Boundedness of the Temperature for the General Frémond Model for Shape Memory Alloys*

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Abstract

This note deals with the evolution of a shape memory material described by the general Frémond model [1, 2]. By establishing a maximum principle, we prove that, under rather weak assumptions on data, the absolute temperature of the body remains almost everywhere bounded. This general result applies to all the eventually simplified versions of the Frémond model that have been accounted for in the literature.

Key words and phrases: shape memory alloys, maximum principle, boundedness of the temperature.

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1 Introduction

The present analysis is concerned with a system of PDEs governing the evolution of the four unknown fields $\vartheta, \boldsymbol{u}, \chi_1$, and χ_2. Namely, we deal with the following equations and condition

$$\partial_t\big(c_0\vartheta - L\chi_1\big)(x,t) + \partial_t\big((\alpha(\vartheta) - \vartheta\alpha'(\vartheta))\chi_2 \operatorname{div}\boldsymbol{u}\big)(x,t) - h\Delta\vartheta(x,t)$$
$$= F(x,t,\vartheta(x,t)) + \big(\alpha(\vartheta)\chi_2 \operatorname{div}\boldsymbol{u}_t\big)(x,t), \quad \text{for a.e. } (x,t)\in Q, \tag{1.1}$$

$$\boldsymbol{u}_{tt} - \operatorname{div}\Big(\big(-\nu\Delta(\operatorname{div}\boldsymbol{u}) + \lambda\operatorname{div}\boldsymbol{u} + \alpha(\vartheta)\chi_2\big)J + 2\mu\varepsilon(\boldsymbol{u})\Big) = \boldsymbol{G}, \tag{1.2}$$

$$k\partial_t\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \begin{pmatrix}\ell(\vartheta-\vartheta_*)\\ \alpha(\vartheta)\operatorname{div}\boldsymbol{u}\end{pmatrix} + \partial I_K(\chi_1,\chi_2) \ni \begin{pmatrix}0\\0\end{pmatrix}, \tag{1.3}$$

where Ω is an open and bounded subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$, $T>0$ stands for some final time, $Q := \Omega\times(0,T)$, and the relations (1.2)-(1.3) are

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satisfied almost everywhere in Q. In the system above, α is a smooth function with properties to be specified later, ε is the linearized strain tensor, J is the identity matrix in $\mathbb{R}^3$, and $c_0, L, h, \nu, \lambda, \mu, k, \ell$, and ϑ_* are positive parameters (see [3] for their physical meaning). Moreover, $F : Q \times \mathbb{R} \to \mathbb{R}$ and $\boldsymbol{G} : Q \to \mathbb{R}^3$ are given functions, and we denote by K the triangle

$$K := \big\{(\gamma_1, \gamma_2) \in \mathbb{R}^2 \text{ such that } |\gamma_2| \leq \gamma_1 \leq 1\big\}. \tag{1.4}$$

The symbol I_K in (1.3) represents the indicator function of K (namely $I_K(\gamma_1, \gamma_2) = 0$ if $(\gamma_1, \gamma_2) \in K$ and $I_K(\gamma_1, \gamma_2) = +\infty$ elsewhere) and $\partial I_K : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ stands for the subdifferential of I_K, that turns out to be a maximal monotone operator in $\mathbb{R}^2$ (we refer to [4] for the details).

The model (1.1)-(1.3) has been introduced by M. Frémond [1, 2] in order to describe the thermomechanical evolution of a shape memory alloy. The latter alloy may be deformed, avoiding fractures, and then be forced to recover its original shape just by thermal means.

From the microscopic viewpoint, this phenomenon may be interpreted as a *structural phase transition* between different configurations of the metallic lattices, namely the *austenite* (the high temperature phase) and its shared counterparts termed *martensites* (prevailing at low temperatures) [2].

Among the various *macroscopic models* for shape memory alloys that have been proposed in the last fifteen years the Frémond model [1, 2], which is here taken into account, is of a relevant interest. Indeed, by allowing the phases to coexist at each point of the body and by dealing with just two martensitic variants besides one austenite (actually, in three space dimensions, 24 different martensitic variants have been detected), the Frémond model refers to any dimension of space (in particular 3-D, which is the natural setting of the problem). In the framework of the Frémond model, ϑ stands for the absolute temperature, $\boldsymbol{u}$ for the actual displacement of the body and χ_1 and χ_2 are related to the phase parameters. In particular, let β_1, β_2, and β_3 denote the local proportions of the two martensitic phases and the austenite, respectively. Then, the following conditions have to be fulfilled

$$0 \leq \beta_i \leq 1 \quad \text{for } i = 1, 2, 3, \quad \text{and} \quad \beta_1 + \beta_2 + \beta_3 = 1. \tag{1.5}$$

By eliminating β_3 in the equation above and letting $\chi_1 := \beta_1 + \beta_2$, and $\chi_2 := \beta_2 - \beta_1$, it turns out that (1.5) reduces to (see (1.4))

$$(\chi_1, \chi_2) \in K.$$

Moreover, F and $\boldsymbol{G}$ stand for the density of the heat source and the body force, respectively, while α is a smooth function representing the *thermal expansion* of the system. Referring to [3] for the details about the physical interest and behavior of α, we only stress that the latter is known to be vanishing for any temperature ϑ larger then a critical temperature $\vartheta_C > \vartheta_*$, usually called *Curie point* [2].

As far as we are interested in a mathematical study of (1.1)-(1.3), the system has to be complemented with initial and boundary conditions. Namely, denoting by $\boldsymbol{n}$ the outer unit normal vector to the boundary $\partial\Omega$ and letting $\{\Gamma_0, \Gamma_N\}$ be a partition of $\partial\Omega$ into two measurable subsets of positive surface measure, we prescribe

$$\vartheta(\cdot,0) = \vartheta_0, \quad \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(\cdot,0) = \boldsymbol{w}_0,$$
$$\chi_1(\cdot,0) = \chi_{1,0}, \quad \chi_2(\cdot,0) = \chi_{2,0} \quad \text{a.e. in } \Omega, \tag{1.6}$$
$$h\partial_{\boldsymbol{n}}\vartheta + \eta(\vartheta - f) = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.7}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0 \times (0,T), \tag{1.8}$$
$$\big(\big(-\nu\Delta(\operatorname{div}\boldsymbol{u}) + \lambda \operatorname{div}\boldsymbol{u} + \alpha(\vartheta)\chi_2\big)J + 2\mu\varepsilon(\boldsymbol{u})\big)\,\boldsymbol{n} = \boldsymbol{g} \quad \text{on } \Gamma_N \times (0,T), \tag{1.9}$$
$$\partial_{\boldsymbol{n}}(\operatorname{div}\boldsymbol{u}) = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{1.10}$$

Here η is a positive parameter, while $f : \partial\Omega\times(0,T) \to \mathbb{R}$ and $\boldsymbol{g} : \Gamma_N\times(0,T) \to \mathbb{R}^3$ represent the interactions with the medium surrounding the domain, in particular an external temperature and a traction on Γ_N, respectively.

The problem (1.1)-(1.3), (1.6)-(1.10) has been studied under different assumptions and simplifications in many articles (see [5] for a review). Let us just point out that the existence of a weak solution to the *quasistationary* three-dimensional problem (thus omitting the inertial term $\boldsymbol{u}_{tt}$) complete in all its nonlinearities has been proved in [5]. Moreover, dealing with just one space dimension, one has that the full 1-D problem admits a weak solution [6]. Indeed, the uniqueness of the solutions referred above is well-known [7] and their almost everywhere positivity has been proved in [8].

Up to now, it has been as open question if the boundedness of the temperature, although physically desirable, could be directly recovered from the model. The main result of this paper is that of giving an affirmative answer to this question. Indeed, accounting for suitable data, we prove that any weak solution to the system (1.1)-(1.3), (1.6)-(1.10), in a sense that will be later specified, remains almost everywhere bounded as the evolution occurs. This proof is achieved by deducing a *maximum principle* which is satisfied by the temperature ϑ. The key step of our argument is that of making a careful use of the relation (1.3) rewritten as a pointwise variational inequality as

$$(\chi_1, \chi_2) \in K \quad \text{a.e. in } Q,$$
$$k\sum_{j=1}^{2} \partial_t\chi_j(\chi_j - \gamma_j) + \ell(\vartheta - \vartheta_*)(\chi_1 - \gamma_1) + \alpha(\vartheta)\operatorname{div}\boldsymbol{u}(\chi_2 - \gamma_2) \ \leq\ 0$$
$$\text{a.e. in } Q, \quad \forall\,(\gamma_1, \gamma_2) \in K. \tag{1.11}$$

This is the plan of the paper, Section 2 contains the assumptions on data and the statement of the main result, while Section 3 brings to its rigorous proof.

2 Main result

We start by fixing some notation. Let $(\cdot,\cdot)$, $\|\cdot\|$ be the standard inner product and the norm either in $L^2(\Omega)$ or in $(L^2(\Omega))^3$. Moreover, the symbol $(\cdot,\cdot)$ stands also for the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. We set

$$\boldsymbol{V} := \big\{\boldsymbol{v} \in (H^1(\Omega))^3 \ \text{ such that } \ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0 \ \text{ and } \ \nu \operatorname{div} \boldsymbol{v} \in H^1(\Omega)\big\},$$

endowed with the norm $\|\boldsymbol{v}\|^2_{\boldsymbol{V}} := \|\boldsymbol{v}\|^2 + \sum_{i=1}^3 \|\nabla v_i\|^2 + \nu\|\nabla(\operatorname{div} \boldsymbol{v})\|^2$, for all $\boldsymbol{v} \in \boldsymbol{V}$. Let us use the symbol $\langle\cdot,\cdot\rangle$ for the duality pairing between $\boldsymbol{V}'$ and $\boldsymbol{V}$. Moreover, we introduce the bilinear form

$$a(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \Big(\nu\nabla(\operatorname{div}\boldsymbol{u})\cdot\nabla(\operatorname{div}\boldsymbol{v}) + \lambda \operatorname{div}\boldsymbol{u} \operatorname{div}\boldsymbol{v} + 2\mu \sum_{i,j=1}^3 \varepsilon_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\Big)\,dx, \quad \forall\, \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V},$$

and the nonempty, bounded, convex, and closed subset of $(L^2(\Omega))^2$

$$\mathcal{K} := \big\{(\chi_1,\chi_2) \in (L^2(\Omega))^2 \text{ such that } (\chi_1,\chi_2) \in K \text{ a.e. in } \Omega\big\}.$$

Next, we list our assumptions on data

$$\alpha \in C^2(\mathbb{R}), \tag{2.1}$$

F is a Carathéodory function such that

$$\forall u \in L^2(Q) \ \text{ one has } \ F\big(\cdot,\cdot,u(\cdot,\cdot)\big) \in L^2(Q), \tag{2.2}$$

$$f \in L^2(\partial\Omega \times (0,T)), \tag{2.3}$$

$$\boldsymbol{G} \in L^2(0,T;(L^2(\Omega))^3), \quad \boldsymbol{g} \in L^2(0,T;(L^2(\Gamma_N))^3), \tag{2.4}$$

$$\vartheta_0 \in L^2(\Omega), \quad \boldsymbol{u}_0 \in \boldsymbol{V}, \quad \boldsymbol{w}_0 \in (L^2(\Omega))^3, \quad (\chi_{1,0},\chi_{2,0}) \in \mathcal{K}. \tag{2.5}$$

Then, it is worth stating our notion of weak solution to the problem (1.1)-(1.3), (1.6)-(1.10)

Definition 2.1. *A quadruple $(\vartheta,\boldsymbol{u},\chi_1,\chi_2)$ is called a weak solution of (1.1)-(1.3), (1.6)-(1.10) if the following are fulfilled*

$$(\chi_1,\chi_2) \in \mathcal{K} \quad a.e. \ in \ (0,T), \tag{2.6}$$

$$\begin{aligned}
&\int_0^t \big(c_0\vartheta_t - L\chi_{1,t},\varphi\big)\,ds + h\int_0^t\int_\Omega \nabla\vartheta\cdot\nabla\varphi\,dx\,ds + \eta\int_0^t\int_{\partial\Omega}(\vartheta - f)\varphi\,d\Gamma\,ds \\
&\quad = \int_0^t \Big(\vartheta\alpha''(\vartheta)\vartheta_t\,\chi_2 \operatorname{div}\boldsymbol{u},\varphi\Big)ds + \int_0^t \Big(\big(\vartheta\alpha'(\vartheta) - \alpha(\vartheta)\big)\chi_{2,t}\operatorname{div}\boldsymbol{u},\varphi\Big)ds \\
&\qquad + \int_0^t \Big(\vartheta\alpha'(\vartheta)\chi_2 \operatorname{div}\boldsymbol{u}_t,\varphi\Big)ds + \int_0^t \big(F(\cdot,s,\vartheta(\cdot,s)),\varphi(s)\big)\,ds \\
&\qquad\qquad \forall\varphi \in L^2(0,T;H^1(\Omega)),\ 0 \le t \le T,
\end{aligned} \tag{2.7}$$

$$\int_0^t \Big(\langle \boldsymbol{u}_{tt}, \boldsymbol{v} \rangle + a(\boldsymbol{u}, \boldsymbol{v}) + \big(\alpha(\vartheta)\chi_2,\, \operatorname{div} \boldsymbol{v}\big)\Big) ds$$
$$= \int_0^t \int_\Omega \boldsymbol{G} \cdot \boldsymbol{v} \, dx \, ds + \int_0^t \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v} \, d\Gamma \, ds$$
$$\forall \boldsymbol{v} \in L^2(0,T;\boldsymbol{V}),\ 0 \le t \le T, \tag{2.8}$$

$$\int_0^t \int_\Omega \left(k \sum_{j=1}^{2} \partial_t \chi_j (\chi_j - \gamma_j) \right) dx \, ds + \ell \int_0^t \int_\Omega (\vartheta - \vartheta_*)(\chi_1 - \gamma_1) \, dx \, ds$$
$$+ \int_0^t \int_\Omega \alpha(\vartheta) \operatorname{div} \boldsymbol{u} (\chi_2 - \gamma_2) \, dx \, ds \ \le\ 0 \qquad \forall\, (\gamma_1, \gamma_2) \in (L^2(Q))^2$$
$$\textit{such that } (\gamma_1, \gamma_2) \in K \textit{ a.e. in } Q, \quad 0 \le t \le T, \tag{2.9}$$

$$\vartheta(\cdot, 0) = \vartheta_0, \quad \chi_1(\cdot, 0) = \chi_{1,0}, \quad \chi_2(\cdot, 0) = \chi_{2,0} \quad \textit{a.e. in } \Omega, \tag{2.10}$$

$$\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0 \quad \textit{in } (L^2(\Omega))^3, \quad \boldsymbol{u}_t(\cdot, 0) = \boldsymbol{w}_0 \quad \textit{in } \boldsymbol{V}'. \tag{2.11}$$

Note that, if the following conditions hold

$$\vartheta \in H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^1(\Omega)) \quad \big(\subset C^0([0,T];L^2(\Omega))\big), \tag{2.12}$$
$$\boldsymbol{u} \in H^2(0,T;\boldsymbol{V}') \cap L^2(0,T;\boldsymbol{V}) \quad \big(\subset C^0([0,T];(L^2(\Omega))^3)\big), \tag{2.13}$$
$$\chi_1, \chi_2 \in H^1(0,T;L^2(\Omega)), \tag{2.14}$$
$$\alpha(\vartheta) \operatorname{div} \boldsymbol{u},\ \alpha(\vartheta)\chi_2 \in L^2(Q), \tag{2.15}$$
$$\vartheta \alpha''(\vartheta) \vartheta_t \, \chi_2 \operatorname{div} \boldsymbol{u} \ + \ \big(\vartheta \alpha'(\vartheta) - \alpha(\vartheta)\big) \chi_{2,t} \operatorname{div} \boldsymbol{u}$$
$$+ \ \vartheta \alpha'(\vartheta) \chi_2 \operatorname{div} \boldsymbol{u}_t \ \in\ L^2(0,T;(H^1(\Omega))')), \tag{2.16}$$

then expressions (2.6)-(2.11) make sense. Obviously, if we are dealing with a reduced energy balance equation (1.1) where some of the nonlinear terms are omitted, then we will conveniently modify the assumption in (2.16) as well. Similarly, when the *quasistationary* problem in taken into account by omitting the term $\boldsymbol{u}_{tt}$ from (1.2) and (2.8), respectively, then we will just require $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})$ in (2.13) and will not prescribe the initial condition (2.11).

Remark 2.2. It is worth recalling that (1.3), (1.11), and (2.9) are completely equivalent. Indeed, the equivalence between (1.3) and (1.11) relies in the definition of subdifferential (see, e.g., [4, Exemple 2.3.4, p. 25]), while the equivalence between (1.11) and (2.9) is straightforward.

We are now in the position to state our main result.

Theorem 2.3. *Let assumptions (2.1)-(2.5), (2.12)-(2.16) (or the corresponding simplified conditions, respectively) hold. Moreover, let*

$$\vartheta_0 \in L^\infty(\Omega), \quad f \in L^\infty(\partial\Omega \times (0,T)), \tag{2.17}$$
$$F(x,t,r) \le 0 \ \textit{ for a.e. } (x,t) \in Q, \ \ \forall r \ge \vartheta_C, \tag{2.18}$$
$$\alpha(r) = 0 \quad \forall r \in [\vartheta_C, +\infty). \tag{2.19}$$

Then, for any weak solution $(\vartheta, \boldsymbol{u}, \chi_1, \chi_2)$ *of the problem (1.1)-(1.3), (1.6)-(1.10) we have that*

$$\vartheta \;\le\; \max\left\{\|\vartheta_0\|_{L^\infty(\Omega)}; \|f\|_{L^\infty(\partial\Omega)}; \vartheta_C\right\} \quad \textit{a.e. in } \; Q. \tag{2.20}$$

3 Boundedness of the temperature

Proof of Theorem 2.3. Let us start by setting (see (2.17))

$$\vartheta_{\max} := \max\left\{\|\vartheta_0\|_{L^\infty(\Omega)}; \|f\|_{L^\infty(\partial\Omega)}; \vartheta_C\right\} > 0. \tag{3.1}$$

Next, we take (cf. (2.12)) $\varphi = (\vartheta - \vartheta_{\max})^+ = \max\left\{0, \vartheta - \vartheta_{\max}\right\}$ as a test function in (2.7). Owing to condition (2.10) and assumptions (2.18)-(2.19) and (3.1), we easily get

$$\begin{aligned} &\frac{c_0}{2}\|(\vartheta-\vartheta_{\max})^+(t)\|^2 + h\int_0^t \|\nabla((\vartheta-\vartheta_{\max})^+)(s)\|^2 ds \\ &+\eta\int_0^t\int_{\partial\Omega} |(\vartheta-\vartheta_{\max})^+|^2 d\Gamma\, ds \;\le\; L\int_0^t\int_\Omega \chi_{1,t}(\vartheta-\vartheta_{\max})^+ dx\, ds, \end{aligned} \tag{3.2}$$

for all $t \in (0,T)$. Our next aim is to prove that the right hand side above is nonpositive. To this end, let the following subsets of $\mathbb{R}^2$ be defined by

$$\begin{aligned} M_0 &:= \left\{(0,0)\right\}, \\ M_1 &:= \left\{(\gamma_1,\gamma_2) \in \mathbb{R}^2 \text{ such that } 0<\gamma_1<1,\ \gamma_1=\gamma_2\right\}, \\ M_2 &:= \left\{(\gamma_1,\gamma_2) \in \mathbb{R}^2 \text{ such that } 0<\gamma_1<1,\ \gamma_1=-\gamma_2\right\}, \\ M_3 &:= \left\{(\gamma_1,\gamma_2) \in \mathbb{R}^2 \text{ such that } \gamma_1=1,\ -1\le\gamma_2\le 1\right\}, \\ M_4 &:= \operatorname{int}(K) = \left\{(\gamma_1,\gamma_2) \in \mathbb{R}^2 \text{ such that } |\gamma_2|<\gamma_1<1\right\}, \end{aligned}$$

where $\operatorname{int}(K)$ obviously stands for the interior of K. Moreover, fix any $t \in (0,T)$ and consider that (see (2.6))

$$L\int_0^t\int_\Omega \chi_{1,t}(\vartheta-\vartheta_{\max})^+ dx\, ds \;=\; \sum_{i=0}^4 I_i(t) \tag{3.3}$$

where

$$I_i(t) := L\iint_{\mathcal{M}_i(t)} \chi_{1,t}(\vartheta-\vartheta_{\max})\, dx\, ds$$

with $\mathcal{M}_i(t) \subseteq \Omega\times(0,t)$ such that $(\chi_1,\chi_2) \in M_i$ and $\vartheta > \vartheta_{\max}$ a.e in $\mathcal{M}_i(t)$.

Let us start by considering the term $I_0(t)$. Due to the assumption (2.14), it is straightforward to achieve that, since $\chi_1 = 0$ a.e. in $\mathcal{M}_0(t)$ we have that $\chi_{1,t} = 0$ a.e. in $\mathcal{M}_0(t)$ and, consequently, $I_0(t) = 0$.

As regards $I_1(t)$ and $I_2(t)$, arguing as above we achieve that $\chi_{1,t} = \chi_{2,t}$ a.e. in $\mathcal{M}_1(t)$ and $\chi_{1,t} = -\chi_{2,t}$ a.e. in $\mathcal{M}_2(t)$. Moreover, it is a standard matter to observe that, due to (2.19), whenever ϑ is larger than ϑ_C the order parameters χ_1 and χ_2 solve the almost everywhere variational relation

$$k\partial_t \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + \partial I_K(\chi_1, \chi_2) \ni \begin{pmatrix} -\ell(\vartheta - \vartheta_*) \\ 0 \end{pmatrix}, \tag{3.4}$$

or, equivalently, the variational inequality

$$(\chi_1, \chi_2) \in K \quad \text{a.e. in } Q, \quad k\sum_{j=1}^{2} \partial_t \chi_j (\chi_j - \gamma_j) \le -\ell(\vartheta - \vartheta_*)(\chi_1 - \gamma_1)$$
$$\text{a.e. in } Q \cap \{\vartheta > \vartheta_{\max}\}, \ \forall (\gamma_1, \gamma_2) \in K. \tag{3.5}$$

Thus, choosing $\gamma_1 = \chi_1$ in the previous inequality, and accounting for the assumption (1.4) we easily obtain that $\chi_{2,t} \le 0$ a.e. in $\mathcal{M}_1(t)$ while $\chi_{2,t} \ge 0$ a.e. in $\mathcal{M}_2(t)$. Finally, the considerations above ensure that both $I_1(t)$ and $I_2(t)$ are nonpositive.

Again, due to the special form of K, by choosing $\chi_1 = 1$ and $\gamma_2 = \chi_2$ in (3.5) it is straightforward to conclude that $\chi_{1,t} \le 0$ a.e. in $\mathcal{M}_3(t)$, whence, it easily follows that $I_3(t) \le 0$.

As regards $I_4(t)$, one observes that, whenever the set $\mathcal{M}_4(t)$ is taken into account, the pointwise variational inequality (3.5) reduces to the almost everywhere equality

$$k\partial_t \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} -\ell(\vartheta - \vartheta_*) \\ 0 \end{pmatrix} \quad \text{a.e. in } \mathcal{M}_4(T).$$

Indeed, from the definition of subdifferential one easily checks that $\partial I_K\big((\gamma_1, \gamma_2)\big) \equiv (0, 0)$ whenever $(\gamma_1, \gamma_2) \in M_4$. Thus, due to (3.1) and the fact that $\vartheta_* < \vartheta_C$ [3], we obtain

$$I_4(t) = -\frac{L\ell}{k} \iint_{\mathcal{M}_4(t)} (\vartheta - \vartheta_*)(\vartheta - \vartheta_{\max}) dx\, ds \le 0 \quad \text{a.e. in } (0, T). \tag{3.6}$$

Finally, accounting for the above considerations, the inequality (3.2) ensures that

$$(\vartheta - \vartheta_{\max})^+ = 0 \quad \text{a.e. in } Q,$$

and the assertion of Theorem 2.3 follows. □

We conclude by noting that the previous argument applies as well to the case when the energy balance equation (1.1) is simplified by omitting some of its nonlinear terms. Moreover, our proof is completely independent of the form of the momentum balance equation (1.2). In particular, this result applies to all the models discussed in [3, 5, 8, 9, 10, 11].

Remark 3.1. Let us point out that the idea of dealing with the form of K in order to achieve the boundedness of the temperature ϑ has an interest that goes beyond the Frémond model. Indeed, accounting for some intricacy, we may apply a suitably modified version of our argument to a wide class of *multi-phase* relaxation problems, possibly including the system studied in [12].

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