A macroscopic model for magnetic shape-memory single crystals

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Abstract

A rate-independent model for the quasi-static magneto-elastic evolution of a magnetic shape memory single crystal is presented. In particular, the purely mechanical Souza-Auricchio model for shape memory alloys is here combined with classical micro-magnetism by suitably associating magnetization and inelastic strain. By balancing the effect of conservative and dissipative actions, a nonlinear evolution PDE system of rate-independent type is obtained. We prove the existence of so-called *energetic solutions* to this system. Moreover, we discuss several limits for the model corresponding to parameter asymptotics by means of a rigorous Γ-convergence argument.

Keywords: magnetic shape memory alloys, micro-magnetism, energetic solution, existence, Γ-convergence for rate-independent processes.

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1 Introduction

Magnetic shape memory alloys (MSMAs) are active materials exhibiting an amazing behavior for comparably large recoverable strains can be induced by thermo-mechanical and/or magnetic treatment. As ordinary shape memory alloys, MSMAs are *super-elastic* at low temperatures as strains up to 8% are recovered without plasticization. At higher temperatures they show the classical *shape-memory effect*: after a loading cycle residual inelastic strains can be recovered by a thermal treatment. This behavior is the effect of a structural phase-change in the material from a high symmetry crystallographic variant called *austenite* (predominant at high temperature and low stresses) and many low symmetry variants called *martensites* (predominant at low temperatures or high stresses) [19].

A further property of MSMAs with respect to ordinary SMAs is that they can be mechanically activated *at distance* by applying moderate magnetic fields. This is the so-called *magnetic shape-memory effect* and is caused by the ferromagnetic nature of the crystallographic variants in MSMAs. In particular, the martensitic phase of MSMAs is ferromagnetic: the magnetization is organized in domains in which the magnetization vector tends to align with a preferred direction, the so-called *easy axis*. Upon the application of a magnetic field, a redistribution occurs as domains with specific easy axes are more favorably oriented towards the applied field. This redistribution is operated by domain wall motions, magnetization vector rotation (growth of domains with magnetization direction close to the field direction), and martensitic variant reorientation. This last mechanism is specific of MSMAs and consists in the nucleation of a more favorably magnetically oriented martensitic variant at the expense of others. This, in particular, couples the mechanics of the material with its micro-magnetic state and it is responsible for the occurrence of recoverable magnetically-induced inelastic strains.

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The interest in the possible applications of the unique material behavior of MSMAs is evident. As such, a vast Engineering literature is nowadays available on MSMAs. We shall mention, without claim of completeness, the papers [23, 25, 27, 34]. One has to mention that all models proposed (as well as all applications developed) so far deal specifically with MSMAs single crystals. In fact, the possibility of producing a device (actuator) based on a MSMA polycrystal seems critical due to the extreme brittleness of the materials investigated so far.

In [13] a three-dimensional phenomenological model for MSMA single crystals is advanced. The idea there is to extend to MSMAs the well-admitted Souza-Auricchio model for SMAs [7, 8, 9, 45]. The latter is a phenomenological, internal-variable-type model model capable of describing both the shape memory and the superelastic effect. The model has been originally proposed in the small-strain regime by Souza, Mamiya, & Zouain [45] and then combined with finite elements by Auricchio & Petrini [7, 9]. The interest in this model is motivated by its robustness with respect to parameters and discretizations despite its *simplicity*: in the three-dimensional situation, the constitutive behavior of the specimen is determined by the knowledge of just 8 material parameters (note that linearized thermo-plasticity with linear hardening already requires 5 material parameters). These parameters are directly available for they can be easily fitted from experimental data. The Souza-Auricchio model has been analyzed from the viewpoint of existence and approximation of solutions of the three-dimensional quasi-static evolution problem in [6]. Later on, convergence rates for space-time discretization of the problem were obtained in [37, 38]. Extensions of the original Souza-Auricchio model to incorporate more refined material descriptions in the direction of non-symmetric material behavior [12], residual plasticity [10, 11, 16], finite strains [1, 17, 18], thermal evolution [28, 29, 39, 36], and space discretizations [37, 38] are also available.

The MSMA model in [13] relies on directly linking the magnetization M of the body with its inelastic strain $z = \varepsilon(u) - \mathbb{C}^{-1}\sigma$ (here $\varepsilon(u)$ is the total strain, $\mathbb{C}$ is the elasticity tensor, σ is the stress). In particular, an affine relation $z \mapsto A_z$ which associates with a strain z the corresponding easy axis A_z is introduced. Then, the magnetization is explicitly given by the position $M = \alpha\, m_{\rm sat}\, A_z$ where α is the scalar proportion of domains oriented in the positive direction with respect to A_z. Namely, M is given in terms of other variables and hence does not appear into the final PDE system. This modeling choice may be motivated by invoking some *high magnetic anisotropic behavior* of the MSMA crystal forcing the magnetization vectors to be strongly attached to the easy axes. In particular, magnetization rotation is neglected in [13]. Some mathematical analysis on this model is reported in [13] whereas numerical simulations are presented in [3, 4, 5].

In the present paper, we extend the MSMA model of [13] by allowing for magnetization rotations. In particular, we keep M as an independent state variable and directly combine the Souza-Auricchio model with the classical micro-magnetics theory [14, 32, 33]. The thermomechanical coupling of the model is rendered through the anisotropy magnetic energy term which favors (though not imposing) the alignment of the magnetization M with the specific local easy axis of the martensitic phase. Moreover, dissipative evolution mechanisms are included for both mechanical (as in the Souza-Auricchio model) and magnetic variables (as in ferromagnetism).

Our interest in the present model extension is twofold. On the one hand, we aim at presenting a complete description of the MSMA behavior, taking into account all the relevant phenomenology. In particular, we directly include the treatment on the demagnetization field instead of assuming the existence of a suitable demagnetization tensor as in [13]. This was indeed a major drawback of the simplification from [13] where the demagnetization response of the material was taken to be independent of the mechanical behavior of the body. We complement our modelization from Section 2 with the existence analysis for a suitably weak (*energetic*) solution of the related quasi-static evolution problem in Section 3.

On the other hand, we aim at providing a sort of *cross-validation* of the proposed MSMA model by explicitly proving that the well-validated mechanical Souza-Auricchio and micro-magnetic models can be rigorously obtained from the present complete one by means of parameters asymptotics. This, together with the discussions in [3, 4, 5, 13], shall provide significant evidence of the interest of this perspective. The parameters asymptotics analysis is performed in Section 4 by means of a specific Γ-convergence tool for rate-independent evolution developed in [41].

Before moving on we shall recall the phenomenological models of internal-variable-type for MSMA single crystals by Hirsinger & Lexcellent [22] and Kiefer & Lagoudas [27]. These two models, albeit basically derived from same principles, differ from ours as they are essentially restricted to two dimensions (or two martensitic variants). Both these models assume the *scalar* local proportion of one martensitic variant with respect to the other as an internal variable. Moreover, the choice for the specific energy is these models is different from ours and comparably more complex. One has also to mention that no mathematical results are presently available for these models.

2 Model description

We shall provide in this section some detail on the proposed MSMA model. Let the reference configuration $\Omega \subset \mathbb{R}^3$ be an open bounded set with smooth boundary $\partial\Omega$. We shall use the symbol : for the standard contraction product between matrices whereas $\cdot$ denotes the scalar product of vectors and $|\cdot|$ stands for the corresponding norms.

Let $u : \Omega \to \mathbb{R}^3$ be the displacement of the body from the reference configuration. Working within the small-strain regime, we decompose the linearized strain $\varepsilon(u) = (\varepsilon(u))_{ij} = (u_{i,j} + u_{j,i})/2 \in \mathbb{R}^{3\times 3}_{\rm sym}$ into the elastic part $\varepsilon_{\rm el} \in \mathbb{R}^{3\times 3}_{\rm sym}$ and the inelastic (or transformation) part $z \in \mathbb{R}^{3\times 3}_{\rm dev}$ as

$$\varepsilon = \varepsilon_{\rm el} + z. \tag{2.1}$$

Here, $\mathbb{R}^{3\times 3}_{\rm sym}$ denotes the set of symmetric 3-tensors whereas $\mathbb{R}^{3\times 3}_{\rm dev} \subset \mathbb{R}^{3\times 3}_{\rm sym}$ is the subset of deviatoric symmetric tensors. The inelastic strain z describes the strain associated with the transformations between the *parent phase* (austenite and twinned martensite) and the *product phase* (detwinned or single-variant martensite).

The magnetization of the body is described by $M : \Omega \to \mathbb{R}^3$ subject to the constraint

$$|M(x)| = m_{\rm sat} \quad \text{a.e. } x \in \Omega,$$

where $m_{\rm sat} > 0$ is the saturation magnetization of martensites, assumed to be constant for all variants.

The phase state of the material is described by means of a vector of phase proportions $\lambda \in \mathbb{R}^m$ whose components are subject to the simplicial constraint

$$\lambda \in \Lambda = \big\{\lambda \in \mathbb{R}^m \ : \ \lambda_i \geq 0, \ \lambda_1 + \cdots + \lambda_m = 1\big\}.$$

The scalar λ_i represents the proportion of the i-th martensitic phase. Note that, for simplicity, by assuming a suitable low temperature, no austenite will be considered in this model. We focus here on a single crystal of an alloy presenting a *magnetically uniaxial* martensitic structure. Namely, we assume that each martensite presents a single easy axis. This is common to cubic-to-tetragonal ($m = 3$) and cubic-to-orthorhombic ($m = 6$) systems such as Ni_2MnGa, FePd, and FePt among many others.

2.1 Mechanical modeling

We recall here the basic assumptions of the purely mechanical constitutive model for shape memory alloys introduced by Souza, Mamiya, & Zouain [45] and later refined by Auricchio & Petrini [7, 8, 9].

In the Souza-Auricchio model, for suitably low temperatures (that we consider to be fixed throughout) the mechanical free energy of the body is simply given by the plasticity-like expression

$$\psi_{\rm SA}(\varepsilon, z) = \frac{1}{2}(\varepsilon - z){:}\mathbb{C}{:}(\varepsilon - z) + \frac{h}{2}|z|^2.$$

Here, $\mathbb{C}$ stands for the fourth-order symmetric elasticity tensor, $h > 0$ is a hardening modulus, and $\varepsilon_L > 0$ is the maximal strain modulus obtained by alignment of martensitic variants. Given the phase proportion $\lambda \in \Lambda$, we shall impose the transformation strain to take the explicit form

$$z_\lambda = \lambda_1 z_1 + \cdots + \lambda_m z_m \tag{2.2}$$

where z_i are the transformation strains related to pure variants. In particular, for cubic-to-tetragonal systems, by fixing a frame aligned to the reference austenitic cubic structure, we have

$$z_1 = \frac{\varepsilon_L}{\sqrt{6}} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z_2 = \frac{\varepsilon_L}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z_3 = \frac{\varepsilon_L}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

whereas cubic-to-orthorhombic variants correspond to

$$z_1 = \frac{\varepsilon_L}{c_\gamma} \begin{pmatrix} 1+\gamma & 0 & 1-\gamma \\ 0 & -2-2\gamma & 0 \\ 1-\gamma & 0 & 1+\gamma \end{pmatrix}, \quad z_2 = \frac{\varepsilon_L}{c_\gamma} \begin{pmatrix} 1+\gamma & 0 & \gamma-1 \\ 0 & -2-2\gamma & 0 \\ \gamma-1 & 0 & 1+\gamma \end{pmatrix},$$

$$z_3 = \frac{\varepsilon_L}{c_\gamma} \begin{pmatrix} 1+\gamma & 1-\gamma & 0 \\ 1-\gamma & 1+\gamma & 0 \\ 0 & 0 & -2-2\gamma \end{pmatrix}, \quad z_4 = \frac{\varepsilon_L}{c_\gamma} \begin{pmatrix} 1+\gamma & \gamma-1 & 0 \\ \gamma-1 & 1+\gamma & 0 \\ 0 & 0 & -2-2\gamma \end{pmatrix},$$

$$z_5 = \frac{\varepsilon_L}{c_\gamma} \begin{pmatrix} -2-2\gamma & 0 & 0 \\ 0 & 1+\gamma & 1-\gamma \\ 0 & 1-\gamma & 1+\gamma \end{pmatrix}, \quad z_6 = \frac{\varepsilon_L}{c_\gamma} \begin{pmatrix} -2-2\gamma & 0 & 0 \\ 0 & 1+\gamma & \gamma-1 \\ 0 & \gamma-1 & 1+\gamma \end{pmatrix}$$

where $\gamma > 0$ is a specific alloy-dependent crystallographic parameter and $c_\gamma = \sqrt{8(1+\gamma+\gamma^2)}$ is a normalization constant letting $|z_i| = \varepsilon_L$. Note that, by assuming $|z_i| = \varepsilon_L$, one readily deduces that $|z_\lambda| = |\lambda_1 z_1 + \cdots + \lambda_m z_m| \leq \varepsilon_L$ and that equality in the latter occurs for pure variants only.

The dissipative mechanical behavior of the body for the Souza-Auricchio model is described by the dissipation (pseudo-)potential $R_{\mathrm{SA}}|\dot{z}_\lambda|$ where $R_{\mathrm{SA}} > 0$ is a transformation radius. By exploiting the constraint (2.2) we hence specify the mechanical dissipation of our model as

$$R^\lambda|\dot{\lambda}|$$

where R^λ is directly computed from R_{SA} (in three dimensions) as $R^\lambda = \sqrt{3/2}\varepsilon_L R_{\mathrm{SA}}$.

Eventually, the purely mechanical constitutive equations for the material are given by the system

$$\sigma = \mathbb{C}(\varepsilon(u) - z_\lambda), \tag{2.3}$$

$$\partial R^\lambda|\dot{\lambda}| + \partial_\lambda \psi_{\mathrm{SA}}(\varepsilon(u), z_\lambda) \ni 0. \tag{2.4}$$

Here and in the following the symbol ∂ stands for the classical (and possibly partial) subdifferential in the sense of convex analysis.

2.2 Micro-magnetic modeling

In the classical theory of micro-magnetism [14, 31, 32, 33] a specific energy term is associated with each phenomenon occurring in the magnetization of the material. In particular, given the external magnetic field $H(t)$, we let the *magnetic energy* be defined as

$$\Psi_{\mathrm{magn}}(t, M) = \frac{\mu_0}{2} \int_{\mathbb{R}^3} |\nabla v_M|^2 \mathrm{d}x - \mu_0 \int_\Omega H(t) \cdot M \mathrm{d}x + K_M \int_\Omega |\nabla M|^2 \mathrm{d}x. \tag{2.5}$$

The first term in the above right-hand side is the *magneto-static energy* and corresponds to the energy of the *demagnetization field* ∇v_M generated by the magnetization M in all of $\mathbb{R}^3$. The constant μ_0 is the vacuum magnetic permeability and the *potential* v_M is governed by the Maxwell equation

$$\mathrm{div}(-\mu_0\nabla v_M + M\chi_\Omega) = 0 \ \text{ in } \mathbb{R}^3 \tag{2.6}$$

where $\chi_\Omega : \mathbb{R}^3 \to \{0, 1\}$ is the characteristic function of Ω (namely $\chi_\Omega = 1$ in Ω and $\chi_\Omega = 0$ elsewhere). The magnetic body tries to minimize it by arranging its domain structure in such a way as to produce a minimal exterior field. In the sequel, (2.6) is considered in the weak form

$$v \in H^1(\mathbb{R}^3) \ \text{ is such that } \ \mu_0 \int_{\mathbb{R}^3} \nabla v{\cdot}\nabla\phi \,\mathrm{d}x = \int_\Omega M{\cdot}\nabla\phi \,\mathrm{d}x \ \ \forall\,\phi \in H^1(\mathbb{R}^3). \tag{2.7}$$

The second term in the right-hand side of (2.5) is the *Zeeman energy* encoding the interaction between the applied external field H and the magnetization M. Finally, the last term in (2.5) is the *exchange energy* ($K_M > 0$ is the *exchange* constant) and describes the ability of the material to create domain structure by penalizing spatial changes of M.

The magnetization M is assumed to present a dissipative dynamics driven by the (pseudo-)potential of dissipation on $L^1(\Omega, \mathbb{R}^3)$

$$R^M \int_\Omega |\dot M| \mathrm{d}x$$

where $R^M > 0$ is a suitable activation radius. Eventually, the purely mechanical constitutive relation for the materials is given by the inclusion

$$\partial \left(R^M \int_\Omega |\dot M| \mathrm{d}x \right) + \partial_M \Psi_{\mathrm{magn}}(t, M) \ni 0.$$

where the subdifferential is now taken with respect to the $L^2(\Omega; \mathbb{R}^3)$ metric.

2.3 Magneto-mechanical coupling

The anisotropic behavior of the magnetization in MSMAs plays the role of coupling magnetic and mechanical effects. Each martensitic variant is associated with a preferred magnetization direction and is hence favored by the application of a specific magnetic field. This results in a magnetically induced martensitic reorientation effect. On the contrary, by mechanically reorienting the martensitic phase pattern of the medium one is changing its anisotropic magnetic response.

Given the phase proportion $\lambda \in \Lambda$, we let the (directed) easy axis of the material be defined as

$$A_\lambda = A_1\lambda_1 + \cdots + A_m\lambda_m$$

where A_i corresponds to the easy axis of the i-th variant.

The *anisotropy magnetic energy* is minimized by developing a domain structure such that the magnetization is preferably aligned with the easy axis. Hence, the anisotropy energy term is chosen as

$$-\mu_0 K_{\mathrm{ani}} \int_\Omega (M{\cdot}A_\lambda)^2 \mathrm{d}x,$$

$K_{\mathrm{ani}} > 0$ being the anisotropy constant. Note that $|M{\cdot}A_\lambda| \le |M||A_\lambda| \le |M|$ so that the anisotropy magnetic energy term is minimized for $M = m_{\mathrm{sat}}\, A_i$ and $\lambda = \lambda_i$.

2.4 Total energy and dissipation

We shall be concerned with a quasi-static evolution of the MSMA sample, namely

$$\nabla \cdot \sigma + f = 0 \quad \text{in} \quad \Omega \tag{2.8}$$

for a given body force $f : \Omega \to \mathbb{R}^3$ along with the boundary conditions

$$u = 0 \quad \text{on} \ \ \Gamma_0, \qquad \sigma n = g \quad \text{on} \ \ \Gamma_{\text{tr}}. \tag{2.9}$$

Here, $\Gamma_0 \subset \partial\Omega$ has a positive surface measure, n is the unit external normal to the boundary $\partial\Omega$, $\Gamma_{\text{tr}} = \partial\Omega \setminus \Gamma_0$, and $g : \Gamma_{\text{tr}} \to \mathbb{R}^3$ is a given traction.

To this aim, we consider the admissible set of displacements

$$\mathcal{U} := \big\{ u \in H^1(\Omega, \mathbb{R}^3) : u = 0 \text{ on } \Gamma_0 \big\},$$

we denote by $\mathcal{U}'$ its dual, and by $\langle \cdot, \cdot \rangle$ the corresponding duality pairing. As for phase proportions, we will let λ belong to

$$\mathcal{L} := \big\{ \lambda \in BV(\Omega, \mathbb{R}^m) : \lambda \in \Lambda \text{ a.e. in } \Omega \big\}.$$

Then, we associate with $\lambda \in \mathcal{L}$ the inelastic strain z_λ defined by (2.2). We also consider the admissible set for the magnetizations to be defined by

$$\mathcal{M} := \big\{ M \in H^1(\Omega, \mathbb{R}^3) : |M| = m_{\text{sat}} \text{ a.e. in } \Omega \big\}.$$

By letting $t \in [0,T] \mapsto f(t) \in L^2(\Omega; \mathbb{R}^3)$ and $t \in [0,T] \mapsto g(t) \in L^2(\Gamma_{\text{tr}}; \mathbb{R}^3)$ be given, we classically define the total load $\ell : [0,T] \to \mathcal{U}'$ by

$$\langle \ell(t), u \rangle := \int_\Omega f(t) \cdot u \, \mathrm{d}x + \int_{\Gamma_{\text{tr}}} g(t) \cdot u \, \mathrm{d}\Gamma \quad \forall u \in \mathcal{U}, \ t \in [0,T].$$

Moreover, we assume to be given the magnetic field $t \in [0,T] \mapsto H(t) \in L^2(\Omega; \mathbb{R}^3)$.

According to the above discussion, the *total energy* we are concerned with is given by the functional $\Psi : [0,T] \times \mathcal{U} \times \mathcal{L} \times \mathcal{M} \longrightarrow \mathbb{R}$ as

$$\begin{aligned}
\Psi(t, u, \lambda, M) = & \int_\Omega \psi_{\text{SA}}(\varepsilon(u), z_\lambda) \mathrm{d}x - \langle \ell(t), u \rangle \\
& + \frac{\mu_0}{2} \int_{\mathbb{R}^3} |\nabla v_M|^2 \mathrm{d}x - \mu_0 \int_\Omega H(t) \cdot M \mathrm{d}x + K_M \int_\Omega |\nabla M|^2 \mathrm{d}x \\
& - \mu_0 K_{\text{ani}} \int_\Omega (M \cdot A_\lambda)^2 \mathrm{d}x + K_\lambda \int_\Omega |\nabla \lambda|.
\end{aligned}$$

The total variation of $\nabla\lambda$ has been explicitly included in the energy in order to give rise to a scale effect (modulated by the positive constant K_λ) with the aim of penalizing martensitic phase boundaries and possibly describing the occurrence of a specific twinning length scale. Let us remark that, clearly, λ needs not be continuous and sharp interfaces are admitted. The occurrence of gradient terms in phenomenological models for SMAs is not new and the reader is referred to Frémond [19] Arndt, Griebel, & Roubíček [2], Fried & Gurtin [20], [30], Mielke & Roubíček [40], Roubíček [43, 44], and [21] for examples and discussions on non-local energy contributions.

The dissipation (pseudo-)potential is given by

$$R(\dot{\lambda}, \dot{M}) = R^\lambda \int_\Omega |\dot{\lambda}| \mathrm{d}x + R^M \int_\Omega |\dot{M}| \mathrm{d}x$$

and is naturally related to the dissipation distance defined by

$$\mathcal{D}(q, \widehat{q}) := R^\lambda \int_\Omega |\lambda - \widehat{\lambda}| \mathrm{d}x + R^M \int_\Omega |M - \widehat{M}| \mathrm{d}x,$$

for all q, $\widehat{q} \in \mathcal{L} \times \mathcal{M}$ where, in order to shorten notation (here and in the following), we set $q := (\lambda, M)$ for all $(\lambda, M) \in \mathcal{L} \times \mathcal{M}$.

We shall mention that the dissipation associated with domain wall motion and magnetization rotation is usually very small in MSMAs and hence sometimes neglected [15, 26] when compared with the mechanical dissipation due to variant reorientation. Namely R^M is often taken to be 0 in the latter. We shall explicitly consider this situation in Section 4.3.

Before closing this section, let us record that the material constitutive relations system (2.3)-(2.4), (2.2) together with quasi-static equilibrium (2.8)-(2.9) for the complete MSMA model can be rewritten in the compact form

$$\partial_u \Psi(t, u, \lambda, M) = 0 \tag{2.10}$$

$$\partial_{\dot{\lambda}} R(\dot{\lambda}, \dot{M}) + \partial_\lambda \Psi(t, u, \lambda, M) \ni 0. \tag{2.11}$$

$$\partial_{\dot{M}} R(\dot{\lambda}, \dot{M}) + \partial_M \Psi(t, u, \lambda, M) \ni 0 \tag{2.12}$$

where subdifferentials are here taken with respect to the corresponding L^2 metrics.

3 Quasi-static evolution problem

We shall be concerned here with the solvability of the quasi-static equilibrium problem associated with the above introduced material model. In particular, upon suitable data qualification, we focus on a suitably weak notion of solution: the so-called *energetic formulation* (see [42]). Being given $t \in [0, T] \mapsto (f(t), g(t), H(t))$ and an initial datum (u^0, λ^0, M^0) an *energetic solution* of the quasi-static problem is a function $t \in [0, T] \mapsto (u(t), \lambda(t), M(t)) \in \mathcal{U} \times \mathcal{L} \times \mathcal{M}$ such that $(u(0), z(0), M(0)) = (u^0, z^0, M^0)$ and, for every $t \in [0, T]$,

$$\Psi(t, u(t), \lambda(t), M(t)) \leq \Psi(t, \widehat{u}, \widehat{\lambda}, \widehat{M}) + \mathcal{D}(q(t), \widehat{q}) \quad \forall\, (\widehat{u}, \widehat{\lambda}, \widehat{M}) \in \mathcal{U} \times \mathcal{L} \times \mathcal{M} \tag{S}$$

$$\begin{aligned} &\Psi(t, u(t), \lambda(t), M(t)) + \mathrm{Diss}_{\mathcal{D}}(q, [0, t]) \\ &\quad = \Psi(0, u^0, \lambda^0, M^0) - \int_0^t \langle \dot{\ell}(s), u(s) \rangle \mathrm{d}s - \mu_0 \int_0^t \int_\Omega \dot{H}(s) \cdot M(s) \mathrm{d}x \, \mathrm{d}s \end{aligned} \tag{E}$$

where $\mathrm{Diss}_{\mathcal{D}}(q, [0, t])$ is the total dissipation on $[0, t]$ defined by

$$\mathrm{Diss}_{\mathcal{D}}(q, [0, t]) := \sup \left\{ \sum_{i=1}^N \mathcal{D}(q(t^i), q(t^{i-1})) : \{0 = t^0 < t^1 < \ldots < t^N = t\} \right\}, \tag{3.1}$$

the supremum being taken over all the partitions of $[0, t]$.

For later convenience, we define the set of *stable states* at time $t \in [0, T]$ as

$$\mathcal{S}(t) := \Big\{ (u, \lambda, M) \in \mathcal{U} \times \mathcal{L} \times \mathcal{M} : \ \Psi(t, u, \lambda, M) \leq \Psi(t, \widehat{u}, \widehat{\lambda}, \widehat{M}) + \mathcal{D}(q, \widehat{q}) \quad \forall\, (\widehat{u}, \widehat{\lambda}, \widehat{M}) \in \mathcal{U} \times \mathcal{L} \times \mathcal{M} \Big\}.$$

Our existence result reads as follows.

Theorem 3.1 (Existence for the quasi-static evolution)**.** *By assuming $f \in W^{1,1}(0, T; L^2(\Omega, \mathbb{R}^3))$, $g \in W^{1,1}(0, T; L^2(\Gamma_{\rm tr}, \mathbb{R}^3))$, $H \in W^{1,1}(0, T; L^2(\Omega, \mathbb{R}^3))$, and $(u^0, \lambda^0, M^0) \in \mathcal{S}(0)$, there exists an energetic solution for the quasi-static evolution problem.*

We shall not report here a full proof of Theorem 3.1 as it may be readily obtained in the frame of the by now classical existence theory for energetic solutions by Mielke & Theil [35, 42]. We limit ourselves in remarking that the sublevels of the energy $\Psi(t, \cdot)$ are bounded in

$$H^1(\Omega; \mathbb{R}^3) \times \big(BV(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m) \big) \times \big(H^1(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3) \big)$$

hence compact with respect to the strong topology in $L^2(\Omega;\mathbb{R}^3)\times L^p(\Omega;\mathbb{R}^m)\times L^p(\Omega;\mathbb{R}^3)$ for all $p<\infty$. In particular, the magneto-mechanical coupling term turns out to be continuous along energy-bounded sequences converging with respect to the latter strong topology. The same holds true for the magneto-static energy. Indeed, fix $M\in\mathcal{M}$ and let $\varphi_k\in C_0^\infty(\mathbb{R}^3)$ converge to v_M strongly in $H^1(\mathbb{R}^3)$. We have

$$\begin{aligned}\mu_0\|\nabla v_M\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} &= \lim_{k\to\infty}\mu_0\int_{\mathbb{R}^3}\nabla v_M\cdot\nabla\varphi_k\mathrm{d}x=\lim_{k\to\infty}\int_\Omega M\cdot\nabla\varphi_k\mathrm{d}x\\ &\leq \lim_{k\to\infty}\|M\|_{L^2(\Omega;\mathbb{R}^3)}\|\nabla\varphi_k\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}=\|M\|_{L^2(\Omega;\mathbb{R}^3)}\|\nabla v_M\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}.\end{aligned}$$

Hence, the linear mapping $M\in\mathcal{M}\subset L^2(\Omega;\mathbb{R}^3)\mapsto\nabla v_M\in L^2(\mathbb{R}^3;\mathbb{R}^3)$ is continuous. In particular, if $M_n\in\mathcal{M}\to M$ almost everywhere then $\nabla v_{M_n}\to\nabla v_M$ strongly in $L^2(\Omega;\mathbb{R}^3)$ and the lower-semicontinuity of the magneto-static energy term follows.

It is well-known that the energetic formulation of a rate-independent evolution has to be handled cautiously out of the strictly convex energy case. Still, we aim at motivating our choice by stressing the relevance of the energetic solubility notion as the natural limit within time-discrete approximation procedures. In particular, we identify time partitions of $[0,T]$ with vectors of positive time steps $\boldsymbol{\tau}$. More precisely, given $\boldsymbol{\tau}=(\tau_1,\dots,\tau_{N_{\boldsymbol{\tau}}})$, the partition is

$$P_{\boldsymbol{\tau}}=\{0=t^0_{\boldsymbol{\tau}}<t^1_{\boldsymbol{\tau}}<\ldots<t^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}}=T\}\ \text{ with }\ t^0_{\boldsymbol{\tau}}:=0,\quad t^i_{\boldsymbol{\tau}}:=t^{i-1}_{\boldsymbol{\tau}}+\tau_i,\quad i=1,\dots,N_{\boldsymbol{\tau}}.$$

For all vectors $(w^0,\dots,w^{N_{\boldsymbol{\tau}}})$, we define the constant interpolation function $w_{\boldsymbol{\tau}}$ on $[0,T]$ as

$$w_{\boldsymbol{\tau}}(t):=w^i\ \text{ for } t\in[t^i_{\boldsymbol{\tau}},t^{i+1}_{\boldsymbol{\tau}}),\ i=0,\dots,N_{\boldsymbol{\tau}}-1.$$

Assume now to be given a sequence of partitions $\boldsymbol{\tau}_k$ with diameters $|\boldsymbol{\tau}_k|\to0$. The classical time-discretization scheme for (2.10)-(2.12) is the following

$$\begin{aligned}&(u^0_k,\lambda^0_k,M^0_k)=(u^0,\lambda^0,M^0)\\ &(u^i_k,\lambda^i_k,M^i_k)\in\operatorname{Arg\,min}\Big\{\Psi(t^i_{\boldsymbol{\tau}_k},\widehat{u},\widehat{\lambda},\widehat{M})+\mathcal{D}(q^{i-1}_k,\widehat{q})\ :\ (\widehat{u},\widehat{\lambda},\widehat{M})\in\mathcal{U}\times\mathcal{L}\times\mathcal{M}\Big\}\\ &\qquad\qquad\text{for }\ i=1,\dots,N_{\boldsymbol{\tau}_k}\end{aligned}$$

The above incremental problems are solvable as the functional under consideration is, as already commented, both coercive and lower semicontinuous with respect to the strong topology in $L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^m\times\mathbb{R}^3)$. We shall remark that a by-product of the existence proof for energetic solutions is the following convergence result for the above time-discrete scheme.

Corollary 3.2 (Convergence of time-discretizations)**.** *Under the assumptions of Theorem* 3.1 *we have that, for all* $t\in[0,T]$,

(i) $u_{\boldsymbol{\tau}_k}(t)\to u(t)$ *weakly in* $H^1(\Omega,\mathbb{R}^3)$,

(ii) $\lambda_{\boldsymbol{\tau}_k}(t)\to\lambda(t)$ *weakly star in* $BV(\Omega,\mathbb{R}^m)\cap L^\infty(\Omega,\mathbb{R}^m)$ *and strongly in* $L^p(\Omega,\mathbb{R}^m)$ *for all* $p<\infty$,

(iii) $M_{\boldsymbol{\tau}_k}(t)\to M(t)$ *weakly in* $H^1(\Omega,\mathbb{R}^3)$, *weakly star in* $L^\infty(\Omega,\mathbb{R}^3)$, *and strongly in* $L^p(\Omega,\mathbb{R}^3)$ *for all* $p<\infty$,

(iv) $\mathrm{Diss}_{\mathcal{D}}(q_{\boldsymbol{\tau}_k},[0,t])\to\mathrm{Diss}_{\mathcal{D}}(q,[0,t])$,

(v) $\Psi(t,u_{\boldsymbol{\tau}_k}(t),\lambda_{\boldsymbol{\tau}_k}(t),M_{\boldsymbol{\tau}_k}(t))\to\Psi(t,u(t),\lambda(t),M(t))$,

(vi) $\langle\dot{\ell},u_{\boldsymbol{\tau}_k}\rangle+\mu_0\int_\Omega\dot{H}\cdot M_{\boldsymbol{\tau}_k}\mathrm{d}x\to\langle\dot{\ell},u\rangle+\mu_0\int_\Omega\dot{H}\cdot M\mathrm{d}x$ *in* $L^1(0,T)$

where $t\in[0,T]\mapsto(u(t),\lambda(t),M(t))$ *is an energetic solution of the quasi-static evolution problem.*

4 Model asymptotics

We shall present here a sort of cross-validation of the proposed MSMA model by proving rigorously some asymptotic result on parameters. In particular, we focus on the limits $R^M \to \infty$, $R^\lambda \to \infty$. By letting the activation radius R^M grow to ∞ we prove that the complete MSMA model reduces to the mechanical Souza-Auricchio model. On the other hand, the limit $R^\lambda \to \infty$ gives back the micro-magnetic model.

Eventually, as the mechanical dissipation in MSMA is generally much more relevant than the magnetic one [46], we explicitly consider the limit $R^M \to 0$ in Subsection 4.3.

The asymptotic analysis is performed within the general frame of the Γ-convergence theory adapted to rate-independent systems from [41]. Loosely speaking, the main result of [41] consists in a convergence theorem for energetic solutions of approximating problems driven by energy and dissipation functionals $(\Psi_n, \mathcal{D}_n)$ to an energetic solution of the limiting problem $(\Psi_\infty, \mathcal{D}_\infty)$. It is rather easy to observe that the sole Γ-convergence of energies and dissipations is not sufficient in order to conclude for the convergence of the respective energetic solutions and one is forced to ask for the upper semicontinuity of the sets of stable states. This latter semicontinuity condition turns out to be the restrictive condition with respect to applications and one is generally asked to construct a so-called *mutual recovery sequence*. We shall provide the necessary details of these constructions in the following subsections. In particular, some slight modification of the original argument of [41] is reported in Subsection 4.3.

In the forthcoming of this section, given any parameter-dissipation distance $\mathcal{D}_k : (\mathcal{L} \times \mathcal{M})^2 \to [0, \infty]$ for $k \in \mathbb{N}$, we define the set of *stable states related to* $(\Psi, \mathcal{D}_k)$ *at time* $t \in [0, T]$ as the set

$$\Big\{(u, \lambda, M) \in \mathcal{U} \times \mathcal{L} \times \mathcal{M} : \ \Psi(t, u, \lambda, M) < \infty$$
$$\text{and} \ \ \Psi(t, u, \lambda, M) \le \Psi(t, \widehat{u}, \widehat{\lambda}, \widehat{M}) + \mathcal{D}_k(q, \widehat{q}) \quad \forall (\widehat{u}, \widehat{\lambda}, \widehat{M}) \in \mathcal{U} \times \mathcal{L} \times \mathcal{M}\Big\}.$$

Moreover, for any $l \mapsto k_l$ non-decreasing, we say that $(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l})$ is a *stable sequence related to* $(\Psi, \mathcal{D}_k)$ if

$$(u_{k_l}, \lambda_{k_l}, M_{k_l}) \ \text{ is stable related to } (\Psi, \mathcal{D}_{k_l}) \text{ at } t_l \ \text{ and } \ \sup_{l \in \mathbb{N}} \Psi(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l}) < \infty.$$

Finally, we say that (u, λ, M) is an *energetic solution related to* $(\Psi, \mathcal{D}_k)$ *and the initial datum* (u^0, λ^0, M^0) if $(u(0), \lambda(0), M(0)) = (u^0, \lambda^0, M^0)$ and, for all $t \in [0, T]$, $(u(t), \lambda(t), M(t))$ is stable related to $(\Psi, \mathcal{D}_k)$ and the energy equality

$$\Psi(t, u(t), \lambda(t), M(t)) + \mathrm{Diss}_{\mathcal{D}_k}(q, [0, t])$$
$$= \Psi(0, u^0, \lambda^0, M^0) - \int_0^t \langle \dot{\ell}(s), u(s) \rangle \mathrm{d}s - \mu_0 \int_0^t \int_\Omega \dot{H}(s) \cdot M(s) \mathrm{d}x \, \mathrm{d}s$$

holds where $\mathrm{Diss}_{\mathcal{D}_k}(q, [0, t])$ is the total dissipation on $[0, t]$ defined by (3.1) with $\mathcal{D}_k$ instead of $\mathcal{D}$.

4.1 Mechanical limit: $R^M \to \infty$

Let $(u^0, \lambda^0, M^0) \in \mathcal{S}(0)$. We recall that $t \mapsto (u(t), \lambda(t))$ is an energetic solution for the mechanical Souza-Auricchio model (for some fixed magnetization M^0) if $(u(0), \lambda(0)) = (u^0, \lambda^0)$ and, for every $t \in [0, T]$, the following stability condition and energy balance hold

$$\begin{aligned}
&\int_\Omega \psi_{\rm SA}(u(t),z_\lambda(t)){\rm d}x-\mu_0K_{\rm ani}\int_\Omega (M^0{\cdot}A_\lambda)^2{\rm d}x+K_\lambda\int_\Omega|\nabla\lambda(t)|-\langle\ell(t),u(t)\rangle\\
&\leq\int_\Omega \psi_{\rm SA}(\widehat u,z_{\widehat\lambda})-\mu_0K_{\rm ani}\int_\Omega (M^0{\cdot}A_{\widehat\lambda})^2{\rm d}x+K_\lambda\int_\Omega|\nabla\widehat\lambda|-\langle\ell(t),\widehat u\rangle+\mathcal{D}_{\rm mech}(\lambda(t),\widehat\lambda)\\
&\qquad\forall\,(\widehat u,\widehat\lambda)\in\mathcal{U}\times\mathcal{L} \qquad\qquad ({\rm S}_{\rm mech})\\
&\int_\Omega \psi_{\rm SA}(u(t),z_\lambda(t))-\mu_0K_{\rm ani}\int_\Omega (M^0{\cdot}A_{\lambda(t)})^2{\rm d}x+K_\lambda\int_\Omega|\nabla\lambda(t)|-\langle\ell(t),u(t)\rangle+{\rm Diss}_{\mathcal{D}_{\rm mech}}(\lambda,[0,t])\\
&=\int_\Omega \psi_{\rm SA}(u^0,\lambda^0)-\mu_0K_{\rm ani}\int_\Omega (M^0{\cdot}A_{\lambda^0})^2{\rm d}x+K_\lambda\int_\Omega|\nabla\lambda^0|-\langle\ell(0),u^0\rangle-\int_0^t\langle\dot\ell(s),u(s)\rangle{\rm d}s \qquad ({\rm E}_{\rm mech})
\end{aligned}$$

where we have denoted the *mechanical* dissipation distance by

$$\mathcal{D}_{\rm mech}(\lambda,\widehat\lambda):=R^\lambda\int_\Omega|\lambda-\widehat\lambda|{\rm d}x,$$

and defined the total dissipation ${\rm Diss}_{\mathcal{D}_{\rm mech}}$ as in (3.1) with $\mathcal{D}_{\rm mech}$ instead of $\mathcal{D}$.

Let us denote by $\mathcal{D}^k_\lambda$ the dissipation distance defined by

$$\mathcal{D}^k_\lambda(q,\widehat q)=R^\lambda\int_\Omega|\lambda-\widehat\lambda|{\rm d}x+k\int_\Omega|M-\widehat M|{\rm d}x$$

for every $q=(\lambda,M)$, $\widehat q=(\widehat\lambda,\widehat M)\in\mathcal{L}\times\mathcal{M}$, and consider the dissipation distance $\mathcal{D}^\infty_\lambda:\mathcal{L}\times\mathcal{M}\longrightarrow[0,\infty]$ defined by

$$\mathcal{D}^\infty_\lambda(q,\widehat q):=\begin{cases}\mathcal{D}_{\rm mech}(\lambda,\widehat\lambda) & \text{if } M=\widehat M\\ \infty & \text{else.}\end{cases}$$

Let f, g, and H be given as in Theorem 3.1 and denote by $\mathcal{S}^k_\lambda$ the set of stable states related to $(\Psi,\mathcal{D}^k_\lambda)$. As we have assumed that $(u^0,\lambda^0,M^0)\in\mathcal{S}(0)\subset\mathcal{S}^k_\lambda(0)$ (definitely in k), Theorem 3.1 entails the existence of an energetic solution (u_k,λ_k,M_k) related to $(\Psi,\mathcal{D}^k_\lambda)$ and the initial datum (u^0,λ^0,M^0) for all k large enough. The main result of this subsection is the following.

Theorem 4.1 (Mechanical limit). *Under the assumptions of Theorem 3.1, let* $(u^0,\lambda^0,M^0)\in\mathcal{S}(0)$ *and* (u_k,λ_k,M_k) *be an energetic solution related to* $(\Psi,\mathcal{D}^k_\lambda)$ *and the initial datum* (u^0,λ^0,M^0). *Then,* (u_k,λ_k,M_k) *converges pointwise (up to some not relabeled subsequence) to* (u,λ,M^0) *and* $t\mapsto(u(t),\lambda(t))$ *is an energetic solution of the mechanical Souza-Auricchio problem, namely, a solution of* (${\rm S}_{\rm mech}$)-(${\rm E}_{\rm mech}$).

Proof. The assertion follows by an application of [41, Thm. 3.1] once we prove the following three facts:

- Γ-liminf inequality for Ψ:

$$\Psi(t,u,\lambda,M)\leq\inf\Big\{\liminf_{l\to\infty}\Psi(t_l,u_{k_l},\lambda_{k_l},M_{k_l})\ :\ (t_l,u_{k_l},\lambda_{k_l},M_{k_l})\text{ is a stable sequence related to }(\Psi,\mathcal{D}^{k_l}_\lambda)\text{ and }(t_l,u_{k_l},\lambda_{k_l},M_{k_l})\to(t,u,\lambda,M)\Big\}. \tag{4.1}$$

- Γ-liminf inequality for $\mathcal{D}^k_\lambda$:

$$\mathcal{D}^\infty_\lambda(q,\widehat q)\leq\inf\Big\{\liminf_{l\to\infty}\mathcal{D}^{k_l}_\lambda(q_{k_l},\widehat q_{k_l})\ :\ (t_l,q_{k_l})\to(t,q),\ (\widehat t_l,\widehat q_{k_l})\to(\widehat t,\widehat q)\Big\}. \tag{4.2}$$

- Upper semicontinuity of stable states:

$$\begin{aligned}
&\text{For all stable sequence } (t_l, u_{k_l}, \lambda_{k_l}, M_{k_l}) \text{ related to } (\Psi, \mathcal{D}_\lambda^{k_l}) \text{ such that}\\
&(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l}) \to (t, u, \lambda, M),\ \forall\, (\widehat{u}, \widehat{\lambda}, \widehat{M}),\\
&\text{there exists a \textit{mutual recovery sequence} } (\widehat{u}_{k_l}, \widehat{\lambda}_{k_l}, \widehat{M}_{k_l}) \text{ such that}\\
&\limsup_{l\to\infty} \Big[\Psi(t_l, \widehat{u}_{k_l}, \widehat{\lambda}_{k_l}, \widehat{M}_{k_l}) + \mathcal{D}_\lambda^{k_l}(q_{k_l}, \widehat{q}_{k_l}) - \Psi(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l})\Big]\\
&\le \Psi(t, \widehat{u}, \widehat{\lambda}, \widehat{M}) + \mathcal{D}_\lambda^\infty(q, \widehat{q}) - \Psi(t, u, \lambda, M).
\end{aligned} \tag{4.3}$$

Ad (4.1): this follows from the lower semi-continuity of Ψ.

Ad (4.2): let $(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l})$ and $(\widehat{t}_l, \widehat{u}_{k_l}, \widehat{\lambda}_{k_l}, \widehat{M}_{k_l})$ be two stable sequences related to $(\Psi, \mathcal{D}_\lambda^{k_l})$ such that $(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l}) \to (t, u, \lambda, M)$ and $(\widehat{t}_l, \widehat{u}_{k_l}, \widehat{\lambda}_{k_l}, \widehat{M}_{k_l}) \to (\widehat{t}, \widehat{u}, \widehat{\lambda}, \widehat{M})$. Two situations may occur. If $M \neq \widehat{M}$, then $k \int_\Omega |M_{k_l} - \widehat{M}_{k_l}| \mathrm{d}x \to \infty$ so that we have

$$\mathcal{D}_\lambda^\infty(q, \widehat{q}) \le \liminf_{l\to\infty} \mathcal{D}_\lambda^{k_l}(q_{k_l}, \widehat{q}_{k_l}) = \infty.$$

If $M = \widehat{M}$, then $\mathcal{D}_\lambda^\infty(q, \widehat{q}) = R^\lambda \int_\Omega |z_\lambda - z_{\widehat{\lambda}}| \mathrm{d}x = \mathcal{D}_{\mathrm{mech}}(\lambda, \widehat{\lambda})$ and we have that

$$\liminf_{l\to\infty} \mathcal{D}_\lambda^{k_l}(q_{k_l}, \widehat{q}_{k_l}) \ge \liminf_{l\to\infty} R^\lambda \int_\Omega |\lambda_{k_l} - \widehat{\lambda}_{k_l}| \mathrm{d}x = R^\lambda \int_\Omega |\lambda - \widehat{\lambda}| \mathrm{d}x = \mathcal{D}_\lambda^\infty(q, \widehat{q}).$$

Ad (4.3): let $(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l})$ be a stable sequence related to $(\Psi, \mathcal{D}_\lambda^{k_l})$ and $(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l}) \to (t, u, \lambda, M)$ and let $(\widehat{u}, \widehat{\lambda}, \widehat{M}) \in \mathcal{U} \times \mathcal{L} \times \mathcal{M}$. We need to construct a *mutual recovery sequence*, namely a sequence $(\widehat{u}_{k_l}, \widehat{\lambda}_{k_l}, \widehat{M}_{k_l})$ such that

$$\begin{aligned}
&\limsup_{l\to\infty} \Big[\Psi(t_l, \widehat{u}_{k_l}, \widehat{\lambda}_{k_l}, \widehat{M}_{k_l}) + \mathcal{D}_\lambda^{k_l}(q_{k_l}, \widehat{q}_{k_l}) - \Psi(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l})\Big]\\
&\le \Psi(t, \widehat{u}, \widehat{\lambda}, \widehat{M}) + \mathcal{D}_\lambda^\infty(q, \widehat{q}) - \Psi(t, u, z, M).
\end{aligned}$$

If $M \neq \widehat{M}$, then $\mathcal{D}_\lambda^\infty(q, \widehat{q}) = \infty$ and there is nothing to prove. Let us then assume that $M = \widehat{M}$. Consider the sequence $(\widehat{u}_{k_l}, \widehat{\lambda}_{k_l}, \widehat{M}_{k_l}) = (\widehat{u}, \widehat{\lambda}, M_{k_l})$. Then, we have that

$$\begin{aligned}
&\limsup_{l\to\infty} \Big[\Psi(t_l, \widehat{u}_{k_l}, \widehat{\lambda}_{k_l}, \widehat{M}_{k_l}) + \mathcal{D}_\lambda^{k_l}(q_{k_l}, \widehat{q}_{k_l}) - \Psi(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l})\Big]\\
&\quad = \limsup_{l\to\infty} \bigg(\int_\Omega \psi_{\mathrm{SA}}(\widehat{u}, z_{\widehat{\lambda}}) \mathrm{d}x - \mu_0 K_{\mathrm{ani}} \int_\Omega (M_{k_l} \cdot A_{\widehat{\lambda}})^2 \mathrm{d}x + K_\lambda \int_\Omega |\nabla \widehat{\lambda}| + R^\lambda \int_\Omega |z_{\lambda_{k_l}} - z_{\widehat{\lambda}}| \mathrm{d}x\\
&\qquad - \langle \ell(t_l), \widehat{u} - u_{k_l}\rangle - \int_\Omega \psi_{\mathrm{SA}}(u_{k_l}, z_{\lambda_{k_l}}) \mathrm{d}x + \mu_0 K_{\mathrm{ani}} \int_\Omega (M_{k_l} \cdot A_{\lambda_{k_l}})^2 \mathrm{d}x - K_\lambda \int_\Omega |\nabla \lambda_{k_l}|\bigg)\\
&\quad \le \int_\Omega \psi_{\mathrm{SA}}(\widehat{u}, z_{\widehat{\lambda}}) \mathrm{d}x - \mu_0 K_{\mathrm{ani}} \int_\Omega (M \cdot A_{\widehat{\lambda}})^2 \mathrm{d}x + K_\lambda \int_\Omega |\nabla \widehat{\lambda}| + R^\lambda \int_\Omega |z_\lambda - z_{\widehat{\lambda}}| \mathrm{d}x\\
&\qquad - \langle \ell(t), \widehat{u} - u\rangle - \int_\Omega \psi_{\mathrm{SA}}(u, z_\lambda) \mathrm{d}x + \mu_0 K_{\mathrm{ani}} \int_\Omega (M \cdot A_\lambda)^2 \mathrm{d}x - K_\lambda \int_\Omega |\nabla \lambda|\\
&\quad = \Psi(t, \widehat{u}, \widehat{\lambda}, \widehat{M}) + \mathcal{D}_\lambda^\infty(q, \widehat{q}) - \Psi(t, u, \lambda, M).
\end{aligned}$$

Given the convergences (4.1)-(4.3) it is straightforward to check the remaining structure conditions, equi-coercivity, and semicontinuity assumptions of [41, Thm. 3.1] and obtain that (u, λ, M) is an energetic solution related to $(\Psi, \mathcal{D}_\lambda^\infty)$ and the initial datum (u^0, λ^0, M^0). Moreover, as (u_k, λ_k, M_k) is an energetic solution related to $(\Psi, \mathcal{D}_\lambda^k)$ and the initial datum (u^0, λ^0, M^0), one can prove that

$$\sup_{k\in\mathbb{N}} \mathrm{Diss}_{\mathcal{D}_\lambda^k}(q_k; [0, T]) < \infty.$$

In particular, for all $t \in [0,T]$ we have

$$k\int_\Omega |M_k(t) - M^0|\mathrm{d}x \le \sup_k \mathcal{D}^k_\lambda(q_k,[0,T]) < \infty$$

so that $M_k(t) \to M^0$ strongly in $L^1(\Omega;\mathbb{R}^3)$. In particular, $M \equiv M^0$. Hence, we deduce that, (u,λ) solves ($\mathrm{S}_{\mathrm{mech}}$)-($\mathrm{E}_{\mathrm{mech}}$), namely, it is an energetic solution of the mechanical Souza-Auricchio model (along with a fixed magnetization M^0). □

4.2 Micro-magnetic limit: $R^\lambda \to \infty$

Following the same strategy as in the previous subsection, we aim here at proving that the complete MSMA model reduces to the micro-magnetic model by letting the mechanical activation radius R^λ tend to ∞.

We shall be considering the *magneto-elastic energy* Ψ_{me} defined for every $(u,M) \in \mathcal{U}\times\mathcal{M}$ as

$$\Psi_{\mathrm{me}}(t,u,M) := \frac{1}{2}\int_\Omega \varepsilon(u){:}\mathbb{C}{:}\varepsilon(u)\mathrm{d}x - \mu_0 K_{\mathrm{ani}}\int_\Omega (M\cdot A_{\lambda^0})^2\mathrm{d}x + K_M\int_\Omega |\nabla M|^2 + \frac{\mu_0}{2}\int_{R^3}|\nabla v_M|^2\mathrm{d}x \\ - \mu_0\int_\Omega H(t)\cdot M(t)\mathrm{d}x - \langle \ell(t), u(t)\rangle.$$

We say that $t \mapsto (u(t),M(t))$ is an energetic solution for the magneto-elastic problem if $(u(0),M(0)) = (u^0,M^0)$ and if, for every $t\in[0,T]$, the following stability condition and energy balance hold,

$$\Psi_{\mathrm{me}}(t,u(t),M(t)) \le \Psi_{\mathrm{me}}(t,\widehat{u},\widehat{M}) + \mathcal{D}_{\mathrm{magn}}(M(t),\widehat{M}) \quad \forall\,(\widehat{u},\widehat{M}) \in \mathcal{U}\times\mathcal{M}, \tag{S_{me}}$$

$$\Psi_{\mathrm{me}}(t,u(t),M(t)) + \mathrm{Diss}_{\mathcal{D}_{\mathrm{magn}}}(M,[0,t]) \\ = \Psi_{\mathrm{me}}(0,u^0,M^0) - \int_0^t \langle \dot\ell(s), u(s)\rangle\mathrm{d}s - \mu_0\int_0^t\int_\Omega \dot H(s)\cdot M(s)\mathrm{d}x\,\mathrm{d}s \tag{E_{me}}$$

where the *magnetic* dissipation distance is defined as

$$\mathcal{D}_{\mathrm{magn}}(M,\widehat{M}) = R^M\int_\Omega |M-\widehat{M}|\mathrm{d}x,$$

and the total dissipation $\mathrm{Diss}_{\mathcal{D}_{\mathrm{magn}}}$ is as in (3.1) with $\mathcal{D}_{\mathrm{magn}}$ instead of $\mathcal{D}$.

Note that in the latter formulation (S_{me})-(E_{me}) the two variables u and M are completely decoupled. In particular, if (u,M) is an energetic solution of (S_{me})-(E_{me}), then M is an energetic solution of the classical micro-magnetic model, namely, for all $t\in[0,T]$,

$$\Psi_{\mathrm{magn}}(t,M(t)) - \mu_0 K_{\mathrm{ani}}\int_\Omega (M(t)\cdot A_{\lambda^0})^2\mathrm{d}x \\ \le \Psi_{\mathrm{magn}}(t,\widehat{M}) - \mu_0 K_{\mathrm{ani}}\int_\Omega (\widehat{M}\cdot A_{\lambda^0})^2\mathrm{d}x + \mathcal{D}_{\mathrm{magn}}(M(t),\widehat{M}) \quad \forall\,\widehat{M}\in\mathcal{M}, \tag{$\mathrm{S}_{\mathrm{magn}}$}$$

$$\Psi_{\mathrm{magn}}(t,M(t)) - \mu_0 K_{\mathrm{ani}}\int_\Omega (M(t)\cdot A_{\lambda^0})^2\mathrm{d}x + \mathrm{Diss}_{\mathcal{D}_{\mathrm{magn}}}(M,[0,t]) \\ = \Psi_{\mathrm{magn}}(0,M^0) - \mu_0 K_{\mathrm{ani}}\int_\Omega (M^0\cdot A_{\lambda^0})^2\mathrm{d}x - \mu_0\int_0^t\int_\Omega \dot H(s)\cdot M(s)\mathrm{d}x\,\mathrm{d}s. \tag{$\mathrm{E}_{\mathrm{magn}}$}$$

Set now

$$\mathcal{D}^k_M(q,\widehat{q}) = k\int_\Omega |\lambda-\widehat{\lambda}|\mathrm{d}x + R^M\int_\Omega |M-\widehat{M}|\mathrm{d}x$$

for all $q = (\lambda, M)$, $\widehat{q} = (\widehat{\lambda}, \widehat{M}) \in \mathcal{L} \times \mathcal{M}$ and

$$\mathcal{D}^{\infty}_M(q, \widehat{q}) := \begin{cases} \mathcal{D}_{\rm magn}(M, \widehat{M}) & \text{if } \lambda = \widehat{\lambda} \\ \infty & \text{else.} \end{cases}$$

By letting $\mathcal{S}^k_M$ be the set of stable states related to $(\Psi, \mathcal{D}^k_M)$, we have that $(u^0, \lambda^0, M^0) \in S(0) \subset S^k_M(0)$ for k large enough. Then, Theorem (3.1) ensures that there exists (u_k, λ_k, M_k) energetic solution related to $(\Psi, \mathcal{D}^k_M)$ and the initial datum (u^0, λ^0, M^0). We have the following.

Theorem 4.2 (Micro-magnetic limit)**.** *Under the assumptions of Theorem 3.1, let* (u_k, λ_k, M_k) *be an energetic solution related to* $(\Psi, \mathcal{D}^k_M)$ *and the initial datum* (u^0, λ^0, M^0)*. Then* (u_k, λ_k, M_k) *converges pointwise (up to some not relabeled subsequence) to* (u, λ^0, M) *and* (u, M) *is an energetic solution of the* magneto-elastic *problem, namely a solution of the system* ($\mathrm{S}_{\rm me}$)-($\mathrm{E}_{\rm me}$)*. Hence, M is an energetic solution of the* micro-magnetic *problem, namely a solution of the system* ($\mathrm{S}_{\rm magn}$)-($\mathrm{E}_{\rm magn}$)

Proof. Exactly as for the proof of Theorem 4.1, we shall exploit the general convergence theory of [41]. As the Γ-liminf inequality for Ψ (4.1) still holds, one is left with the proof of

- Γ-liminf inequality for $\mathcal{D}^k_M$:

$$\mathcal{D}^{\infty}_M(q, \widehat{q}) \leq \inf \Big\{ \liminf_{l\to\infty} \mathcal{D}^{k_l}_M(q_{k_l}, \widehat{q}_{k_l}) : (t_l, q_{k_l}) \to (t, q),\ (\widehat{t}_l, \widehat{q}_{k_l}) \to (\widehat{t}, \widehat{q}) \Big\}. \tag{4.4}$$

- Upper semicontinuity of stable states:

For all stable sequence $(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l})$ related to $(\Psi, \mathcal{D}^k_M)$ such that
$(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l}) \to (t, u, \lambda, M)$, $\forall\ (\widehat{u}, \widehat{\lambda}, \widehat{M})$,
there exists a *mutual recovery sequence* $(\widehat{u}_{k_l}, \widehat{\lambda}_{k_l}, \widehat{M}_{k_l})$ such that

$$\begin{aligned} &\limsup_{l\to\infty} \Big[\Psi(t_l, \widehat{u}_{k_l}, \widehat{\lambda}_{k_l}, \widehat{M}_{k_l}) + \mathcal{D}^{k_l}_M(q_{k_l}, \widehat{q}_{k_l}) - \Psi(t_l, u_{k_l}, \lambda_{k_l}, M_{k_l}) \Big] \\ &\leq \Psi(t, \widehat{u}, \widehat{\lambda}, \widehat{M}) + \mathcal{D}^{\infty}_M(q, \widehat{q}) - \Psi(t, u, \lambda, M). \end{aligned} \tag{4.5}$$

Now, the Γ-liminf inequality (4.4) follows analogously to (4.2). As for the upper semicontinuity check (4.5) one may use the mutual recovery sequence $(\widehat{u}_{k_l}, \widehat{\lambda}_{k_l}, \widehat{M}_{k_l}) = (\widehat{u}, \lambda_{k_l}, \widehat{M})$.

Hence, we are again in the setting of [41, Thm. 3.1]. In particular, (u_k, λ_k, M_k) converges pointwise to (u, λ, M) which in turn is an energetic solution related to $(\Psi, \mathcal{D}^{\infty}_M)$ and the initial datum (u^0, λ^0, M^0). Exactly as in the proof of Theorem 4.1 this amounts to say that $\lambda \equiv \lambda^0$ so that indeed (u, M) is an energetic solution of ($\mathrm{S}_{\rm me}$)-($\mathrm{E}_{\rm me}$). □

4.3 Non-dissipative magnetization limit: $R^M \to 0$

In MSMAs the dissipation related to domain wall motion and magnetization rotation is usually observed to be small compared to mechanical dissipation [15, 26]. We shall hence conclude this asymptotics section by focusing in the limit of non-dissipative magnetic behavior by letting $R^M \to 0$.

Letting $\delta = 1/k$ or $\delta = 0$ we define the dissipation distance $\mathcal{D}^{\delta}_{\lambda}$ as

$$\mathcal{D}^{\delta}_{\lambda}(q, \widehat{q}) = R^{\lambda} \int_{\Omega} |\lambda - \widehat{\lambda}| \mathrm{d}x + \delta \int_{\Omega} |M - \widehat{M}| \mathrm{d}x$$

for all $q = (\lambda, M)$, $\widehat{q} = (\widehat{\lambda}, \widehat{M}) \in \mathcal{L} \times \mathcal{M}$. Let $S^{\delta}_{\lambda}(t)$ denote the set of stable states related to $(\Psi, \mathcal{D}^{\delta}_{\lambda})$ at time t. We have the following.

Theorem 4.3 (Non-dissipative magnetization limit)**.** *Under the assumptions of Theorem 3.1, let* $(u^0,\lambda^0,M^0)\in S^0_\lambda(0)$ *and* $(u_\delta,\lambda_\delta,M_\delta)$ *be an energetic solution related to* $(\Psi,\mathcal{D}^\delta_\lambda)$ *and the initial datum* (u^0,λ^0,M^0)*. Then,* $(u_\delta,\lambda_\delta,M_\delta)$ *converges pointwise to* (u,λ,M) *(up to some not relabeled subsequence) which is an energetic solution related to* $(\Psi,\mathcal{D}_{mech})$ *and the initial datum* (u^0,λ^0,M^0)*, namely the model with no dissipation associated with* M*.*

Proof. Once again, we aim at making use of the Γ-convergence theory for rate-independent processes of [41]. As the semicontinuity (4.2) still holds, the analogue in this setting to (4.2)-(4.3) reads

- Γ-liminf inequality for $\mathcal{D}^\delta_\lambda$:

$$\mathcal{D}_{\rm mech}(q,\widehat{q})\leq\inf\Big\{\liminf_{l\to\infty}\mathcal{D}^{\delta_l}_\lambda(q_{\delta_l},\widehat{q}_{\delta_l}):(t_l,q_{\delta_l})\to(t,q),\ (\widehat{t}_l,\widehat{q}_{\delta_l})\to(\widehat{t},\widehat{q})\Big\}. \tag{4.6}$$

- Upper semicontinuity of stable states:

$$\begin{aligned}&\text{For all stable sequence } (t_l,\widehat{u}_{\delta_l},\widehat{\lambda}_{\delta_l},\widehat{M}_{\delta_l}) \text{ related to } (\Psi,\mathcal{D}^\delta_\lambda) \text{ such that}\\ &(t_l,\widehat{u}_{\delta_l},\widehat{\lambda}_{\delta_l},\widehat{M}_{\delta_l})\to(t,u,\lambda,M),\ \forall\ (\widehat{u},\widehat{\lambda},\widehat{M}),\\ &\text{there exists a \textit{mutual recovery sequence} } (\widehat{u}_{\delta_l},\widehat{\lambda}_{\delta_l},\widehat{M}_{\delta_l}) \text{ such that}\\ &\limsup_{l\to\infty}\Big[\Psi(t_l,\widehat{u}_{\delta_l},\widehat{\lambda}_{\delta_l},\widehat{M}_{\delta_l})+\mathcal{D}^{\delta_l}_\lambda(q_{\delta_l},\widehat{q}_{\delta_l})-\Psi(t_l,u_{\delta_l},\lambda_{\delta_l},M_{\delta_l})\Big]\\ &\leq\Psi(t,\widehat{u},\widehat{\lambda},\widehat{M})+\mathcal{D}_{\rm mech}(q,\widehat{q})-\Psi(t,u,\lambda,M).\end{aligned} \tag{4.7}$$

The Γ-liminf inequality for $\mathcal{D}^\delta_\lambda$ is immediate and a possible choice for the mutual recovery sequence is the constant sequence $(\widehat{u}_{\delta_l},\widehat{\lambda}_{\delta_l},\widehat{M}_{\delta_l})=(\widehat{u},\widehat{\lambda},\widehat{M})$ for which we get

$$\delta_l\int_\Omega|M_{\delta_l}-\widehat{M}|{\rm d}x\longrightarrow 0$$

when $\delta_l\to 0$.

The degenerate character of $\mathcal{D}_{\rm mech}$ is such that the general result [41, Thm. 3.1] is not directly applicable in this case. In fact, the theory of [41] assumes that no degeneracy would be developed in the limit. Here instead, the magnetization M passes from a dissipative to a non dissipative behavior. This indeed requires a (minor) modification of the argument of [41, Thm. 3.1] that we sketch here for completeness, referring however the reader to [41] for all missing details.

First of all, independently of δ one can prove that

$$\sup_{t\in[0,T]}\big[\Psi(t,u_\delta(t),\lambda_\delta(t),M_\delta(t))+{\rm Diss}_{\mathcal{D}_{\rm mech}}(\lambda,[0,t])\big]<C.$$

Hence, by a generalization of Helly's selection principle [41, Thm. A.1] we have that, at least for some not relabeled subsequence, $\lambda_\delta(t)\to\lambda(t)$ weakly-star in $BV(\Omega;\mathbb{R}^m)$ and strongly in $L^p(\Omega;\mathbb{R}^m)$ for $p<\infty$ and all $t\in[0,T]$. Moreover, further extracting t-dependent subsequences $(u_{\delta^t_k}(t),\lambda_{\delta^t_k}(t),M_{\delta^t_k}(t))$ we find limits

$$\begin{aligned}&u_{\delta^t_k}(t)\to u(t)\quad\text{weakly in}\quad H^1(\Omega;\mathbb{R}^3),\\ &M_{\delta^t_k}(t)\to M(t)\quad\text{weakly in}\quad H^1(\Omega;\mathbb{R}^3)\ \text{ and strongly in }\ L^p(\Omega;\mathbb{R}^3)\ \forall p<\infty.\end{aligned}$$

Note that the everywhere defined functions $t\mapsto u(t)$ and $t\mapsto M(t)$ need not be continuous nor measurable. Still, the stability of $(u_\delta,\lambda_\delta,M_\delta)$ and the upper semicontinuity (4.7) entail that indeed (u,λ,M) is a stable state related to $(\Psi,\mathcal{D}_{\rm mech})$ for all times. For all fixed $t\in[0,T]$, the upper energy estimate

$$\begin{aligned}&\Psi(t,u(t),\lambda(t),M(t))+{\rm Diss}_{\mathcal{D}_{\rm mech}}(\lambda,[0,t])\\ &\qquad\leq\Psi(0,u^0,\lambda^0,M^0)-\int_0^t\langle\dot{\ell}(s),u(s)\rangle{\rm d}s-\mu_0\int_0^t\int_\Omega\dot{H}(s)\cdot M(s){\rm d}x\,{\rm d}s\end{aligned}$$

follows at once by the analogous estimate at level δ, the lower semicontinuity of Ψ, and the Γ-liminf inequality (4.7). Finally, the converse lower energy estimate follows by stability and [35, Prop. 5.7]. $\square$

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References

[1] J. Arghavani, F. Auricchio, R. Naghdabadi, A. Reali, S. Sohrabpour. A 3-D phenomenological model for shape memory alloys under multiaxial loadings. *Internat. J. Plast.*, (2010), to appear.

[2] M. Arndt, M. Griebel, and T. Roubíček. Modelling and numerical simulation of martensitic transformation in shape memory alloys. *Contin. Mech. Thermodyn.*, 15 (2003), 5:463–485.

[3] F. Auricchio, A.-L. Bessoud, A. Reali and U. Stefanelli. A phenomenological model for ferromagnetism in shape-memory materials. *In preparation*, (2010).

[4] F. Auricchio, A.-L. Bessoud, A. Reali and U. Stefanelli. Macroscopic moldeing of magnetic shape memory alloys. *Oberwolfach Report*, 14 (2010), 1:771-773.

[5] F. Auricchio, A.-L. Bessoud, A. Reali and U. Stefanelli. A three-dimensional phenomenological model for Magnetic Shape Memory Alloys, *GAMM Mitt.*, (2010), to appear.

[6] F. Auricchio, A. Mielke, and U. Stefanelli. A rate independent model for the isothermal quasi-static evolution of shape-memory materials, *Math. Models Methods Appl. Sci.*, 18 (2008), 125–164.

[7] F. Auricchio and L. Petrini. Improvements and algorithmical considerations on a recent three-dimensional model describing stress-induced solid phase transformations, *Internat. J. Numer. Methods Engrg.*, 55 (2002), 1255–1284.

[8] F. Auricchio and L. Petrini. A three-dimensional model describing stress-temperature induced solid phase transformations. Part I: solution algorithm and boundary value problems. *Internat. J. Numer. Methods Engrg.*, 61 (2004), 807–836.

[9] F. Auricchio and L. Petrini. A three-dimensional model describing stress-temperature induced solid phase transformations. Part II: thermomechanical coupling and hybrid composite applications, *Internat. J. Numer. Methods Engrg.*, 61 (2004), 716–737.

[10] F. Auricchio, A. Reali, and U. Stefanelli. A phenomenological 3D model describing stress-induced solid phase transformations with permanent inelasticity, in *Topics on Mathematics for Smart Systems (Rome, 2006)*, 1–14, World Sci. Publishing, 2007.

[11] F. Auricchio, A. Reali, and U. Stefanelli. A three-dimensional model describing stress-induced solid phase transformation with permanent inelasticity, *Internat. J. Plast.*, 23 (2007), 207–226.

[12] F. Auricchio, A. Reali, and U. Stefanelli. A macroscopic 1D model for shape memory alloys including asymmetric behaviors and transformation-dependent elastic properties, *Comput. Methods Appl. Mech. Engrg.*, 198 (2009), 1631–1637.

[13] A-L. Bessoud and U. Stefanelli. Magnetic Shape Memory Alloys: three-dimensional modeling and analysis. *Math. Models Methods Appl. Sci.*, (2010), to appear.

[14] W. F. Brown. *Micromagnetics,* Wiley, New-York, 1963.

[15] B. D. Cullity and C. D. Graham. *Introduction to magnetic materials,* Second ed., Wiley, 2008.

[16] M. Eleuteri, L. Lussardi, and U. Stefanelli. A rate-independent model for permanent inelastic effects in shape memory materials. Preprint IMATI-CNR, 14PV10/14/0 (2010).

[17] V. Evangelista, S. Marfia, and E. Sacco, Phenomenological 3D and 1D consistent models for shape-memory alloy materials. *Comput. Mech.* 44 (2009), 3:405–421.

[18] V. Evangelista, S. Marfia, and E. Sacco, A 3D SMA constitutive model in the framework of finite strain, *Internat. J. Numer. Methods Engrg.*, 81 (2010), 6:761–785.

[19] M. Frémond. *Non-smooth Thermomechanics.* Springer-Verlag, Berlin, 2002.

[20] E. Fried and M. E. Gurtin. Dynamic solid-solid transitions with phase characterized by an order parameter. *Phys. D*, 72 (1994), 4:287–308.

[21] S. Frigeri, P. Krejčí, and U. Stefanelli. Quasistatic isothermal evolution of shape memory alloys. Preprint IMATI-CNR, 1PV10/1/0, (2010).

[22] L. Hirsinger and C. Lexcellent, Internal variable model for magneto-mechanical behaviour of ferromagnetic shape memory alloys Ni-Mn-Ga, *J. Phys. IV*, 112 (2003), 977–980.

[23] R. D. James and M. Wuttig. Magnetostriction of martensite, *Phil. Mag. A,* 77 (1998), 1273-1299.

[24] H. E. Karaca, I. Karaman, B. Basaran, Y. I. Chumlyakov, and H. J. Maier, Magnetic field and stress induced martensite reorientation in NiMnGa ferromagnetic shape memory alloy single crystals, *Acta Mat.* **54** (2006), 233–245.

[25] J. Kiang and L. Tong. Modelling of magneto-mechanical behaviour of NiMnGa single crytals, *J. Magn. Magn. Mater.*, 292 (2005), 394–412.

[26] B. Kiefer. A phenomelogical model for magnetic shape memory alloys. PhD Thesis, Texas A&M University, 2006.

[27] B. Kiefer and D.C. Lagoudas. Modeling the coupled strain and magnetization response of magnetic shape memory alloys under magnetomechanical loading. *J. Intell. Mater. Syst. Struct.*, 20 (2009), 143–170.

[28] P. Krejčí and U. Stefanelli. Existence and nonexistence for the full thermomechanical Souza-Auricchio model of shape memory wires, *Math. Mech. Solids*, (2010), to appear.

[29] P. Krejčí and U. Stefanelli. Well-posedness of a thermo-mechanical model for shape memory alloys under tension, *M2AN Math. Model. Anal. Numer.*, (2010), to appear.

[30] M. Kružík, A. Mielke, and T. Roubíček. Modelling of microstructure and its evolution in shape-memory-alloy single cristals, in particular in CuAlNi. *Meccanica*, 40 (2005), 4-6:389-418.

[31] M. Kružík and A. Prohl. Recent developments in modeling, analysis and numerics of ferromagnetism. *SIAM Review*, 48 (2006), 439–483.

[32] L. D. Landau and E. M. Lifshitz. On the theory of the dispersion of magnetic permeability of ferromagnetic bodies. *Physik Z. Sowjetunion*, 8 (1935), 153–169.

[33] L. D. Landau and E. M. Lifshitz. *Course of theoretical physics,* Pergamon Press, Oxford, 1960.

[34] A. A. Likhachev and K. Ullakko. Magnetic-field-controlled twin boundaries motion and giant magneto-mechanical effects in NiMnGa shape memory alloy, *Phys. Lett. A*, 275 (2000), 142–151.

[35] A. Mielke. *Evolution in rate-independent systems (ch. 6).* In C. Dafermos and E. Feireisl, editors, Handbook of Differential Equations, Evolutionary Equations, **2**, 461-559. Elsevier B.V., 2005.

[36] A. Mielke, L. Paoli, and A. Petrov. On existence and approximation for a 3D model of thermally-induced phase transformations in shape-memory alloys, *SIAM J. Math. Anal.*, 41 (2009), 1388–1414.

[37] A. Mielke, L. Paoli, A. Petrov, and U. Stefanelli. Error estimates for discretizations of a rate-independent variational inequality, *SIAM J. Numer. Anal.*, 8 (2010), 5:1625–1646.

[38] A. Mielke, L. Paoli, A. Petrov, and U. Stefanelli. Error bounds for space-time discretizations of a 3D model for shape-memory alloys, Proceedings of the IUTAM Symposium on *Variational Concepts with Applications to the Mechanics of Materials* (Bochum 2008), IUTAM Bookseries, Springer, 2009.

[39] A. Mielke and A. Petrov. Thermally driven phase transformation in shape memory alloys, *Adv. Math. Sci. Appl.*, 17 (2007), 160–182.

[40] A. Mielke and T. Roubíček. A rate-independent model for inelastic behavior of shape-memory alloys, *Multiscale Model. Simul.*, 1 (2003), 4:571–597.

[41] A. Mielke, T. Roubíček, and U. Stefanelli. Γ-imits and relaxations for rate-independent evolutionary problems, *Calc. Var. Partial Differential Equations*, 31 (2008), 3:387–416.

[42] A. Mielke and F. Theil. On rate-independent hysteresis model, *NoDEA Nonlinear Diff. Equations Applications*, 11 (2004), 151–189.

[43] T. Roubíček. Evolution model for martensitic phase transformation in shape-memory alloys, *Interfaces Free Bound.* 4 (2002), 2:111–136.

[44] T. Roubíček. Models of microstructure evolution in shape memory alloys. In: Nonlinear Homogenization and its Appl.to Composites, Polycrystals and Smart Materials. (Eds. P. Ponte Castaneda, J. J. Telega, B. Gambin), NATO Sci. Series II/170, Kluwer, Dordrecht, 2004, pp.269-304.

[45] A. C. Souza, E. N. Mamiya, and N. Zouain. Three-dimensional model for solids undergoing stress-induces transformations, *Eur. J. Mech. A Solids*, 17 (1998), 789-806.

[46] R. Tickle and R. D. James. Magnetic and magnetomechanical properties of Ni_2MnGa, *J. Magn. Magn. Mater.*, 195 (1999) 627–638.