leq \liminf_{m \rightarrow \infty} W(\tau_{\tau_m}(t), \mathbf{z}_{\tau_m}(t)) + \Delta_\infty(t) \\ &= \liminf_{m \rightarrow \infty} \left[W(\tau_{\tau_m}(t), \mathbf{z}_{\tau_m}(t)) + \text{Diss}_D(\mathbf{z}_{\tau_m}, [0, \tau_{\tau_m}(t)]) \right] \\ &\leq W(0, \mathbf{z}_0) - \int_0^t \left(\dot{\sigma}(s) : \mathbf{z}(s) + F'_{\text{M}}(\mathbf{H}(s) \cdot \mathcal{A}^* \mathbf{z}(s)) \dot{\mathbf{H}}(s) \cdot \mathcal{A}^* \mathbf{z}(s) \right) ds. \quad (3.4) \end{aligned}$$

16 *A.-L. Bessoud and U. Stefanelli*

The converse inequality follows from stability. Let $P_{\tau_n} := \{0 = s_{\tau_n}^0 < s_{\tau_n}^1 < \dots < s_{\tau_n}^{N_{\tau_n}} = t\}$ be a partition of $[0, t]$. To shorten notations, in what follows, we will denote by \mathbf{z}_n^j (resp. $\boldsymbol{\sigma}_n^j$ and \mathbf{H}_n^j) the quantities $\mathbf{z}(s_{\tau_n}^j)$ (resp. $\boldsymbol{\sigma}(s_{\tau_n}^j)$ and $\mathbf{H}(s_{\tau_n}^j)$). Since $\mathbf{z}(s) \in S(s)$ for all $s \in [0, t]$, given any $j = 1, \dots, N_{\tau_n}$, we have

$$\begin{aligned} & F_{\text{SA}}(\mathbf{z}_n^{j-1}) - F_{\text{M}}(\mathbf{H}_n^{j-1} \cdot \mathcal{A}^* \mathbf{z}_n^{j-1}) - \boldsymbol{\sigma}_n^{j-1} : \mathbf{z}_n^{j-1} \\ & \leq F_{\text{SA}}(\mathbf{z}_n^j) - F_{\text{M}}(\mathbf{H}_n^{j-1} \cdot \mathcal{A}^* \mathbf{z}_n^j) - \boldsymbol{\sigma}_n^{j-1} : \mathbf{z}_n^j + D(\mathbf{z}_n^{j-1} - \mathbf{z}_n^j) \end{aligned}$$

so that

$$\begin{aligned} & F_{\text{SA}}(\mathbf{z}_n^{j-1}) - F_{\text{M}}(\mathbf{H}_n^{j-1} \cdot \mathcal{A}^* \mathbf{z}_n^{j-1}) - \boldsymbol{\sigma}_n^{j-1} : \mathbf{z}_n^{j-1} \\ & - (\boldsymbol{\sigma}_n^j - \boldsymbol{\sigma}_n^{j-1}) : \mathbf{z}_n^j - (F_{\text{M}}(\mathbf{H}_n^j \cdot \mathcal{A}^* \mathbf{z}_n^j) - F_{\text{M}}(\mathbf{H}_n^{j-1} \cdot \mathcal{A}^* \mathbf{z}_n^j)) \\ & \leq F_{\text{SA}}(\mathbf{z}_n^j) - F_{\text{M}}(\mathbf{H}_n^j \cdot \mathcal{A}^* \mathbf{z}_n^j) - \boldsymbol{\sigma}_n^j : \mathbf{z}_n^j + D(\mathbf{z}_n^{j-1} - \mathbf{z}_n^j). \end{aligned}$$

By summing over $j = 1, \dots, N_{\tau_n}$, we obtain

$$\begin{aligned} W(0, \mathbf{z}_0) - \sum_{j=1}^{N_{\tau_n}} (\boldsymbol{\sigma}_n^j - \boldsymbol{\sigma}_n^{j-1}) : \mathbf{z}_n^j - \sum_{j=1}^{N_{\tau_n}} (F_{\text{M}}(\mathbf{H}_n^j \cdot \mathcal{A}^* \mathbf{z}_n^j) - F_{\text{M}}(\mathbf{H}_n^{j-1} \cdot \mathcal{A}^* \mathbf{z}_n^j)) \\ \leq W(t, \mathbf{z}(t)) + \text{Diss}_D(\mathbf{z}, [0, t]). \end{aligned} \quad (3.5)$$

Moreover, we can write that

$$\sum_{j=1}^{N_{\tau_n}} (\boldsymbol{\sigma}_n^j - \boldsymbol{\sigma}_n^{j-1}) : \mathbf{z}_n^j = \int_0^t M_{\tau_n}(\dot{\boldsymbol{\sigma}})(s) : \bar{\mathbf{z}}_{\tau_n}(s) \, ds.$$

Clearly $\bar{\mathbf{z}}_{\tau_n} \rightarrow \mathbf{z}$ and $M_{\tau_n}(\dot{\boldsymbol{\sigma}}) \rightarrow \dot{\boldsymbol{\sigma}}$ almost everywhere on $[0, t]$. Thus, from the Dominated Convergence Theorem we get

$$\sum_{j=1}^{N_{\tau_n}} (\boldsymbol{\sigma}_n^j - \boldsymbol{\sigma}_n^{j-1}) : \mathbf{z}_n^j \longrightarrow \int_0^t \dot{\boldsymbol{\sigma}}(s) : \mathbf{z}(s) \, ds.$$

On the other hand, from the Lipschitz continuity of F_{M} we have

$$\begin{aligned} & \sum_{j=1}^{N_{\tau_n}} (F_{\text{M}}(\mathbf{H}_n^j \cdot \mathcal{A}^* \mathbf{z}_n^j) - F_{\text{M}}(\mathbf{H}_n^{j-1} \cdot \mathcal{A}^* \mathbf{z}_n^j)) \\ & = \sum_{j=1}^{N_{\tau_n}} (F'_{\text{M}}(\mathbf{H}(\xi_n^j) \cdot \mathcal{A}^* \mathbf{z}_n^j) (\mathbf{H}_n^j - \mathbf{H}_n^{j-1}) \cdot \mathcal{A}^* \mathbf{z}_n^j) \\ & = \int_0^t F'_{\text{M}}(\mathbf{H}(\xi_n(s)) \cdot \mathcal{A}^* \bar{\mathbf{z}}_{\tau_n}(s)) M_{\tau_n}(\dot{\mathbf{H}})(s) \cdot \mathcal{A}^* \bar{\mathbf{z}}_{\tau_n}(s) \, ds, \end{aligned}$$

where

$\mathbf{H} \circ \xi_n \rightarrow \mathbf{H}$ almost everywhere on $[0, t]$ and is uniformly bounded;

$M_{\tau_n}(\dot{\mathbf{H}}) \rightarrow \dot{\mathbf{H}}$ almost everywhere on $[0, t]$;

$F'_{\text{M}}(\mathbf{H}(\xi_n(s)) \cdot \mathcal{A}^* \bar{\mathbf{z}}_{\tau_n}(s)) \rightarrow F'_{\text{M}}(\mathbf{H}(s) \cdot \mathcal{A}^* \mathbf{z}(s))$ for almost every $s \in [0, t]$ and is uniformly bounded.

Thus, again the Dominated Convergence Theorem entails

$$\sum_{j=1}^{N_{\tau_n}} (F_M(\mathbf{H}_n^j \cdot \mathcal{A}^* \mathbf{z}_n^j) - F_M(\mathbf{H}_n^{j-1} \cdot \mathcal{A}^* \mathbf{z}_n^j)) \longrightarrow \int_0^t F'_M(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}) \dot{\mathbf{H}}(s) \cdot \mathcal{A}^* \mathbf{z}(s) \, ds$$

and the converse inequality with respect to (3.4) follows by passing to the limit in (3.5). \square

Corollary 3.2 (Convergence of time-discretizations). *Under the assumptions of Theorem 3.1, the following convergences hold:*

- i) $\forall t \in [0, T] : \mathbf{z}_{\tau_m}(t) \rightarrow \mathbf{z}(t)$,*
- ii) $\forall t \in [0, T] : \text{Diss}_D(\mathbf{z}_{\tau_m}, [0, t]) \rightarrow \text{Diss}_D(\mathbf{z}, [0, t])$,*
- iii) $\forall t \in [0, T] : W(t, \mathbf{z}_{\tau_m}(t)) \rightarrow W(t, \mathbf{z}(t))$,*
- iv) $\dot{\boldsymbol{\sigma}} : \mathbf{z}_{\tau_m} + F'_M(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}_{\tau_m}) \dot{\mathbf{H}} \cdot \mathcal{A}^* \mathbf{z}_{\tau_m} \longrightarrow \dot{\boldsymbol{\sigma}} : \mathbf{z} + F'_M(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}) \dot{\mathbf{H}} \cdot \mathcal{A}^* \mathbf{z}$ in $L^1(0, T)$.*

Proof. Let us prove that $\Delta_\infty(t) = \text{Diss}_D(\mathbf{z}, [0, t])$ which then entails *ii*). This is a consequence of

$$\begin{aligned} W(0, \mathbf{z}_0) - \int_0^t (\dot{\boldsymbol{\sigma}}(s) : \mathbf{z}(s) + F'_M(\mathbf{H}(s) \cdot \mathcal{A}^* \mathbf{z}(s)) \dot{\mathbf{H}}(s) \cdot \mathcal{A}^* \mathbf{z}(s)) \, ds \\ \stackrel{\text{(E)}}{=} W(t, \mathbf{z}(t)) + \text{Diss}_D(\mathbf{z}, [0, t]) \\ \leq W(t, \mathbf{z}(t)) + \Delta_\infty(t) \\ \stackrel{\text{(3.4)}}{\leq} W(0, \mathbf{z}_0) - \int_0^t (\dot{\boldsymbol{\sigma}}(s) : \mathbf{z}(s) + F'_M(\mathbf{H}(s) \cdot \mathcal{A}^* \mathbf{z}(s)) \dot{\mathbf{H}}(s) \cdot \mathcal{A}^* \mathbf{z}(s)) \, ds. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} F_{\text{SA}}(\mathbf{z}(t)) - F_M(\mathbf{H}(t) \cdot \mathcal{A}^* \mathbf{z}(t)) - \boldsymbol{\sigma}(t) : \mathbf{z}(t) \\ = \lim_{m \rightarrow \infty} \left[F_{\text{SA}}(\mathbf{z}(t)) - F_M(\mathbf{H}(\tau_{\tau_m}(t)) \cdot \mathcal{A}^* \mathbf{z}(t)) \right. \\ \left. - \boldsymbol{\sigma}(\tau_{\tau_m}(t)) : \mathbf{z}(t) + D(\mathbf{z}(t) - \mathbf{z}_{\tau_m}(t)) \right] \\ \stackrel{\text{(S)}}{\geq} \limsup_{m \rightarrow \infty} \left[F_{\text{SA}}(\mathbf{z}_{\tau_m}(t)) - F_M(\mathbf{H}(\tau_{\tau_m}(t)) \cdot \mathcal{A}^* \mathbf{z}_{\tau_m}(t)) - \boldsymbol{\sigma}(\tau_{\tau_m}(t)) : \mathbf{z}_{\tau_m}(t) \right] \\ \geq F_{\text{SA}}(\mathbf{z}(t)) - F_M(\mathbf{H}(t) \cdot \mathcal{A}^* \mathbf{z}(t)) - \boldsymbol{\sigma}(t) : \mathbf{z}(t) \end{aligned}$$

which proves assertion *iii*).

Assertion *iv*) is easily obtained by applying the Dominated Convergence Theorem to the sequence of functions $\dot{\boldsymbol{\sigma}} : \mathbf{z}_{\tau_m} + F'_M(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}_{\tau_m}) \dot{\mathbf{H}} \cdot \mathcal{A}^* \mathbf{z}_{\tau_m}$. \square

3.2. Quasi-static evolution problem

Let $\Omega \subset \mathbb{R}^3$ be a non-empty, bounded, and open set with smooth boundary $\partial\Omega$. Assume that the body is clamped ($\mathbf{U} = \mathbf{0}$) on the part $\Gamma_0 \subset \partial\Omega$ with $\text{meas}(\Gamma_0) >$

0, it is subject to some surface traction \mathbf{G} on the remainder of its boundary $\Gamma_{\text{tr}} = \partial\Omega \setminus \Gamma_0$, and \mathbf{F} is some body given force. We set

$$\begin{aligned} \mathcal{U} &:= \{ \mathbf{V} \in H^1(\Omega, \mathbb{R}^3) : \mathbf{V} = \mathbf{0} \text{ on } \Gamma_0 \} \\ \mathcal{Z} &:= \{ \mathbf{z} \in BV(\Omega, \mathbb{R}^{3 \times 3}) \cap L^2(\Omega, \mathbb{R}^{3 \times 3}) : \mathbf{z} \in \mathcal{Z} \text{ a.e. in } \Omega \}. \end{aligned}$$

In particular, we have *Korn's inequality*¹⁵

$$\|\mathbf{U}\|_{\mathcal{U}} \leq c_{\text{Korn}} \|\boldsymbol{\varepsilon}(\mathbf{U})\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})}$$

for some $c_{\text{Korn}} > 0$ depending only on Ω and Γ_0 .

By letting $\mathbf{F} \in W^{1,1}(0, T; L^2(\Omega, \mathbb{R}^3))$ and $\mathbf{G} \in W^{1,1}(0, T; L^2(\Gamma_{\text{tr}}, \mathbb{R}^3))$, we define the total load $\boldsymbol{\ell} \in W^{1,1}(0, T; \mathcal{U}')$ (the prime denotes the dual) by

$$\langle \boldsymbol{\ell}(t), \mathbf{U} \rangle := \int_{\Omega} \mathbf{F} \cdot \mathbf{U} \, dx + \int_{\Gamma_{\text{tr}}} \mathbf{G} \cdot \mathbf{U} \, d\Gamma \quad \forall \mathbf{U} \in \mathcal{U}, \, t \in [0, T]$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between \mathcal{U}' and \mathcal{U} . Let $\mathbf{H} \in W^{1,1}(0, T; L^1(\Omega, \mathbb{R}^3))$. We are concerned with the energy functional $\mathcal{W} : [0, T] \times \mathcal{U} \times \mathcal{Z} \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{W}(t, \mathbf{U}, \mathbf{z}) &:= \frac{1}{2} \int_{\Omega} (\boldsymbol{\varepsilon}(\mathbf{U}) - \mathbf{z}) : \mathbb{C} : (\boldsymbol{\varepsilon}(\mathbf{U}) - \mathbf{z}) \, dx + \int_{\Omega} F_{\text{SA}}(\mathbf{z}) \, dx \\ &\quad - \int_{\Omega} F_{\text{M}}(\mathbf{H}(t) \cdot \mathcal{A}^* \mathbf{z}) \, dx + \text{Var}(\mathbf{z}) - \langle \boldsymbol{\ell}(t), \mathbf{U} \rangle, \end{aligned}$$

where $\text{Var}(\mathbf{z}) = \int_{\Omega} |\mathbf{D}\mathbf{z}|$ stands for the total variation of \mathbf{z} .

The dissipation functional $\mathcal{D} : L^1(\Omega, \mathbb{R}^{3 \times 3}) \times L^1(\Omega, \mathbb{R}^{3 \times 3}) \rightarrow [0, \infty)$ is defined by

$$\mathcal{D}(\mathbf{a}, \mathbf{b}) := \int_{\Omega} D(\mathbf{a} - \mathbf{b}) \, dx = \int_{\Omega} R|\mathbf{a} - \mathbf{b}| \, dx. \quad (3.6)$$

It is worth pointing out that the dissipation functional \mathcal{D} defined in (3.6) is continuous with respect to the strong topology of $L^1(\Omega, \mathbb{R}^{3 \times 3}) \times L^1(\Omega, \mathbb{R}^{3 \times 3})$.

Being given \mathbf{F} , \mathbf{G} , and \mathbf{H} , an energetic solution of the quasi-static problem is a function $t \in [0, T] \mapsto (\mathbf{U}(t), \mathbf{z}(t)) \in \mathcal{U} \times \mathcal{Z}$ such that $(\mathbf{U}(0), \mathbf{z}(0)) = (\mathbf{U}_0, \mathbf{z}_0)$ and, for every $t \in [0, T]$,

$$\mathcal{W}(t, \mathbf{U}(t), \mathbf{z}(t)) \leq \mathcal{W}(t, \hat{\mathbf{U}}, \hat{\mathbf{z}}) + \mathcal{D}(\mathbf{z}(t), \hat{\mathbf{z}}) \quad \forall (\hat{\mathbf{U}}, \hat{\mathbf{z}}) \in \mathcal{U} \times \mathcal{Z} \quad (\text{S}')$$

$$\begin{aligned} &\mathcal{W}(t, \mathbf{U}(t), \mathbf{z}(t)) + \text{Diss}_{\mathcal{D}}(\mathbf{z}, [0, t]) \\ &= \mathcal{W}(0, \mathbf{U}_0, \mathbf{z}_0) - \int_0^t \left(\langle \dot{\boldsymbol{\ell}}(s), \mathbf{U}(s) \rangle + \int_{\Omega} F'_{\text{M}}(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}) \dot{\mathbf{H}}(s) \cdot \mathcal{A}^* \mathbf{z}(s) \, dx \right) ds \quad (\text{E}') \end{aligned}$$

where $(\mathbf{U}_0, \mathbf{z}_0)$ is a suitable initial datum and $\text{Diss}_{\mathcal{D}}(\mathbf{z}, [0, t])$ is the analogous of the previous section for the functional \mathcal{D} .

We define the set of *stable states* at time $t \in [0, T]$ as

$$\mathcal{S}(t) := \{ (\mathbf{U}, \mathbf{z}) \in \mathcal{U} \times \mathcal{Z} : \forall (\hat{\mathbf{U}}, \hat{\mathbf{z}}) \in \mathcal{U} \times \mathcal{Z}, \mathcal{W}(t, \mathbf{U}, \mathbf{z}) \leq \mathcal{W}(t, \hat{\mathbf{U}}, \hat{\mathbf{z}}) + \mathcal{D}(\mathbf{z}, \hat{\mathbf{z}}) \}$$

The following theorem is the main result of this section.

Theorem 3.2 (Existence for the quasi-static evolution). *For all $(\mathbf{U}_0, \mathbf{z}_0) \in \mathcal{S}(0)$, there exists an energetic solution for the quasi-static evolution problem.*

Proof. As in the previous section, the proof is based on a time-discretization.

The map $\mathbf{z} \mapsto \int_{\Omega} F_M(\mathbf{H}(t) \cdot \mathcal{A}^* \mathbf{z}) dx$ is continuous with respect to the strong topology in $L^1(\Omega, \mathbb{R}^{3 \times 3})$. Hence, for all $t \in [0, T]$, the functional $\mathcal{W}(t, \cdot, \cdot)$ is lower-semicontinuous with respect to the weak topology of $H^1(\Omega, \mathbb{R}^3) \times (BV(\Omega, \mathbb{R}^{3 \times 3}) \cap L^2(\Omega, \mathbb{R}^{3 \times 3}))$. Consider $P_{\tau_m} := \{0 = t_{\tau_m}^0 < t_{\tau_m}^1 < \dots < t_{\tau_m}^{N_{\tau_m}} = T\}$, $m \in \mathbb{N}$, a sequence of partitions of $[0, T]$ with diameters $|\tau_m| \rightarrow 0$. Then, the incremental problems associated with the partition P_{τ_m} given by

$$\begin{cases} \text{Let } (\mathbf{U}_m^0, \mathbf{z}_m^0) = (\mathbf{U}_0, \mathbf{z}_0), \text{ and for all } k = 1, \dots, N_{\tau_m}, \text{ find} \\ (\mathbf{U}_m^k, \mathbf{z}_m^k) \in \arg \min \{ \mathcal{W}(t_{\tau_m}^k, \mathbf{U}, \mathbf{z}) + \mathcal{D}(\mathbf{z}_m^{k-1}, \mathbf{z}) \mid (\mathbf{U}, \mathbf{z}) \in \mathcal{U} \times \mathcal{Z} \} \end{cases}$$

admit a solution.

Let $k \in \{1, \dots, N_{\tau_m}\}$ be fixed. For every $i = 1, \dots, k$, one has

$$\begin{aligned} & \mathcal{W}(t_{\tau_m}^i, \mathbf{U}_m^i, \mathbf{z}_m^i) + \mathcal{D}(\mathbf{z}_m^i, \mathbf{z}_m^{i-1}) \\ & \leq \mathcal{W}(t_{\tau_m}^{i-1}, \mathbf{U}_m^{i-1}, \mathbf{z}_m^{i-1}) - \langle \boldsymbol{\ell}(t_{\tau_m}^i) - \boldsymbol{\ell}(t_{\tau_m}^{i-1}), \mathbf{U}_m^{i-1} \rangle \\ & \quad - \int_{\Omega} F'_M(\mathbf{H}(\xi_m^i) \cdot \mathcal{A}^* \mathbf{z}_m^{i-1}) (\mathbf{H}(t_{\tau_m}^i) - \mathbf{H}(t_{\tau_m}^{i-1})) \cdot \mathcal{A}^* \mathbf{z}_m^{i-1} dx \end{aligned}$$

where $\xi_m^i \in (t_{\tau_m}^{i-1}, t_{\tau_m}^i)$. By summing over $i = 1, \dots, k$, we obtain

$$\begin{aligned} & \mathcal{W}(t_{\tau_m}^k, \mathbf{U}_m^k, \mathbf{z}_m^k) + \sum_{i=1}^k \mathcal{D}(\mathbf{z}_m^i, \mathbf{z}_m^{i-1}) \\ & \leq \mathcal{W}(0, \mathbf{U}_0, \mathbf{z}_0) - \sum_{i=1}^k \langle \boldsymbol{\ell}(t_{\tau_m}^i) - \boldsymbol{\ell}(t_{\tau_m}^{i-1}), \mathbf{U}_m^{i-1} \rangle \\ & \quad - \sum_{i=1}^k \int_{\Omega} F'_M(\mathbf{H}(\xi_m^i) \cdot \mathcal{A}^* \mathbf{z}_m^{i-1}) (\mathbf{H}(t_{\tau_m}^i) - \mathbf{H}(t_{\tau_m}^{i-1})) \cdot \mathcal{A}^* \mathbf{z}_m^{i-1} dx. \end{aligned}$$

One easily gets that

$$\begin{aligned} \left| \sum_{i=1}^k \langle \boldsymbol{\ell}(t_{\tau_m}^i) - \boldsymbol{\ell}(t_{\tau_m}^{i-1}), \mathbf{U}_m^{i-1} \rangle \right| &= \left| \int_0^{t_{\tau_m}^k} \langle M_{\tau_m}(\dot{\boldsymbol{\ell}})(s), \mathbf{U}_{\tau_m}(s) \rangle ds \right| \\ &\leq \int_0^{t_{\tau_m}^k} \|M_{\tau_m}(\dot{\boldsymbol{\ell}})\|_{\mathcal{U}'} \|\mathbf{U}_{\tau_m}\|_{\mathcal{U}} ds \\ &\leq C \int_0^{t_{\tau_m}^k} \|\mathbf{U}_{\tau_m}\|_{\mathcal{U}} ds \\ &\stackrel{\text{Korn}}{\leq} C \int_0^{t_{\tau_m}^k} \|\boldsymbol{\varepsilon}(\mathbf{U}_{\tau_m})\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})} ds. \end{aligned}$$

20 *A.-L. Bessoud and U. Stefanelli*

Moreover, since F_M is Lipschitz continuous and $|z_m| \leq \varepsilon_L$ almost everywhere, we can write

$$\begin{aligned} & \left| \sum_{i=1}^k \int_{\Omega} F'_M(\mathbf{H}(\xi_m^i) \cdot \mathcal{A}^* z_{\tau_m}^{i-1}) (\mathbf{H}(t_{\tau_m}^i) - \mathbf{H}(t_{\tau_m}^{i-1})) \cdot \mathcal{A}^* z_{\tau_m}^{i-1} dx \right| \\ &= \left| \int_0^{t_{\tau_m}^k} \int_{\Omega} F'_M(\mathbf{H}(\xi_m(s)) \cdot \mathcal{A}^* z_{\tau_m}(s)) M_{\tau_m}(\dot{\mathbf{H}})(s) \cdot \mathcal{A}^* z_{\tau_m}(s) dx ds \right| \\ &\leq \int_0^{t_{\tau_m}^k} C \|M_{\tau_m}(\dot{\mathbf{H}})\|_{L^1(\Omega, \mathbb{R}^3)} \|\mathcal{A}^*\|_{\text{Lin}} \varepsilon_L ds \leq C \int_0^{t_{\tau_m}^k} \|\dot{\mathbf{H}}\|_{L^1(\Omega, \mathbb{R}^3)} ds \leq C. \end{aligned}$$

Thus, from Gronwall's inequality, we get

$$\sup_{t \in [0, T]} \left(\|\mathbf{U}_{\tau_m}(t)\|_{\mathcal{U}} + \|z_{\tau_m}(t)\|_{BV(\Omega, \mathbb{R}^{3 \times 3}) \cap L^\infty(\Omega, \mathbb{R}^{3 \times 3})} \right) \leq C \quad \forall m \in \mathbb{N}$$

and then

$$\text{Diss}_{\mathcal{D}}(z_{\tau_m}, [0, T]) \leq C \quad \forall m \in \mathbb{N}.$$

By Helly's principle, we deduce that there exist $z \in BV(0, T; L^1(\Omega, \mathbb{R}^{3 \times 3}))$, a function $\Delta_\infty : [0, T] \rightarrow [0, \infty)$, and a subsequence (not relabelled) such that, for every $t \in [0, T]$,

$$\begin{aligned} z_{\tau_m}(t) &\rightarrow z(t) \text{ strongly in } L^1(\Omega, \mathbb{R}^{3 \times 3}) \\ &\text{and weakly in } BV(\Omega, \mathbb{R}^{3 \times 3}) \cap L^2(\Omega, \mathbb{R}^{3 \times 3}), \\ \text{Diss}_{\mathcal{D}}(z_{\tau_m}, [0, t]) &\rightarrow \Delta_\infty(t). \end{aligned}$$

Moreover, for all $t \in [0, T]$, $\mathbf{U}_{\tau_m}(t)$ is the unique (weak) solution of

$$\int_{\Omega} \varepsilon(\mathbf{U}) : \mathbb{C} : \varepsilon(\mathbf{V}) dx = \langle \ell(\tau_{\tau_m}(t)), \mathbf{V} \rangle + \int_{\Omega} z_{\tau_m} : \mathbb{C} : \varepsilon(\mathbf{V}) dx \quad \forall \mathbf{V} \in \mathcal{U}. \quad (3.7)$$

Hence, $\mathbf{U}_{\tau_m}(t)$ depends linearly on $\ell(\tau_{\tau_m}(t))$ and $z_{\tau_m}(t)$ and we have that $\mathbf{U}_{\tau_m}(t) \rightarrow \mathbf{U}(t)$ weakly in $H^1(\Omega, \mathbb{R}^3)$ where $\mathbf{U}(t)$ solves

$$\int_{\Omega} \varepsilon(\mathbf{U}) : \mathbb{C} : \varepsilon(\mathbf{V}) dx = \langle \ell(t), \mathbf{V} \rangle + \int_{\Omega} z : \mathbb{C} : \varepsilon(\mathbf{V}) dx \quad \forall \mathbf{V} \in \mathcal{U}.$$

Finally, by the lower semicontinuity property of \mathcal{W} , we have that

$$\mathcal{W}(t, \mathbf{U}(t), z(t)) \leq \liminf_{m \rightarrow \infty} \mathcal{W}(\tau_{\tau_m}(t), \mathbf{U}_{\tau_m}(t), z_{\tau_m}(t)) < \infty.$$

Let us prove that the limit $(\mathbf{U}(t), z(t))$ is stable for every t in $[0, T]$. This comes again from the closure of $\mathcal{S}_{[0, T]} := \cup_{t \in [0, T]} \mathcal{S}(t)$ with respect to the weak topology of $H^1(\Omega, \mathbb{R}^3) \times (L^2(\Omega, \mathbb{R}^{3 \times 3}) \cap BV(\Omega, \mathbb{R}^{3 \times 3}))$. Let $(t_\ell, \mathbf{U}_\ell, z_\ell)_{\ell \in \mathbb{N}}$ be a sequence in $\mathcal{S}_{[0, T]}$ converging to some $(\tilde{t}, \tilde{\mathbf{U}}, \tilde{z})$, i.e.,

$$\begin{cases} t_\ell \rightarrow \tilde{t}, \\ \mathbf{U}_\ell \rightarrow \tilde{\mathbf{U}} \text{ weakly in } H^1(\Omega, \mathbb{R}^3), \\ z_\ell \rightarrow \tilde{z} \text{ weakly in } L^2(\Omega, \mathbb{R}^{3 \times 3}) \cap BV(\Omega, \mathbb{R}^{3 \times 3}). \end{cases}$$

Then, we have that

$$\begin{aligned} \mathcal{W}(\tilde{t}, \tilde{\mathbf{U}}, \tilde{\mathbf{z}}) &\leq \liminf_{\ell \rightarrow \infty} \mathcal{W}(t_\ell, \mathbf{U}_\ell, \mathbf{z}_\ell) \\ &\leq \liminf_{\ell \rightarrow \infty} \left[\mathcal{W}(t_\ell, \hat{\mathbf{U}}, \hat{\mathbf{z}}) + \mathcal{D}(\mathbf{z}_\ell, \hat{\mathbf{z}}) \right] = \mathcal{W}(\tilde{t}, \hat{\mathbf{U}}, \hat{\mathbf{z}}) + \mathcal{D}(\tilde{\mathbf{z}}, \hat{\mathbf{z}}) \end{aligned}$$

$\forall (\hat{\mathbf{U}}, \hat{\mathbf{z}}) \in \mathcal{U} \times \mathcal{Z}$. In particular, $(\tilde{\mathbf{U}}, \tilde{\mathbf{z}}) \in \mathcal{S}(\tilde{t})$.

In order to establish the energy balance (E') for (\mathbf{U}, \mathbf{z}) , we argue as in the previous section. Let $t \in [0, T]$ be fixed. Then, we have

$$\begin{aligned} &\mathcal{W}(\tau_{\tau_m}(t), \mathbf{U}_{\tau_m}(t), \mathbf{z}_{\tau_m}(t)) + \text{Diss}_{\mathcal{D}}(\mathbf{z}_{\tau_m}, [0, \tau_{\tau_m}(t)]) \\ &\leq \mathcal{W}(0, \mathbf{U}_0, \mathbf{z}_0) - \int_0^{\tau_{\tau_m}(t)} \langle M_{\tau_m}(\dot{\ell})(s), \mathbf{U}_{\tau_m}(s) \rangle ds \\ &\quad - \int_0^{\tau_{\tau_m}(t)} \int_{\Omega} F'_M(\mathbf{H}(\xi_m(s)) \cdot \mathcal{A}^* \mathbf{z}_{\tau_m}(s)) M_{\tau_m}(\dot{\mathbf{H}})(s) \cdot \mathcal{A}^* \mathbf{z}_{\tau_m}(s) dx ds \end{aligned}$$

for some suitable $t \mapsto \xi_m(t)$. By passing to the liminf for $m \rightarrow \infty$, we obtain

$$\begin{aligned} &\mathcal{W}(t, \mathbf{U}(t), \mathbf{z}(t)) + \text{Diss}_{\mathcal{D}}(\mathbf{z}, [0, t]) \\ &\leq \mathcal{W}(t, \mathbf{U}(t), \mathbf{z}(t)) dx + \Delta_{\infty}(t) \\ &\leq \liminf_{m \rightarrow \infty} \left[\mathcal{W}(\tau_{\tau_m}(t), \mathbf{U}_{\tau_m}(t), \mathbf{z}_{\tau_m}(t)) + \text{Diss}_{\mathcal{D}}(\mathbf{z}_{\tau_m}, [0, \tau_{\tau_m}(t)]) \right] \\ &\leq \mathcal{W}(0, \mathbf{U}_0, \mathbf{z}_0) - \int_0^t \langle \dot{\ell}(s), \mathbf{U}(s) \rangle ds \\ &\quad - \int_0^t \int_{\Omega} F'_M(\mathbf{H}(s) \cdot \mathcal{A}^* \mathbf{z}(s)) \dot{\mathbf{H}}(s) \cdot \mathcal{A}^* \mathbf{z}(s) dx ds. \end{aligned}$$

Let us prove now the converse inequality. Let $P_{\tau_n} := \{0 = s_{\tau_n}^0 < s_{\tau_n}^1 < \dots < s_{\tau_n}^{N_{\tau_n}} = t\}$ be a partition of $[0, t]$ with diameter $|\tau_n| \rightarrow 0$. Using the same notations as in the previous section, since $(\mathbf{U}_n^j, \mathbf{z}_n^j) := (\mathbf{U}(s_{\tau_n}^j), \mathbf{z}(s_{\tau_n}^j)) \in \mathcal{S}(s_{\tau_n}^j)$ for all $j = 1, \dots, N_{\tau_n}$, we have that

$$\mathcal{W}(s_{\tau_n}^{j-1}, \mathbf{U}_n^{j-1}, \mathbf{z}_n^{j-1}) \leq \mathcal{W}(s_{\tau_n}^{j-1}, \mathbf{U}_n^j, \mathbf{z}_n^j) + \mathcal{D}(\mathbf{z}_n^{j-1}, \mathbf{z}_n^j).$$

Hence,

$$\begin{aligned} &\mathcal{W}(s_{\tau_n}^{j-1}, \mathbf{U}_n^{j-1}, \mathbf{z}_n^{j-1}) \leq \mathcal{W}(s_{\tau_n}^j, \mathbf{U}_n^j, \mathbf{z}_n^j) + \mathcal{D}(\mathbf{z}_n^{j-1}, \mathbf{z}_n^j) \\ &\quad - \int_{\Omega} (F_M(\mathbf{H}_n^{j-1} \cdot \mathcal{A}^* \mathbf{z}_n^j) - F_M(\mathbf{H}_n^j \cdot \mathcal{A}^* \mathbf{z}_n^j)) dx - \langle \ell_n^{j-1} - \ell_n^j, \mathbf{U}_n^j \rangle. \end{aligned}$$

By summing over $j = 1, \dots, N_{\tau_n}$, we get

$$\begin{aligned} &\mathcal{W}(0, \mathbf{U}_0, \mathbf{z}_0) - \sum_{j=1}^{N_{\tau_n}} \int_{\Omega} (F_M(\mathbf{H}_n^j \cdot \mathcal{A}^* \mathbf{z}_n^j) - F_M(\mathbf{H}_n^{j-1} \cdot \mathcal{A}^* \mathbf{z}_n^j)) dx \\ &\quad - \sum_{j=1}^{N_{\tau_n}} \langle \ell_n^j - \ell_n^{j-1}, \mathbf{U}_n^j \rangle \leq \mathcal{W}(t, \mathbf{U}(t), \mathbf{z}(t)) + \text{Diss}_{\mathcal{D}}(\mathbf{z}, [0, t]). \quad (3.8) \end{aligned}$$

22 *A.-L. Bessoud and U. Stefanelli*

Let us now compute that

$$\begin{aligned} & \sum_{j=1}^{N_{\tau_n}} \int_{\Omega} (F_M(\mathbf{H}_n^j \cdot \mathcal{A}^* \mathbf{z}_n^j) - F_M(\mathbf{H}_n^{j-1} \cdot \mathcal{A}^* \mathbf{z}_n^j)) \, dx \\ &= \int_0^t \int_{\Omega} F'_M(\mathbf{H}(\xi_n(s)) \cdot \mathcal{A}^* \bar{\mathbf{z}}_{\tau_n}(s)) M_{\tau_n}(\dot{\mathbf{H}})(s) \cdot \mathcal{A}^* \bar{\mathbf{z}}_{\tau_n}(s) \, dx \, ds, \end{aligned}$$

for some suitable $t \mapsto \xi_n(t)$. In particular, ξ_n is such that $\mathbf{H} \circ \xi_n(s) \rightarrow \mathbf{H}(s)$ in $L^1(\Omega, \mathbb{R}^3)$ for every $s \in [0, t]$ and is uniformly bounded. Moreover, $M_{\tau_n}(\dot{\mathbf{H}})(s) \rightarrow \dot{\mathbf{H}}(s)$ strongly in $L^1(\Omega, \mathbb{R}^3)$ for almost every $s \in [0, T]$. As $\mathbf{z} \in BV(0, T; L^1(\Omega, \mathbb{R}^{3 \times 3}))$, the convergence $\bar{\mathbf{z}}_{\tau_n}(s) \rightarrow \mathbf{z}(s)$ in $L^1(\Omega, \mathbb{R}^{3 \times 3})$ holds for almost every $s \in [0, t]$. Thus, from the Dominated Convergence Theorem, we obtain

$$\begin{aligned} & \int_0^t \int_{\Omega} F'_M(\mathbf{H}(\xi_n(s)) \cdot \mathcal{A}^* \bar{\mathbf{z}}_{\tau_n}(s)) M_{\tau_n}(\dot{\mathbf{H}})(s) \cdot \mathcal{A}^* \bar{\mathbf{z}}_{\tau_n}(s) \, dx \, ds \\ & \longrightarrow \int_0^t \int_{\Omega} F'_M(\mathbf{H}(s) \cdot \mathcal{A}^* \mathbf{z}(s)) \dot{\mathbf{H}}(s) \cdot \mathcal{A}^* \mathbf{z}(s) \, dx \, ds. \end{aligned}$$

Moreover,

$$\sum_{j=1}^{N_{\tau_n}} \langle \ell_n^j - \ell_n^{j-1}, \mathbf{U}_n^j \rangle = \int_0^t \langle M_{\tau_n}(\dot{\ell})(s), \bar{\mathbf{U}}_{\tau_n}(s) \rangle \, ds$$

which tends to $\int_0^t \langle \dot{\ell}(s), \mathbf{U}(s) \rangle \, ds$ as $|\tau_n| \rightarrow 0$. Then, the claimed energy inequality follows by passing to the limit in (3.8). \square

As in the previous section, we summarize below the convergences of time-discrete quantities which have been deduced along the existence proof.

Corollary 3.3 (Convergence of time-discretizations). *Under the assumptions of Theorem 3.2, the following convergences hold:*

- i) $\forall t \in [0, T] : \mathbf{z}_{\tau_m}(t) \rightarrow \mathbf{z}(t)$ weakly in $BV(\Omega, \mathbb{R}^{3 \times 3}) \cap L^2(\Omega, \mathbb{R}^{3 \times 3})$;
- ii) $\forall t \in [0, T] : \mathbf{U}_{\tau_m}(t) \rightarrow \mathbf{U}(t)$ weakly in $H^1(\Omega, \mathbb{R}^3)$;
- iii) $\forall t \in [0, T] : \text{Diss}_{\mathcal{D}}(\mathbf{z}_{\tau_m}, [0, t]) \rightarrow \text{Diss}_{\mathcal{D}}(\mathbf{z}, [0, t])$;
- iv) $\forall t \in [0, T] : \mathcal{W}(t, \mathbf{U}_{\tau_m}(t), \mathbf{z}_{\tau_m}(t)) \rightarrow \mathcal{W}(t, \mathbf{U}(t), \mathbf{z}(t))$;
- v) $\langle \dot{\ell}, \mathbf{U}_{\tau_m} \rangle + \int_{\Omega} F'_M(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}_{\tau_m}) \dot{\mathbf{H}} \cdot \mathcal{A}^* \mathbf{z}_{\tau_m} \, dx \rightarrow \langle \dot{\ell}, \mathbf{U} \rangle + \int_{\Omega} F'_M(\mathbf{H} \cdot \mathcal{A}^* \mathbf{z}) \dot{\mathbf{H}} \cdot \mathcal{A}^* \mathbf{z} \, dx$ in $L^1(0, T)$.

4. Convergence to the Souza-Auricchio model as $\delta \rightarrow 0$

It is worth noting that in case no magnetic field is applied, then ($\mathbf{H} = \mathbf{0}$ and $\alpha(0) = 0$), and the current model reduces to the original Souza-Auricchio's (without magnetic effects).

In this section, we aim at proving that our model reduces to the original Souza-Auricchio one by rigorously taking the parameter asymptotics $\delta \rightarrow 0$. The latter

corresponds to assume infinite hardening on the magnetic domain proportion α . To the aim of studying the limit $\delta \rightarrow 0$, we shall rely on the general theory for Γ -convergence of rate-independent processes which has been developed in ³⁷. The reader is referred to the above mentioned paper for details.

We start by denoting by G_δ the Gibbs energy in order to make apparent the dependence on δ . In particular, let $G_\delta : [0, T] \times \mathbb{R}^{3 \times 3}_{\text{sym}} \times \mathbb{R}^{3 \times 3}_{\text{dev}} \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$, as

$$G_\delta(t, \boldsymbol{\varepsilon}, \mathbf{z}, \alpha) := \frac{1}{2}(\boldsymbol{\varepsilon} - \mathbf{z}) : \mathbb{C} : (\boldsymbol{\varepsilon} - \mathbf{z}) + F_{\text{SA}}(\mathbf{z}) + \frac{1}{\delta} \frac{\alpha^2}{2} + \frac{1}{\delta} I_{[-1,1]}(\alpha) - \boldsymbol{\sigma}(t) : \boldsymbol{\varepsilon} - \mu_0 \mathbf{H}(t) \cdot \alpha \mathbf{A} \mathbf{z}.$$

Note that the stress $t \mapsto \boldsymbol{\sigma}(t) \in W^{1,1}([0, T], \mathbb{R}^{3 \times 3})$ and the magnetic field $t \mapsto \mathbf{H}(t) \in W^{1,1}([0, T], \mathbb{R}^3)$ are supposed to be given and fixed.

Let $G_0 : [0, T] \times \mathbb{R}^{3 \times 3}_{\text{sym}} \times \mathbb{R}^{3 \times 3}_{\text{dev}} \times \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ be defined by

$$G_0(t, \boldsymbol{\varepsilon}, \mathbf{z}, \alpha) := \begin{cases} \frac{1}{2}(\boldsymbol{\varepsilon} - \mathbf{z}) : \mathbb{C} : (\boldsymbol{\varepsilon} - \mathbf{z}) + F_{\text{SA}}(\mathbf{z}) - \boldsymbol{\sigma}(t) : \boldsymbol{\varepsilon} & \text{if } \alpha = 0 \\ \infty & \text{else.} \end{cases}$$

The sets of stable states $S_\delta(t)$ are now defined for $\delta \geq 0$ as

$$S_\delta(t) := \left\{ (\boldsymbol{\varepsilon}, \mathbf{z}, \alpha) \in \mathbb{R}^{3 \times 3}_{\text{sym}} \times \mathbb{R}^{3 \times 3}_{\text{dev}} \times \mathbb{R} : G_\delta(t, \boldsymbol{\varepsilon}, \mathbf{z}, \alpha) < \infty \right.$$

$$\left. \text{and } G_\delta(t, \boldsymbol{\varepsilon}, \mathbf{z}, \alpha) \leq G_\delta(t, \tilde{\boldsymbol{\varepsilon}}, \tilde{\mathbf{z}}, \tilde{\alpha}) + D(\mathbf{z}, \tilde{\mathbf{z}}) \forall (\tilde{\boldsymbol{\varepsilon}}, \tilde{\mathbf{z}}, \tilde{\alpha}) \in \mathbb{R}^{3 \times 3}_{\text{sym}} \times \mathbb{R}^{3 \times 3}_{\text{dev}} \times \mathbb{R} \right\}.$$

Then, for any $n \mapsto \delta_n$ non increasing, we say that $(t_n, \boldsymbol{\varepsilon}_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n})_{n \in \mathbb{N}}$ is a *stable sequence* if

$$(\boldsymbol{\varepsilon}_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n}) \in S_{\delta_n}(t_n) \quad \text{and} \quad \sup_{n \in \mathbb{N}} G_{\delta_n}(t_n, \boldsymbol{\varepsilon}_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n}) < \infty.$$

For simplicity, we assume that the initial conditions $(\boldsymbol{\varepsilon}_\delta(0), \mathbf{z}_\delta(0), \alpha_\delta(0)) = (\boldsymbol{\varepsilon}^0, \mathbf{z}^0, 0)$ are independent of δ and satisfy the stability conditions $(\boldsymbol{\varepsilon}^0, \mathbf{z}^0, 0) \in S_\delta(0)$ for all $\delta \geq 0$ (more elaborated situations can be considered as well).

Theorem 4.1 (Convergence for the constitutive relation). *For any $\delta > 0$, let $t \mapsto (\boldsymbol{\varepsilon}_\delta(t), \mathbf{z}_\delta(t), \alpha_\delta(t)) \in \mathbb{R}^{3 \times 3}_{\text{sym}} \times \mathbb{R}^{3 \times 3}_{\text{dev}} \times \mathbb{R}$ be an energetic solution associated with the pair (G_δ, D) . Then, up to a not relabeled subsequence, we have that $(\boldsymbol{\varepsilon}_\delta(t), \mathbf{z}_\delta(t), \alpha_\delta(t)) \rightarrow (\boldsymbol{\varepsilon}(t), \mathbf{z}(t), \alpha(t))$ for all $t \in [0, T]$, where $(\boldsymbol{\varepsilon}, \mathbf{z}, \alpha)$ is an energetic solution associated with (G_0, D) .*

Proof. Owing to ³⁷, the proof consists in establishing the three following assertions.

- $G_0(t, \boldsymbol{\varepsilon}, \mathbf{z}, \alpha) \leq \inf \left\{ \liminf_{n \rightarrow \infty} G_{\delta_n}(t_n, \boldsymbol{\varepsilon}_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n}) : (t_n, \boldsymbol{\varepsilon}_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n})_{n \in \mathbb{N}} \text{ is a stable sequence and } (t_n, \boldsymbol{\varepsilon}_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n}) \rightarrow (t, \boldsymbol{\varepsilon}, \mathbf{z}, \alpha) \right\}$
- $D(\mathbf{z} - \tilde{\mathbf{z}}) \leq \inf \left\{ \liminf_{n \rightarrow \infty} D(\mathbf{z}_{\delta_n} - \tilde{\mathbf{z}}_{\delta_n}) : (t_n, \mathbf{z}_{\delta_n}) \rightarrow (t, \mathbf{z}), (t_n, \tilde{\mathbf{z}}_{\delta_n}) \rightarrow (t, \tilde{\mathbf{z}}) \right\}$
- For any stable sequence $(t_n, \boldsymbol{\varepsilon}_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n})_{n \in \mathbb{N}}$:
 $(t_n, \boldsymbol{\varepsilon}_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n}) \rightarrow (t, \boldsymbol{\varepsilon}, \mathbf{z}, \alpha) \Rightarrow (\boldsymbol{\varepsilon}, \mathbf{z}, \alpha) \in S_0(t).$

24 A.-L. Bessoud and U. Stefanelli

Proof of a). Let $(t_n, \varepsilon_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n})_{n \in \mathbb{N}}$ be a stable sequence converging to some $(t, \varepsilon, \mathbf{z}, \alpha)$. From the bound of energy, we have,

$$\alpha_{\delta_n} \in [-1, 1] \quad \text{and} \quad \frac{1}{\delta_n} \alpha_{\delta_n}^2 \leq C.$$

Hence $\alpha_{\delta_n} \rightarrow 0$ and $\alpha = 0$. Then, we can write

$$\begin{aligned} & \liminf_{n \rightarrow \infty} G_{\delta_n}(t_n, \varepsilon_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n}) \\ & \geq \liminf_{n \rightarrow \infty} \left[\frac{1}{2} (\varepsilon_{\delta_n} - \mathbf{z}_{\delta_n}) : \mathbb{C} : (\varepsilon_{\delta_n} - \mathbf{z}_{\delta_n}) + F_{\text{SA}}(\mathbf{z}_{\delta_n}) \right. \\ & \quad \left. - \boldsymbol{\sigma}(t_n) : \varepsilon_{\delta_n} - \mu_0 \mathbf{H}(t_n) \cdot \alpha_{\delta_n} \mathbf{A} \mathbf{z}_{\delta_n} \right] \\ & \geq \frac{1}{2} (\varepsilon - \mathbf{z}) : \mathbb{C} : (\varepsilon - \mathbf{z}) + F_{\text{SA}}(\mathbf{z}) - \boldsymbol{\sigma}(t) : \varepsilon = G_0(t, \varepsilon, \mathbf{z}, \alpha). \end{aligned}$$

Proof of b). Straightforward as D is continuous.

Proof of c). Let $(t_n, \varepsilon_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n})_{n \in \mathbb{N}}$ be a stable sequence converging to $(t, \varepsilon, \mathbf{z}, \alpha)$. Then, $\alpha = 0$ and $G_0(t, \varepsilon, \mathbf{z}, \alpha) < \infty$. We want to show that, for any $(\hat{\varepsilon}, \hat{\mathbf{z}}, \hat{\alpha}) \in \mathbb{R}_{\text{sym}}^{3 \times 3} \times \mathbb{R}_{\text{dev}}^{3 \times 3} \times \mathbb{R}$, one has

$$G_0(t, \varepsilon, \mathbf{z}, \alpha) \leq G_0(t, \hat{\varepsilon}, \hat{\mathbf{z}}, \hat{\alpha}) + D(\mathbf{z}, \hat{\mathbf{z}}).$$

If $\hat{\alpha} \neq 0$ or $\hat{\mathbf{z}} \notin Z$, the above left hand side is ∞ and nothing has to be proved. On the other hand, by assuming $\hat{\alpha} = 0$ and $\hat{\mathbf{z}} \in Z$, then $G_0(t, \hat{\varepsilon}, \hat{\mathbf{z}}, \hat{\alpha}) < \infty$ and we have

$$\begin{aligned} G_0(t, \varepsilon, \mathbf{z}, \alpha) & \leq \liminf_{n \rightarrow \infty} G_{\delta_n}(t_n, \varepsilon_{\delta_n}, \mathbf{z}_{\delta_n}, \alpha_{\delta_n}) \\ & \leq \liminf_{n \rightarrow \infty} \left[G_{\delta_n}(t_n, \hat{\varepsilon}, \hat{\mathbf{z}}, \hat{\alpha}) + D(\mathbf{z}_{\delta_n} - \hat{\mathbf{z}}) \right] = G_0(t, \hat{\varepsilon}, \hat{\mathbf{z}}, \hat{\alpha}) + D(\mathbf{z} - \hat{\mathbf{z}}). \end{aligned}$$

This proves that $(\varepsilon, \mathbf{z}, \alpha) \in S_0(t)$. \square

We conclude this section by stating the corresponding result for the quasi-static evolution problem (in three space dimensions). Set

$$\Lambda := \left\{ \alpha \in L^2(\Omega, \mathbb{R}) : \alpha \in [-1, 1] \right\},$$

and consider the functional $\mathcal{G}_\delta : [0, T] \times L^2(\Omega, \mathbb{R}^3) \times L^1(\Omega, \mathbb{R}^{3 \times 3}) \times L^2(\Omega, \mathbb{R}) \longrightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$\mathcal{G}_\delta(t, \mathbf{U}, \mathbf{z}, \alpha) := \begin{cases} \frac{1}{2} \int_{\Omega} (\varepsilon(\mathbf{U}) - \mathbf{z}) : \mathbb{C} : (\varepsilon(\mathbf{U}) - \mathbf{z}) \, dx + \int_{\Omega} F_{\text{SA}}(\mathbf{z}) \, dx + \frac{1}{\delta} \int_{\Omega} \frac{\alpha^2}{2} \, dx \\ \quad - \mu_0 \int_{\Omega} \mathbf{H}(t) \cdot \alpha \mathbf{A} \mathbf{z} \, dx + \text{Var}(\mathbf{z}) - \langle \boldsymbol{\ell}(t), \mathbf{U} \rangle \\ \quad \text{if } (\mathbf{U}, \mathbf{z}, \alpha) \in \mathcal{U} \times \mathcal{Z} \times \Lambda, \\ \infty \text{ else.} \end{cases}$$

By suitably adapting the argument for the proof of Theorem 4.1 one obtains the following convergence result.

Theorem 4.2 (Convergence for the evolution problem). *For any $\delta > 0$, let $t \mapsto (\mathbf{U}_\delta(t), \mathbf{z}_\delta(t), \alpha_\delta(t))$ be an energetic solution associated with the functionals $(\mathcal{G}_\delta, \mathcal{D})$. Then, up to a not relabeled subsequence, $(\mathbf{U}_\delta(t), \mathbf{z}_\delta(t), \alpha_\delta(t)) \longrightarrow (\mathbf{U}(t), \mathbf{z}(t), \alpha(t))$ strongly in $L^2(\Omega, \mathbb{R}^3) \times L^1(\Omega, \mathbb{R}^{3 \times 3}) \times L^2(\Omega, \mathbb{R})$ for all $t \in [0, T]$, where $(\mathbf{U}, \mathbf{z}, \alpha)$ is an energetic solution associated with $(\mathcal{G}_0, \mathcal{D})$, and $\mathcal{G}_0 : [0, T] \times L^2(\Omega, \mathbb{R}^3) \times L^1(\Omega, \mathbb{R}^{3 \times 3}) \times L^2(\Omega, \mathbb{R}) \longrightarrow \mathbb{R} \cup \{\infty\}$, is defined by*

$$\mathcal{G}_0(t, \mathbf{U}, \mathbf{z}, \alpha) := \begin{cases} \frac{1}{2} \int_{\Omega} (\boldsymbol{\varepsilon}(\mathbf{U}) - \mathbf{z}) : \mathbb{C} : (\boldsymbol{\varepsilon}(\mathbf{U}) - \mathbf{z}) \, dx + \int_{\Omega} F_{SA}(\mathbf{z}) \, dx \\ \quad - \langle \boldsymbol{\ell}(t), \mathbf{U} \rangle + \text{Var}(\mathbf{z}) \text{ if } (\mathbf{U}, \mathbf{z}) \in \mathcal{U} \times \mathcal{Z} \text{ and } \alpha = 0, \\ \infty \text{ else.} \end{cases}$$

References

1. F. Auricchio, A.-L. Bessoud, A. Reali and U. Stefanelli, A phenomenological model for ferromagnetism in shape-memory materials, (2010), in preparation.
2. F. Auricchio, A.-L. Bessoud, A. Reali and U. Stefanelli, A three-dimensional phenomenological model for Magnetic Shape Memory Alloys, *GAMM-Mitt.*, (2010), submitted.
3. F. Auricchio, A. Mielke, and U. Stefanelli, A rate independent model for the isothermal quasi-static evolution of shape-memory materials, *Math. Models Methods Appl. Sci.*, **18** (2008) 125–164.
4. F. Auricchio and L. Petrini, Improvements and algorithmical considerations on a recent three-dimensional model describing stress-induced solid phase transformations, *Internat. J. Numer. Methods Engrg.* **55** (2002) 1255–1284.
5. F. Auricchio and L. Petrini, A three-dimensional model describing stress-temperature induced solid phase transformations. Part I: solution algorithm and boundary value problems, *Internat. J. Numer. Methods Engrg.* **61** (2004) 807–836.
6. F. Auricchio and L. Petrini, A three-dimensional model describing stress-temperature induced solid phase transformations. Part II: thermomechanical coupling and hybrid composite applications, *Internat. J. Numer. Methods Engrg.* **61** (2004) 716–737.
7. F. Auricchio, A. Reali, and U. Stefanelli, A phenomenological 3D model describing stress-induced solid phase transformations with permanent inelasticity, in *Topics on Mathematics for Smart Systems (Rome, 2006)* (World Sci. Publishing, 2007) 1–14.
8. F. Auricchio, A. Reali, and U. Stefanelli, A three-dimensional model describing stress-induced solid phase transformation with permanent inelasticity, *Internat. J. Plast.* **23** (2007) 207–226.
9. F. Auricchio, A. Reali, and U. Stefanelli, A macroscopic 1D model for shape memory alloys including asymmetric behaviors and transformation-dependent elastic properties, *Comput. Methods Appl. Mech. Engrg.* **198** (2009) 1631–1637.
10. F. Auricchio and E. Sacco, A one-dimensional model for superelastic shape-memory alloys with different elastic properties between austenite and martensite, *Int. J. Non-Linear Mech.* **32** (1997) 1101–1114.
11. B. D. Cullity and C. D. Graham, *Introduction to magnetic materials*, (Second ed., Wiley, 2008).
12. A. DeSimone and R. D. James, A constrained theory of magnetoelasticity, *J. Mech. Phys. Solids* **50** (2002) 283–320.
13. T.W. Duerig, K.N. Melton, D. Stökel, C.M. Wayman editors, *Engineering aspects of shape memory alloys* (Butterworth-Heinemann, 1990).

26 A.-L. Bessoud and U. Stefanelli

14. T.W. Duerig, A.R. Pelton editors, *SMST-2003 Proceedings of the International Conference on Shape Memory and Superelastic Technology Conference*, ASM International, 2003.
15. G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, (Springer, 1976).
16. F. Falk and P. Konopka, Three-dimensional Landau theory describing the martensitic phase transformation of shape-memory alloys, *J. Phys. Condens. Matter* **2** (1990) 61–77.
17. M. Frémond, Matériaux à mémoire de forme, *C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre* **304** (1987) 239–244.
18. M. Frémond, *Non-smooth Thermomechanics* (Springer-Verlag, Berlin, 2002).
19. S. Govindjee and C. Miehe, A multi-variant martensitic phase transformation model: formulation and numerical implementation, *Comput. Methods Appl. Mech. Engrg.* **191** (2001) 215–238.
20. D. Helm and P. Haupt, Shape memory behaviour: modelling within continuum thermomechanics, *Internat. J. Solids Structures* **40** (2003) 827–849.
21. L. Hirsinger and C. Lexcellent, Internal variable model for magneto-mechanical behaviour of ferromagnetic shape memory alloys Ni-Mn-Ga, *J. Phys. IV* **112** (2003) 977–980.
22. R. D. James and M. Wuttig, Magnetostriction of martensite, *Phil. Mag. A* **77** (1998) 1273–1299.
23. H. E. Karaca, I. Karaman, B. Basaran, Y. I. Chumlyakov, and H. J. Maier, Magnetic field and stress induced martensite reorientation in NiMnGa ferromagnetic shape memory alloy single crystals, *Acta Mat.* **54** (2006) 233–245.
24. J. Kiang and L. Tong, Modelling of magneto-mechanical behaviour of NiMnGa single crystals, *J. Magn. Magn. Mater.* **292** (2005) 394–412.
25. B. Kiefer, A phenomenological model for magnetic shape memory alloys, PhD Thesis, 2006.
26. B. Kiefer and D.C. Lagoudas, Modeling the coupled strain and magnetization response of magnetic shape memory alloys under magnetomechanical loading, *J. Intell. Mater. Syst. Struct.* **20** (2009) 143–170.
27. P. Krejčí and U. Stefanelli, Existence and nonexistence for the full thermomechanical Souza-Auricchio model of shape memory wires, (2010), submitted. Preprint available at <http://www.imati.cnr.it/ulisse/pubbl.html>
28. P. Krejčí and U. Stefanelli, Well-posedness of a thermo-mechanical model for shape memory alloys under tension, *M2AN Math. Model. Anal. Numer.*, (2010), to appear. Preprint available at <http://www.imati.cnr.it/ulisse/pubbl.html>
29. D. C. Lagoudas, P. B. Entchev, P. Popov, E. Patoor, L. C. Brinson, and X. Gao, Shape memory alloys, Part II: Modeling of polycrystals, *Mech. Materials* **38** (2006) 391–429.
30. V. I. Levitas, Thermomechanical theory of martensitic phase transformations in inelastic materials, *Internat. J. Solids Structures* **35** (1998) 889–940.
31. A. A. Likhachev and K. Ullakko, Magnetic-field-controlled twin boundaries motion and giant magneto-mechanical effects in Ni₂MnGa shape memory alloy, *Phys. Lett. A* **275** (2000) 142–151.
32. A. Mielke, *Evolution in rate-independent systems (ch. 6)*, In C. Dafermos and E. Feireisl, editors, Handbook of Differential Equations, Evolutionary Equations, **2** 461–559 (Elsevier B.V., 2005).
33. A. Mielke, L. Paoli, and A. Petrov, On existence and approximation for a 3D model of thermally-induced phase transformations in shape-memory alloys, *SIAM J. Math. Anal.* **41** (2009) 1388–1414.

34. A. Mielke, L. Paoli, A. Petrov, and U. Stefanelli, Error estimates for discretizations of a rate-independent variational inequality, *SIAM J. Numer. Anal.*, (2010), to appear.
35. A. Mielke, L. Paoli, A. Petrov, and U. Stefanelli, Error bounds for space-time discretizations of a 3D model for shape-memory alloys, *Proceedings of the IUTAM Symposium on Variational Concepts with Applications to the Mechanics of Materials* (Bochum 2008), IUTAM Bookseries, Springer, 2009.
36. A. Mielke and A. Petrov, Thermally driven phase transformation in shape-memory alloys, *Adv. Math. Sci. Appl.* **17** (2007) 667–685.
37. A. Mielke, T. Roubíček, and U. Stefanelli, Γ -limits and relaxations for rate-independent evolutionary problems, *Calc. Var. Partial Differential Equations* **31** (2008) 3:387–416.
38. A. Mielke and F. Theil, On rate-independent hysteresis models, *NoDEA, Nonlinear Diff. Equations Applications*, (2004) 11:151–189.
39. S. J. Murray, M. Marioni, P. G. Tello, S. M. Allen, R. C. O’Handley, Giant magnetic-field-induced strain in Ni-Mn-Ga crystals: experimental results and modeling, *J. Magn. Magn. Mater.* **226–230** (2001) 945–947.
40. R.C. O’Handley, Model for strain and magnetization in magnetic shape-memory alloys, *J. Appl. Phys.* **83** (1998) 3263–3270.
41. B. Peultier, T. Ben Zineb, and E. Patoor, Macroscopic constitutive law for SMA: Application to structure analysis by FEM, *Materials Sci. Engrg. A* **438-440** (2006) 454–458.
42. P. Popov and D. C. Lagoudas, A 3-D constitutive model for shape memory alloys incorporating pseudoelasticity and detwinning of self-accommodated martensite, *Internat. J. Plast.* **23** (2007) 1679–1720.
43. B. Raniecki and Ch. Lexcellent, R_L models of pseudoelasticity and their specification for some shape-memory solids, *Eur. J. Mech. A Solids* **13** (1994) 21–50.
44. S. Reese and D. Christ, Finite deformation pseudo-elasticity of shape memory alloys – Constitutive modelling and finite element implementation, *Internat. J. Plast.* **28** (2008) 455–482.
45. T. Roubíček, Models of microstructure evolution in shape memory alloys, in *Nonlinear Homogenization and its Appl.to Composites, Polycrystals and Smart Materials*, P. Ponte Castaneda, J. J. Telega, B. Gambin eds., NATO Sci. Series II/170, Kluwer, Dordrecht, 2004, 269–304.
46. A. C. Souza, E. N. Mamiya, and N. Zouain, Three-dimensional model for solids undergoing stress-induced transformations, *Eur. J. Mech. A Solids* **17** (1998) 789–806.
47. P. Thamburaja and L. Anand, Polycrystalline shape-memory materials: effect of crystallographic texture, *J. Mech. Phys. Solids* **49** (2001) 709–737.
48. F. Thiebaud, Ch. Lexcellent, M. Collet, and E. Foltete, Implementation of a model taking into account the asymmetry between tension and compression, the temperature effects in a finite element code for shape memory alloys structures calculations, *Comput. Materials Sci.* **41** (2007) 2: 208–221.
49. R. Tickle and R. D. James, Magnetic and magnetomechanical properties of Ni₂MnGa, *J. Magn. Magn. Mater.* **195** (1999) 627–638.
50. R. Tickle, R. D. James, T. Shield, M. Wuttig, and V. V. Kokorin, Ferromagnetic shape memory in the NiMnGa system, *IEEE Trans. Mag.* **35** (1999) 4301–4310.
51. Y. Zhu and G. Dui, Micromechanical modeling of the stress-induced superelastic strain in magnetic shape memory alloy, *Mech. Materials* **39** (2007) 1025–1034.