Analysis of a Variable Time-Step Discretization for the Penrose-Fife Phase Relaxation Problem

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Abstract

In this paper we study a fully implicit time discrete scheme with variable time-steps for a system of field equations of Penrose-Fife type governing the dynamics of phase transitions with a nonconserved order parameter. In particular, Fourier law for the heat flux and zero interfacial energy are assumed. Uniform positivity, existence and uniqueness of the discrete solution is proved, several a priori estimates are estabilished, then passage to the limit is reached via compactness and monotonicity arguments. Optimal order estimates for the discretization error are shown.

Keywords: Penrose-Fife model, phase transitions, phase relaxation, time discretization, convergence results, error estimates.

1 Introduction

Let us consider the initial-boundary value problem

$$(\vartheta + \lambda(\chi))_t(x,t) - \Delta\vartheta(x,t) = g(x,t,\vartheta(x,t),\chi(x,t)) \quad \text{for a. e. } (x,t) \in Q\,, \tag{1.1}$$

$$\chi_t + \partial I(\chi) + \sigma'(\chi) \ni -\lambda'(\chi)/\vartheta \quad \text{a. e. in } Q\,, \tag{1.2}$$

$$\frac{\partial\vartheta}{\partial\nu} = 0 \quad \text{a. e in } \Sigma\,, \tag{1.3}$$

$$\vartheta(\cdot,0) = \vartheta_0\,, \quad \chi(\cdot,0) = \chi_0 \quad \text{a. e. in } \Omega\,. \tag{1.4}$$

Here, $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ denotes some bounded domain with smooth boundary $\partial\Omega$, $\partial/\partial\nu$ is the outward normal derivative to $\partial\Omega$, and we have set $Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$, where $T > 0$ stands for some final time.

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In addition, σ , λ , g are smooth functions, ϑ_0 , χ_0 are given initial data, and ∂I denotes the subdifferential of the indicator function I of the interval $[0,1]$. Namely, we have that $I(\chi) = 0$ if $\chi \in [0,1]$, $I(\chi) = +\infty$ otherwise, and consequently

$$\xi \in \partial I(\chi) \quad \text{if and only if} \quad \chi \in [0,1] \;\; \text{and} \;\; \xi \begin{cases} \in (-\infty, 0] & \text{for } \; \chi = 0 \\ = 0 & \text{for } \; 0 < \chi < 1 \, . \\ \in [0, +\infty) & \text{for } \; \chi = 1 \end{cases}$$

The nonlinear system (1.1)-(1.4) is the system of field equations arising from the Penrose-Fife phase relaxation model of phase transitions. In this system, χ stands for a nonconserved order parameter and ϑ for the absolute temperature. No diffusive effect is assumed for the phase transition, the heat flux obeys the Fourier law, and the free energy has the (nonsmooth) normalized form

$$F(\vartheta, \chi) = \vartheta - \, \vartheta \log \vartheta \, + \, \vartheta \left(I(\chi) \, + \, \sigma(\chi) \right) \, + \, \lambda(\chi) \, . \tag{1.5}$$

Equation (1.1) yields the balance of internal energy, while (1.2) describes the evolution of the order parameter (all physical constants are normalized to unity). For details about the Penrose-Fife model, the reader is referred either to the original papers [15, 16] or to the monograph [5] (cf. especially Chapter 4). This system differs from the standard phase-field model [6, 9] by the presence of the singular factor $1/\vartheta$ in the right hand side of (1.2) and of a nonlinear function $\lambda(\chi)$ in (1.1). Moreover, we may consider the standard phase-field model as a linearization of (1.1)-(1.2) around some equilibrium temperature. The main feature of the actual system (1.1)-(1.4) is that it is consistent with the Second Law of Thermodynamics, as the Clausius-Duhem inequality is satisfied ((1.1) and (1.2) have been tailored exactly with that purpose). In this respect, notice that the inclusion (1.2) can be equivalently rewritten as a pointwise variational inequality, namely

$$\begin{gathered} 0 \leq \chi(x,t) \leq 1 \quad \text{for a.e. } \; (x,t) \in Q \, , \\ \chi_t \, (\chi - r) \leq \, - \left(\sigma'(\chi) \, + \, \lambda'(\chi)/\vartheta \right) (\chi - r) \quad \text{a.e. in } \; Q \, , \quad \forall \, r \in [0,1] \, . \end{gathered} \tag{1.6}$$

The first inequality in (1.6) forces the order parameter χ to attain values only in $[0,1]$ and χ may for instance be regarded as the volume fraction of one of the two phases between which the phase transition occurs.

This problem has some interesting applications. In fact, nonlinearities λ, σ can be chosen either to describe a *solid-liquid phase transition* or to recover a *Ising model for ferromagnets* as is shown in [8]. In particular, the first choice lead to the double obstacle potential considered in [2, 3].

An existence and uniqueness result for (1.1)-(1.4) has been derived in [8] by proving a maximum principle for ϑ and then having recourse to general results for multicomponent systems without singularities (cf. [7]).

Instead, here our purposes are that of investigating (1.1)-(1.4) via direct approach based on a suitable time discretization, and especially that of showing convergence results and error estimates related to the discrete scheme, in which the time step may vary. This kind of inversigation has recently received a good deal of interest as, for instance, the papers [11, 12, 13] addressing analogous problems, show.

Then, it is worth introducing our time discretization. Let $\mathcal{P}$ be a partition of the time interval $[0,T]$,

$$\mathcal{P} = \{0 = t^0, t^1, \dots, t^{N-1}, t^N = T\}, \tag{1.7}$$

with variable step $\tau^i := t^i - t^{i-1}$. No *a priori* constraints are imposed on the time-steps and $\tau := \max_{1\le i\le N} \tau^i$ denotes the maximum of the time-step sizes. The time discrete scheme for problem (1.1)-(1.4) relies on the approximation of (1.1)-(1.2) by

$$\frac{\vartheta^i - \vartheta^{i-1}}{\tau^i} + \lambda'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau^i} - \Delta\vartheta^i = g^i(\cdot, \vartheta^i, \chi^i), \quad \text{a.e. in } \Omega \text{, for } i = 1, \dots, N, \tag{1.8}$$

$$\frac{\chi^i - \chi^{i-1}}{\tau^i} + \partial I(\chi^i) \ni -\sigma'(\chi^i) - \frac{\lambda'(\chi^i)}{\vartheta^i}, \quad \text{a.e. in } \Omega \text{, for } i = 1, \dots, N. \tag{1.9}$$

Here, $g^1, \dots, g^N$ stand for suitable time-independent functions discretizing the nonlinearity g in (1.1). Our arguments run as follows. First, uniform positivity of the N-tuple $(\vartheta^1, \dots, \vartheta^N)$ is proved. Then, we will be able to deduce existence ad uniqueness for the discrete solution. If we term $\widehat{\vartheta}_{\mathcal{P}}, \widehat{\chi}_{\mathcal{P}}$ the piecewise linear in time interpolants of solutions $\{\vartheta^i\}, \{\chi^i\}$ on the grid $\mathcal{P}$, we also prove the strong convergences

$$\begin{aligned} \widehat{\vartheta}_{\mathcal{P}} &\longrightarrow \vartheta \quad \text{in} \quad C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \\ \widehat{\chi}_{\mathcal{P}} &\longrightarrow \chi \quad \text{in} \quad C^0([0,T]; L^2(\Omega)), \end{aligned}$$

obtained by virtue of compactness and direct Cauchy arguments. Convergences allow us to pass to the limit in the equations (1.8)-(1.9). Moreover, under quite reasonable assumpion on g, we derive a discretization error estimate of optimal order (cf. also [12, 13]), namely,

$$\|\vartheta - \widehat{\vartheta}_{\mathcal{P}}\|_{L^2(Q)} + \|\chi - \widehat{\chi}_{\mathcal{P}}\|_{C^0([0,T];L^2(\Omega))} \le C\,\tau.$$

The rest of the paper is organized as follows. In Section 2, we formulate the general assumptions and the main results of the paper. Section 3 brings the proof of existence and uniqueness of the discrete solution along with its positivity. Section 4 contains a collection of uniform estimates which enable us to pass to the limit in section 5. Error estimates are deduced in section 6. Appendix contains a technical convergence result.

2 Statement of the scheme and main results

We make the following assumptions on the data $\lambda, \sigma, g, \chi_0, \vartheta_0$:

$$\lambda\,,\, \sigma \in C^{1,1}[0,1]; \tag{2.1}$$

g is a Carathéodory function satisfying $g(\cdot,\cdot,\varphi,r) \in L^2(Q)$ for all pairs $(\varphi, r) \in \mathbb{R} \times [0,1]$, there exists a constant $C_g > 0$ such that

$$\begin{aligned} &|g(x,t,\varphi_1,r_1) \,-\, g(x,t,\varphi_2,r_2)| \,\le\, C_g\,(|\varphi_1-\varphi_2| \,+\, |r_1-r_2|) \\ &\quad \text{for a.e. } (x,t)\in Q\,, \quad \forall\,\varphi_1\,,\,\varphi_2\in\mathbb{R}\,, \quad \forall\, r_1\,,\, r_2 \in [0,1]\,, \end{aligned} \tag{2.2}$$

and let $g_{00} := g(\cdot,\cdot,0,0) \in L^2(Q)$;

$$\vartheta_0 \in H^1(\Omega), \quad \chi_0 \in L^2(\Omega); \tag{2.3}$$

$$0 \le \chi_0 \le 1 \quad \text{a.e. in } \Omega; \tag{2.4}$$

$$\vartheta_0 > 0 \quad \text{a.e. in } \Omega. \tag{2.5}$$

Moreover, the existence of some constant $\vartheta_* > 0$ is required in order that the three conditions

$$\vartheta_0 \ge \vartheta_* \quad \text{a.e. in } \Omega\,, \tag{2.6}$$

$$g(x,t,\varphi,r) \ge 0 \quad \text{per a.e. } (x,t)\in Q\,, \quad \forall\,\varphi \le \vartheta_*\,, \quad \forall\, r\in[0,1]\,, \tag{2.7}$$

$$|\lambda'(r)|^2 \,+\, \sigma'(r)\,\lambda'(r)\,\vartheta_* \ge 0 \quad \forall\, r\in[0,1]\,. \tag{2.8}$$

are fulfilled at the same time. While (2.6) and (2.7) are rather natural constraints for the initial temperature and the heat source, respectively, inequality (2.8) holds if either $\sigma'\,\lambda'$ has the right sign or if $|\sigma'|$ is not too large when compared with $|\lambda'|$. In regard to applications, let us recall that there are some physically interesting nonlinearities (cf. [8]) which satisfy (2.8). The following result is shown in [8].

Proposition 2.1 *Suppose that the assumptions (2.1)-(2.8) hold. Then there exists a unique triplet* (ϑ, χ, ξ) *with*

$$\vartheta \;\in\; H^1(0,T;L^2(\Omega)) \;\cap\; C^0([0,T];H^1(\Omega)) \;\cap\; L^2(0,T;H^2(\Omega))\,, \tag{2.9}$$

$$\chi \;\in\; H^1(0,T;L^2(\Omega)) \;\cap\; L^\infty(Q)\,, \tag{2.10}$$

$$\xi \;\in\; L^\infty(Q)\,, \tag{2.11}$$

$$\vartheta > 0 \quad a.e. \text{ in } Q, \quad 1/\vartheta \in L^1(Q)\,, \tag{2.12}$$

that satisfies relation (1.1), (1.3)-(1.4) in the sense there specified, and such that

$$\chi_t \,+\, \xi \,+\, \sigma'(\chi) \;=\; -\lambda'(\chi)/\vartheta \quad a.\,e. \text{ in } Q\,, \tag{2.13}$$

$$\xi \in \partial I(\chi) \quad a.\,e. \text{ in } Q\,. \tag{2.14}$$

Moreover, it turns out that

$$\chi_t \;\in\; L^\infty(Q)\,; \tag{2.15}$$

$$\vartheta \ge \theta_* \quad a.\,e. \text{ in } Q\,. \tag{2.16}$$

Observe that in the above setting all terms of (1.1) belong to $L^2(Q)$ and that, by virtue of (2.1) and (2.12), $\lambda'(\chi)/\vartheta \in L^1(Q)$, whence (1.2) and (1.6) make sense. Here, in regard of this result, we move along another direction, being interested to time discretization. In fact, by considering partitions $\mathcal{P}$ like in (1.7), our aim is studying the following discrete scheme

$$\frac{\vartheta^i - \vartheta^{i-1}}{\tau^i} + \lambda'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau^i} - \Delta\vartheta^i = g^i(\cdot, \vartheta^i, \chi^i), \quad \text{a.e. in } \Omega \text{ , for } i = 1, \ldots, N \text{ ,} \tag{2.17}$$

$$\frac{\chi^i - \chi^{i-1}}{\tau^i} + \xi^i = -\sigma'(\chi^i) - \frac{\lambda'(\chi^i)}{\vartheta^i}, \quad \text{a.e. in } \Omega \text{ , for } i = 1, \ldots, N \text{ ,} \tag{2.18}$$

$$0 \le \chi^i \le 1 \ \text{ and } \ \xi^i \in \partial I(\chi^i) \ \text{ a.e. in } \Omega \text{ , for } i = 1, \ldots, N \text{ ,} \tag{2.19}$$

$$\frac{\partial\vartheta^i}{\partial\nu} = 0 \quad \text{a.e. in } \partial\Omega \text{ , for } i = 1, \ldots, N \text{ ,} \tag{2.20}$$

$$\vartheta^0 = \vartheta_{0\mathcal{P}}, \quad \chi^0 = \chi_{0\mathcal{P}} \quad \text{a.e. in } \Omega, \tag{2.21}$$

where, for $i = 1, \ldots, N$, we set

$$g^i(x, \varphi, r) := \frac{1}{\tau^i}\int_{t^{i-1}}^{t^i} g(x, t, \varphi, r)dt, \quad \text{for a.e. } x \in \Omega, \quad \forall\varphi \in \mathbb{R}, \quad \forall r \in [0, 1], \tag{2.22}$$

and $\{\vartheta_{0\mathcal{P}}\}$ and $\{\chi_{0\mathcal{P}}\}$ are two families of approximating initial data which fulfil

$$\vartheta_{0\mathcal{P}} \in H^2(\Omega) \text{ , } \quad \chi_{0\mathcal{P}} \in H^1(\Omega), \tag{2.23}$$

$$0 \le \chi_{0\mathcal{P}} \le 1 \quad \text{a.e. in } \Omega, \tag{2.24}$$

$$\vartheta_{0\mathcal{P}} \ge \vartheta_* \quad \text{a.e. in } \Omega. \tag{2.25}$$

We also ask for a constant $C_d > 0$ independent from $\mathcal{P}$ such that

$$\|\vartheta_{0\mathcal{P}}\|_{H^1(\Omega)} \le C_d, \tag{2.26}$$

$$\|\vartheta_{0\mathcal{P}} - \vartheta_0\|_{L^2(\Omega)} \le C_d\,\tau, \tag{2.27}$$

$$\|\chi_{0\mathcal{P}} - \chi_0\|_{L^2(\Omega)} \le C_d\,\tau \tag{2.28}$$

where τ denotes the diameter of the partition, that is, $\tau = \max_{1\le i\le N}\tau^i$. Due to (2.3), (2.4), and (2.6), it is not difficult to construct examples of $\{\vartheta_{0\mathcal{P}}\}$ and $\{\chi_{0\mathcal{P}}\}$ satisfying (2.23)-(2.28), for instance via convolution.

Let us point out that (2.18)-(2.19) could be conveniently rewritten as a pointwise variational inequality, namely

$$0 \le \chi^i(x) \le 1 \quad \text{for a.e. } x \in \Omega, \tag{2.29}$$

$$\left(\frac{\chi^i - \chi^{i-1}}{\tau^i}\right)(\chi^i - v) \le \left(-\sigma'(\chi^i) - \frac{\lambda'(\chi^i)}{\vartheta^i}\right)(\chi^i - v) \quad \text{a.e. in } \Omega, \quad \forall v \in [0, 1]. \tag{2.30}$$

Definition 2.2 *A N-tuple of triplets* $(\vartheta^i, \chi^i, \xi^i), i = 1, \ldots, N$ *, is said to be a solution to the scheme (2.17)-(2.21) if, for every* $i = 1, \ldots, N$,

$$\vartheta^i \in H^2(\Omega), \quad \vartheta^i > 0 \ \ a.e. \ in \ \ \Omega, \quad \frac{1}{\vartheta^i} \in L^1(\Omega), \tag{2.31}$$

$$\chi^i \in L^\infty(\Omega), \tag{2.32}$$

$$\xi^i \in L^2(\Omega), \tag{2.33}$$

and relations (2.17)-(2.21) are satisfied.

We have the following existence, uniqueness, and uniform positivity result.

Theorem 2.3 *Assume that (2.1),(2.2),(2.7),(2.8),(2.23)-(2.25) hold and let the diameter* τ *of the partition* $\mathcal{P}$ *be small enough. Then the scheme (2.17)-(2.21) has a unique solution* $(\vartheta^i, \chi^i, \xi^i)$, $i = 1, \ldots, N$ *. Moreover, it turns out that*

$$\vartheta^i \geq \vartheta_* \qquad a.e. \ in \, \Omega, \qquad for \; i = 1, \ldots, N. \tag{2.34}$$

Proof of this result can be found in Section 3. By virtue of Theorem **2.3**, we may introduce the piecewise constant and linear in time functions $\vartheta_\mathcal{P}, \chi_\mathcal{P}, \widehat{\vartheta}_\mathcal{P}, \widehat{\chi}_\mathcal{P}, \xi_\mathcal{P}, g_\mathcal{P}$ interpolating the corrisponding values. For this aim, we use the following position

Definition 2.4 *Given* $\mathcal{P} = \{0 = t^0, t^1, \ldots, t^{N-1}, t^N = T\}$, *for any* $\{\varphi^i\}_{i=0}^N \in (L^2(\Omega))^{N+1}$, *for a.e.* $x \in \Omega$, *and for* $t \in]t^{i-1}, t^i]$, *we set*

$$\begin{aligned} \varphi_\mathcal{P}(x,t) &= \varphi^i(x), \\ \widehat{\varphi}_\mathcal{P}(x,t) &= \left(\frac{t - t^{i-1}}{\tau^i}\right) \varphi^i(x) - \left(\frac{t^i - t}{\tau^i}\right) \varphi^{i-1}(x), \end{aligned}$$

recalling that $\tau^i = t^i - t^{i-1}$, $i = 1, \ldots, N$.

Clearly, to specify $\varphi_\mathcal{P}$ we do not need to know the value φ^0. In particular, we remark that $g_\mathcal{P}$ acts as a nonlinear functional and is constructed in the same way, by using the functionals in (2.22).

Thanks to these notations, we may rewrite the scheme (2.17)-(2.21) in terms of $\vartheta_\mathcal{P}, \chi_\mathcal{P}, \widehat{\vartheta}_\mathcal{P}, \widehat{\chi}_\mathcal{P}, \xi_\mathcal{P}, g_\mathcal{P}$ as

$$\frac{\partial \widehat{\vartheta}_\mathcal{P}}{\partial t} + \lambda'(\chi_\mathcal{P}) \frac{\partial \widehat{\chi}_\mathcal{P}}{\partial t} - \Delta \vartheta_\mathcal{P} = g_\mathcal{P}(\cdot, \cdot, \vartheta_\mathcal{P}, \chi_\mathcal{P}) \quad \text{a.e. in } Q, \tag{2.35}$$

$$\frac{\partial \widehat{\chi}_\mathcal{P}}{\partial t} + \xi_\mathcal{P} = -\sigma'(\chi_\mathcal{P}) - \frac{\lambda'(\chi_\mathcal{P})}{\vartheta_\mathcal{P}} \quad \text{a.e. in } Q, \tag{2.36}$$

$$0 \leq \chi_\mathcal{P} \leq 1 \quad \text{and} \quad \xi_\mathcal{P} \in \partial I(\chi_\mathcal{P}) \quad \text{a.e. in } Q, \tag{2.37}$$

$$\frac{\partial \vartheta_\mathcal{P}}{\partial \nu} = 0 \quad \text{a.e. in } \Sigma, \tag{2.38}$$

$$\widehat{\vartheta}_\mathcal{P}(\cdot, 0) = \vartheta_{0\mathcal{P}} \quad , \quad \widehat{\chi}_\mathcal{P}(\cdot, 0) = \chi_{0\mathcal{P}} \quad \text{a.e. in } \Omega. \tag{2.39}$$

In Section 4 we prove the following stability result.

Theorem 2.5 *Let $\vartheta_{\mathcal{P}}, \widehat{\vartheta}_{\mathcal{P}}, \chi_{\mathcal{P}}, \widehat{\chi}_{\mathcal{P}}, \xi_{\mathcal{P}}$ be defined as above. Then, there exists a constant C such that, for every partition $\mathcal{P}$ with diameter τ small enough, the following estimate holds*

$$\begin{aligned} &\|\widehat{\vartheta}_{\mathcal{P}}\|_{H^1(0,T;L^2(\Omega))\,\cap\, C^0([0,T];H^1(\Omega))\,\cap\, L^2(0,T;H^2(\Omega))} \\ +\; &\|\vartheta_{\mathcal{P}}\|_{L^\infty(0,T;H^1(\Omega))\,\cap\, L^2(0,T;H^2(\Omega))} \;+\; \|\widehat{\chi}_{\mathcal{P}}\|_{W^{1,\infty}(Q)} \\ &+\; \|\xi_{\mathcal{P}}\|_{L^\infty(Q)} \;+\; \|\chi_{\mathcal{P}}\|_{L^\infty(Q)} \;\le\; C. \end{aligned} \tag{2.40}$$

In the rest of the paper, the auxiliary scheme in which (2.36) is replaced by

$$\frac{\partial \widehat{\chi}_{\mathcal{P}}}{\partial t} + \xi_{\mathcal{P}} = -\,\sigma'(\chi_{\mathcal{P}}) - \lambda'(\chi_{\mathcal{P}})\rho(\vartheta_{\mathcal{P}}) \quad \text{a.e. in } Q, \tag{2.41}$$

with

$$\rho(\varphi) := \begin{cases} 1/\vartheta_* & \text{if } \varphi \le \vartheta_* \\ 1/\varphi & \text{if } \varphi > \vartheta_* \end{cases}, \tag{2.42}$$

will play an important role. Indeed, if we are able to check (2.34), one is allowed to deal with the scheme (2.35), (2.41), (2.37)-(2.39).

Now, we estabilish two theorems, which entail convergence of the discrete solution to the continuous solution as $\tau \to 0$ and yield estimates for the discretization error. In addition, the first one, applied to some subsequence, could provide an alternative proof of the existence result in [8]. Namely we have

Theorem 2.6 *Let (ϑ, χ, ξ) be the solution to (1.1)-(1.4) given by Proposition* **2.1** *and let $\vartheta_{\mathcal{P}}, \widehat{\vartheta}_{\mathcal{P}}, \chi_{\mathcal{P}}, \widehat{\chi}_{\mathcal{P}}, \xi_{\mathcal{P}}$ be as in Theorem* **2.5**. *Then, we have the following strong ($\to$) and weak star ($\stackrel{*}{\rightharpoonup}$) convergences*

$$\begin{aligned}
\vartheta_{\mathcal{P}} &\to \vartheta && \text{in } L^\infty(0,T;L^2(\Omega)) \,\cap\, L^2(0,T;H^1(\Omega)), && (2.43)\\
\vartheta_{\mathcal{P}} &\stackrel{*}{\rightharpoonup} \vartheta && \text{in } L^\infty(0,T;H^1(\Omega)) \,\cap\, L^2(0,T;H^2(\Omega)), && (2.44)\\
\widehat{\vartheta}_{\mathcal{P}} &\to \vartheta && \text{in } C^0([0,T];L^2(\Omega)) \,\cap\, L^2(0,T;H^1(\Omega)), && (2.45)\\
\widehat{\vartheta}_{\mathcal{P}} &\stackrel{*}{\rightharpoonup} \vartheta && \text{in } H^1(0,T;L^2(\Omega)) \,\cap\, L^\infty(0,T;H^1(\Omega)) && \\
& && \qquad \cap\, L^2(0,T;H^2(\Omega)), && (2.46)\\
\chi_{\mathcal{P}} &\to \chi && \text{in } L^\infty(0,T;L^2(\Omega)), && (2.47)\\
\chi_{\mathcal{P}} &\stackrel{*}{\rightharpoonup} \chi && \text{in } L^\infty(Q), && (2.48)\\
\widehat{\chi}_{\mathcal{P}} &\to \chi && \text{in } C^0([0,T];L^2(\Omega)), && (2.49)\\
\widehat{\chi}_{\mathcal{P}} &\stackrel{*}{\rightharpoonup} \chi && \text{in } W^{1,\infty}(Q), && (2.50)\\
\xi_{\mathcal{P}} &\stackrel{*}{\rightharpoonup} \xi && \text{in } L^\infty(Q) && (2.51)
\end{aligned}$$

for any sequence of partitions $\mathcal{P}$ with diameters tending to 0.

Detailed proof of this result can be found in Section 5. Let us remark that, for the validity of this theorem, conditions (2.27), (2.28) may be replaced by the weaker assumptions

$$\|\vartheta_{0\mathcal{P}} - \vartheta_0\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{and} \quad \|\chi_{0\mathcal{P}} - \chi_0\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{for} \quad \tau \to 0,$$

thus without specifying any order of convergence for the initial data.

The derivation of our error estimates requires a certain time regularity to the function g. For instance we ask, for all $u, v \in BV(0,T;L^2(\Omega))$ such that $0 \le v \le 1$ a.e. in Q,

$$g(\cdot,\cdot,u,v) \in BV(0,T;L^2(\Omega)). \tag{2.52}$$

Besides, letting $Var_{[0,T];L^2(\Omega)}[u]$ denote the total variation of the function u considered from $[0,T]$ to $L^2(\Omega)$, we assume that, for any bounded subset $\mathcal{A}$ of $BV(0,T;L^2(\Omega))$, there exists a positive constant $C_{\mathcal{A}}$ such that, for every $u, v \in \mathcal{A}$ with $0 \le v \le 1$ a.e. in Q, one has

$$Var_{[0,T];L^2(\Omega)}\left[g(\cdot,\cdot,u,v)\right] \le C_{\mathcal{A}}. \tag{2.53}$$

Remark 2.7 Recalling (2.2), let us point out that, in the case when g is uniformly Lipschitz continuous with respect to t as well, then conditions (2.52)-(2.53) are plainly fulfilled. On the other hand, a sufficient condition for the validity of (2.52)-(2.53) is, e.g., the following

$$|g(x,t,\varphi,r) - g(x,s,\varphi,r)| \le C(1+|\varphi|)\,|t-s|$$
$$\forall t,s \in [0,T],\ \forall \varphi \in \mathbb{R},\ \forall r \in [0,1],$$

in which C stands for some positive constant.

We have the following result.

Theorem 2.8 *Let (ϑ,χ,ξ) and $\vartheta_{\mathcal{P}}, \widehat{\vartheta}_{\mathcal{P}}, \chi_{\mathcal{P}}, \widehat{\chi}_{\mathcal{P}}, \xi_{\mathcal{P}}$ be as in Theorem* **2.5** *and assume (2.52)-(2.53). Then there exists a constant C such that, for every partition $\mathcal{P}$ with diameter τ small enough, the following estimates hold*

$$\|\vartheta - \widehat{\vartheta}_{\mathcal{P}}\|_{L^2(Q)} + \|\chi - \widehat{\chi}_{\mathcal{P}}\|_{C^0([0,T];L^2(\Omega))} + \sup_{t\in[0,T]} \left\| \int_0^t (\vartheta - \vartheta_{\mathcal{P}})(\cdot,s)ds \right\|_{H^1(\Omega)} \le C\tau, \tag{2.54}$$

$$\|\vartheta - \widehat{\vartheta}_{\mathcal{P}}\|_{C^0(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le C\sqrt{\tau}. \tag{2.55}$$

We point out that the constant C in (2.54)-(2.55) depends solely on the data. Hence, since the above *a posteriori* estimates impose no constraints between consecutive time-steps, they allow the use of adaptive procedures for the choice of the time-steps. Moreover, let us note that estimate (2.54) is optimal with respect to the order of convergence (see [13, 14]). In conclusion, (2.54)-(2.55) look rather helpful for further numerical investigations.

3 Discrete solution

This section brings the proof of Theorem **2.3** in two steps. First we prove the positivity property (2.34) for all solutions to the scheme (2.17)-(2.21) and then existence and uniqueness of one solution.

3.1 Positivity of ϑ^i

Owing to (2.25), we can reason by induction and show that

$$\vartheta^{i-1} \geq \vartheta_* \quad \text{a.e. in } \Omega \quad \Rightarrow \quad \vartheta^i \geq \vartheta_* \quad \text{a.e. in } \Omega.$$

In fact, multiplying (2.17) by the function

$$-(\vartheta^i - \vartheta_*)^- = \min\{\vartheta^i - \vartheta_*, 0\} \ \in \ L^2(\Omega),$$

and integrating over Ω and by parts, we infer

$$\|(\vartheta^i - \vartheta_*)^-\|^2_{L^2(\Omega)} + \tau^i \int_\Omega |\nabla((\vartheta^i - \vartheta_*)^-)|^2 = \sum_{j=1}^{3} I_j \ , \tag{3.1}$$

where

$$\begin{aligned} I_1 &= -\int_\Omega (\vartheta^{i-1} - \vartheta_*)(\vartheta^i - \vartheta_*)^-, \\ I_2 &= -\int_\Omega g^i(\cdot, \vartheta^i, \chi^i)(\vartheta^i - \vartheta_*)^-, \\ I_3 &= \int_\Omega \lambda'(\chi^i) \frac{\chi^i - \chi^{i-1}}{\tau^i} (\vartheta^i - \vartheta_*)^-. \end{aligned} \tag{3.2}$$

As $\vartheta^{i-1} \geq \vartheta_*$ a.e. in Ω by assumption, we have that $I_1 \leq 0$. Also, it is not difficult to check that $I_2 \leq 0$ because of (2.7) and (2.22). Now, rewrite I_3 as

$$I_3 = I_4 + I_5 + I_6,$$

where

$$\begin{aligned} I_4 &= \int_{\{0<\chi^i<1\}} \lambda'(\chi^i) \frac{\chi^i - \chi^{i-1}}{\tau^i} (\vartheta^i - \vartheta_*)^-, \\ I_5 &= -\int_{\{\chi^i=0\}} \lambda'(0) \frac{\chi^{i-1}}{\tau^i} (\vartheta^i - \vartheta_*)^-, \\ I_6 &= \int_{\{\chi^i=1\}} \lambda'(1) \frac{1 - \chi^{i-1}}{\tau^i} (\vartheta^i - \vartheta_*)^-, \end{aligned}$$

with obvious notations for the measurable subsets of Ω in which $0 < \chi^i < 1$, $\chi^i = 0$, and $\chi^i = 1$. Regarding I_4, note that (2.30) yields the equality

$$\frac{\chi^i - \chi^{i-1}}{\tau^i} = -\sigma'(\chi^i) - \lambda'(\chi^i)/\vartheta^i \quad \text{a.e. in } \{0 < \chi^i < 1\},$$

and consequently

$$I_4 = -\int_{\{0<\chi^i<1\}} \frac{(\vartheta^i - \vartheta_*)^-}{\vartheta^i} (\vartheta^i \sigma'(\chi^i)\lambda'(\chi^i) + |\lambda'(\chi^i)|^2).$$

Taking advantage of (2.8), one verifies that I_4 is nonpositive. Indeed, we have $\vartheta^i \sigma'(\chi^i)\lambda'(\chi^i) \ge \vartheta_* \sigma'(\chi^i)\lambda'(\chi^i)$ as soon as both $\vartheta^i \le \vartheta_*$ and $\sigma'(\chi^i)\lambda'(\chi^i) < 0$, otherwise the inequality $(\vartheta^i - \vartheta_*)^- \vartheta^i \sigma'(\chi^i)\lambda'(\chi^i) \ge 0$ holds. Then, with the help of (2.8) we conclude that

$$\frac{(\vartheta^i - \vartheta_*)^-}{\vartheta^i} \Big(\vartheta^i \sigma'(\chi^i)\lambda'(\chi^i) \ + \ |\lambda'(\chi^i)|^2\Big) \ \ge \ 0 \qquad \text{a.e. in } \ \{0 < \chi^i < 1\},$$

whence $I_4 \le 0$. Notice that I_5 and I_6 have the same sign of I_4 if $\lambda'(0) \ge 0$ and $\lambda'(1) \le 0$. On the other hand, thanks to (2.30) it is straightforward to infer that

$$\begin{aligned} -\frac{\chi^{i-1}}{\tau^i} \ &\ge \ -\sigma'(0) - \frac{\lambda'(0)}{\vartheta^i} \qquad && \text{a.e. in } \ \{\chi^i = 0\}, \\ \frac{1-\chi^{i-1}}{\tau^i} \ &\le \ -\sigma'(1) - \frac{\lambda'(1)}{\vartheta^i} \qquad && \text{a.e. in } \ \{\chi^i = 1\}. \end{aligned}$$

Let $\lambda'(0) < 0$ and $\lambda'(1) > 0$. Then, multiplication by $\lambda'(0)$ and $\lambda'(1)$ implies

$$\begin{aligned} I_5 \ &\le \ -\int_{\{\chi^i=0\}} \frac{(\vartheta^i - \vartheta_*)^-}{\vartheta^i} (\vartheta^i \sigma'(0)\lambda'(0) + |\lambda'(0)|^2), \\ I_6 \ &\le \ -\int_{\{\chi^i=1\}} \frac{(\vartheta^i - \vartheta_*)^-}{\vartheta^i} (\vartheta^i \sigma'(1)\lambda'(1) + |\lambda'(1)|^2), \end{aligned}$$

respectively. We may now argue as for I_4 and conclude that I_5 and I_6 are nonpositive. Therefore, in each of the possible cases it turns out that $I_3 \le 0$. Consequently, from (3.1) we deduce that

$$\|(\vartheta^i - \vartheta_*)^-\|_{L^2(\Omega)} \ = \ 0,$$

and then

$$\vartheta^i \ge \vartheta_* \ \text{ a.e. in } \ \Omega.$$

Slight modifications to this argument entail (2.34) also for the solution to the auxiliary scheme related to (2.41), that is, the scheme in which (2.18) is substituted by

$$\frac{\chi^i - \chi^{i-1}}{\tau^i} + \xi^i = -\sigma'(\chi^i) - \lambda'(\chi^i)\rho(\vartheta^i) \qquad \text{a.e. in } \Omega, \qquad \text{for } i = 1, \ldots, N. \tag{3.3}$$

Indeed, in this case, by (2.42) we even have that

$$(1/\rho(\vartheta^i))\sigma'(\chi^i)\lambda'(\chi^i) = \vartheta_* \sigma'(\chi^i)\lambda'(\chi^i)$$

whenever $\vartheta^i \le \vartheta_*$.

3.2 Existence and uniqueness of $(\vartheta^i, \chi^i, \xi^i)$

Our aim is now proving the existence and uniqueness of the solution $(\vartheta^i, \chi^i, \xi^i)$ to the system (2.17)-(2.20) for a fixed i, where the values ϑ^{i-1} and χ^{i-1} are already known. First, we verify that the problem (2.17), (3.3), (2.19), (2.20) has a unique solution. Then, we show that the found triplet $(\vartheta^i, \chi^i, \xi^i)$ uniquely solves (2.17)-(2.20) as well.

Let us consider the space $(L^2(\Omega))^2$ with scalar product

$$\langle [u,v],[r,s]\rangle_{(L^2(\Omega))^2} \;=\; (u,v) \;+\; (r,s), \quad \forall u,v,r,s \;\in\; L^2(\Omega),$$

where $[\cdot,\cdot]$ denotes the pair and $(\cdot,\cdot)$ stands for the usual scalar product in $L^2(\Omega)$. Set $H = \{u \in H^2(\Omega)$ such that $\partial u/\partial\nu = 0$ a.e. in $\partial\Omega\}$ and $K = \{v \in L^2(\Omega)$ such that $0 \le v \le 1$ a.e. in $\Omega\}$. We define the operators $T_1 : L^2(\Omega) \times K \longrightarrow K$, $T_2 : L^2(\Omega) \times K \longrightarrow H$, and $T : L^2(\Omega) \times K \longrightarrow L^2(\Omega) \times K$ as follows

$$\forall [u,v] \in L^2(\Omega) \times K$$

$$\begin{aligned}
T_1[u,v] &= (I + \tau^i \partial I)^{-1} \Big(\chi^{i-1} - \tau^i \left(\sigma'(v) + \lambda'(v)\rho(u)\right)\Big), \\
T_2[u,v] &= (I - \tau^i \Delta)^{-1} \Big(\vartheta^{i-1} - \lambda'(v)(v - \chi^{i-1}) + \tau^i g^i(\cdot,u,v)\Big), \\
T[u,v] &= [\; T_2[u, T_1[u,v]] \;,\; T_1[u,v] \;].
\end{aligned}$$

Monotonicity and maximality of the operators $-\Delta : L^2(\Omega) \longrightarrow L^2(\Omega)$ and $\partial I : L^2(\Omega) \longrightarrow L^2(\Omega)$, respectively with $D(-\Delta) = H$ e $D(\partial I) = K$ (see, e.g., [4, Exemple 2.3.4]), enable us to achieve

$$\begin{aligned}
\Big\| T[\varphi_1,\psi_1] \;-\; T[\varphi_2,\psi_2]\Big\|^2_{(L^2(\Omega))^2} &\le \Big\| T_2[\varphi_1, T_1[\varphi_1,\psi_1]] - T_2[\varphi_2, T_1[\varphi_2,\psi_2]]\Big\|^2_{L^2(\Omega)} \\
+\; \Big\| T_1[\varphi_1,\psi_1] - T_1[\varphi_2,\psi_2]\Big\|^2_{L^2(\Omega)} &\le N_1^2 + N_2^2,
\end{aligned} \tag{3.4}$$

for all $[\varphi_i,\psi_i] \in L^2(\Omega) \times K$, $i = 1,2$, where

$$\begin{aligned}
N_1 &= \Big\| - T_1[\varphi_1,\psi_1]\; \lambda'(T_1[\varphi_1,\psi_1]) + T_1[\varphi_2,\psi_2]\; \lambda'(T_1[\varphi_2,\psi_2]) \\
&\quad + \chi^{i-1}(\lambda'(T_1[\varphi_1,\psi_1]) - \lambda'(T_1[\varphi_2,\psi_2])) \\
&\quad + \tau^i\Big(g(\cdot,\varphi_1, T_1[\varphi_1,\psi_1]) - g(\cdot,\varphi_2, T_1[\varphi_2,\psi_2])\Big)\Big\|_{L^2(\Omega)}, \\
N_2 &= \Big\| \tau^i \left(\sigma'(\psi_1) - \sigma'(\psi_2) + \lambda'(\psi_1)\rho(\varphi_1) - \lambda'(\psi_2)\rho(\varphi_2)\right)\Big\|_{L^2(\Omega)}.
\end{aligned}$$

For the sake of convenience, we introduce the notations M_h and L_h to specify $\sup_{[0,1]} |h|$ and the Lipschitz constant of a function $h \in C^{0,1}[0,1]$. Relations (2.1), (2.2), (2.29) and definition of K ensure that

$$N_1 \;\le\; \Big\| (T_1[\varphi_1,\psi_1] - T_1[\varphi_2,\psi_2])\lambda'(T_1[\varphi_1,\psi_1])\Big\|_{L^2(\Omega)}$$

$$
\begin{aligned}
& + \left\| T_1[\varphi_2,\psi_2](\lambda'(T_1[\varphi_1,\psi_1]) - \lambda'(T_1[\varphi_2,\psi_2])) \right\|_{L^2(\Omega)} \\
& + \left\| \lambda'(T_1[\varphi_1,\psi_1]) - \lambda'(T_1[\varphi_2,\psi_2]) \right\|_{L^2(\Omega)} + \tau^i C_g \left(\|\varphi_1 - \varphi_2\|_{L^2(\Omega)} + N_2 \right) \\
& \leq \tau^i C_g \|\varphi_1 - \varphi_2\|_{L^2(\Omega)} + \left(M_{\lambda'} + 2L_{\lambda'} + \tau^i C_g \right) \; N_2,
\end{aligned}
$$

$$
\begin{aligned}
N_2 & \leq \tau^i \|(\sigma'(\psi_1) - \sigma'(\psi_2)) + (\lambda'(\psi_1)\rho(\varphi_1) - \lambda'(\psi_2)\rho(\varphi_2))\|_{L^2(\Omega)} \\
& \leq \tau^i \left(L_{\sigma'} + \frac{L_{\lambda'}}{\vartheta_*} \right) \|\psi_1 - \psi_2\|_{L^2(\Omega)} \; + \; \tau^i \frac{M_{\lambda'}}{\vartheta_*^2} \|\varphi_1 - \varphi_2\|_{L^2(\Omega)}.
\end{aligned}
$$

Squaring and substituting in (3.4), it is straightforward to deduce

$$
\left\| T[\varphi_1,\psi_1] - T[\varphi_2,\psi_2] \right\|^2_{(L^2(\Omega))^2} \leq (\tau^i)^2 \left(C_1^2 \|\varphi_1 - \varphi_2\|^2_{L^2(\Omega)} + C_2^2 \|\psi_1 - \psi_2\|^2_{L^2(\Omega)} \right),
$$

where

$$
\begin{aligned}
C_1^2 & = 2 \left(\left(C_g + \frac{M_{\lambda'}}{\vartheta_*^2} \left(2L_{\lambda'} + M_{\lambda'} + TC_g \right) \right)^2 + \left(\frac{M_{\lambda'}}{\vartheta_*^2} \right)^2 \right), \\
C_2^2 & = 2 \left(L_{\sigma'} + \frac{L_{\lambda'}}{\vartheta_*} \right)^2 \left(\left(2L_{\lambda'} + M_{\lambda'} + TC_g \right)^2 + 1 \right).
\end{aligned}
$$

Therefore, by choosing $\tau < (\max\{C_1; C_2\})^{-1}$, it turns out that T is a contraction mapping from $L^2(\Omega) \times K$ to $L^2(\Omega) \times K$. Moreover, the set $L^2(\Omega) \times K$ is closed in $(L^2(\Omega))^2$. Hence, the equation $T[\vartheta^i, \chi^i] = [\vartheta^i, \chi^i]$ admits one and only one solution in $L^2(\Omega) \times K$ and auxiliary scheme (2.17),(3.3),(2.19)-(2.20) has a unique solution provided that τ, diameter of the grid $\mathcal{P}$, is smaller than a fixed constant. Consequently, in view of the property (2.34) holding also for the auxiliary scheme, if we choose

$$
\xi^i = -\sigma'(\chi^i) - \frac{\lambda'(\chi^i)}{\vartheta^i} - \frac{\chi^i - \chi^{i-1}}{\tau^i}, \tag{3.5}
$$

the triplet $(\vartheta^i, \chi^i, \xi^i)$ reduces to be a solution to the scheme (2.17)-(2.21) thanks to (2.42). Morover, since $\vartheta^i \in D(-\Delta) = H$, (2.32) is satisfied. Recalling (2.1) and (2.34), relation (2.33) follows from a comparison in (3.5).

In conclusion, assuming by contradiction the existence of two distinct solutions to the scheme (2.17)-(2.21), both of them have to satisfy (2.34), and then they solve the auxiliary scheme too. Therefore, they must coincide. The proof of Theorem **2.3** is now complete.

4 A priori estimates

In this section we prove Theorem **2.5** by deducing the uniform stability estimate (2.40) from several a priori estimates independent from the grid $\mathcal{P}$.

Remark 4.1 In the rest of the paper, C stands for any constant which may depend on $M_{\sigma'}$, $M_{\lambda'}$, $L_{\lambda'}$, C_g, $\|g_{00}\|_{L^2(Q)}$, ϑ_*, C_d, $|\Omega|$, T, but not on $\mathcal{P}$.

4.1 First a priori estimate

We set

$$\gamma(\varphi, r) = -\sigma'(r) - \lambda'(r)/\varphi \qquad \forall \varphi \in \mathbb{R},\ \forall r \in [0,1],$$
$$\gamma^i = \gamma(\vartheta^i, \chi^i).$$

Due to (2.1) and (2.34) we have that $\|\gamma^i\|_{L^\infty(\Omega)}$ is uniformly bounded with respect to i. Letting $p > 2$, we multiply (2.18) pointwise by $|\xi^i|^{p-2}\xi^i$, obtaining

$$\frac{\chi^i - \chi^{i-1}}{\tau^i}|\xi^i|^{p-2}\xi^i \ + \ |\xi^i|^p \ = \ \gamma^i|\xi^i|^{p-2}\xi^i \quad \text{a.e. in } \Omega.$$

As $I(r) = 0$ for all $r \in [0,1]$, (2.19) and the definition of subdifferential (cf., e.g., [4, Exemple 2.1.4, p. 21]) imply that the first term above is non negative (on account also of (2.21) and (2.24)). Therefore we have

$$|\xi^i|^p \ \leq \ |\gamma^i|\, |\xi^i|^{p-1}.$$

By virtue of Young's inequality one easily derives

$$|\xi^i|^p \ \leq \ |\gamma^i|^p.$$

Hence, integrating over Ω, multiplying by τ^i, and adding for $i = 1, \ldots, N$, one obtains

$$\|\xi_{\mathcal{P}}\|_{L^p(Q)} \leq \|\gamma_{\mathcal{P}}\|_{L^p(Q)},$$

where $\gamma_{\mathcal{P}}$ is the piecewise constant function defined according to Definition **2.4**. Passing to the limit as $p \to \infty$, from the boundedness of $\|\gamma_{\mathcal{P}}\|_{L^\infty(Q)}$ it follows that

$$\|\xi_{\mathcal{P}}\|_{L^\infty(Q)} \leq \ C. \tag{4.1}$$

Then, by (2.36) we have that

$$\left\|\frac{\partial \widehat{\chi}_{\mathcal{P}}}{\partial t}\right\|_{L^\infty(Q)} \leq \ \|\xi_{\mathcal{P}}\|_{L^\infty(Q)} + \|\gamma_{\mathcal{P}}\|_{L^\infty(Q)} \leq \ C,$$

and, recalling (2.29) and Definition **2.4**, we finally obtain

$$\|\chi_{\mathcal{P}}\|_{L^\infty(Q)} + \ \|\widehat{\chi}_{\mathcal{P}}\|_{W^{1,\infty}(Q)} \leq \ C. \tag{4.2}$$

4.2 Second a priori estimate

We multiply (2.17) by $\tau^i \vartheta^i$ and integrate over Ω. Thanks to (2.20), it results that

$$\frac{1}{2}\|\vartheta^i\|^2_{L^2(\Omega)} + \frac{1}{2}\|\vartheta^i - \vartheta^{i-1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\vartheta^{i-1}\|^2_{L^2(\Omega)} + \tau^i \int_\Omega |\nabla \vartheta^i|^2 = I_8 + I_9, \tag{4.3}$$

where

$$\begin{aligned} I_8 &= -\int_\Omega \lambda'(\chi^i)(\chi^i - \chi^{i-1})\vartheta^i, \\ I_9 &= \tau^i \int_\Omega g^i(\cdot, \vartheta^i, \chi^i)\vartheta^i. \end{aligned}$$

We may control I_8 as follows

$$\begin{aligned} I_8 &\le \int_\Omega |\lambda'(\chi^i)|\,|(\chi^i - \chi^{i-1})|\,|\vartheta^i| \le M_{\lambda'}\|\chi^i - \chi^{i-1}\|_{L^2(\Omega)}\|\vartheta^i\|_{L^2(\Omega)} \\ &\le (M_{\lambda'})^2\,|\Omega|\,\tau^i \left\| \frac{\chi^i - \chi^{i-1}}{\tau^i} \right\|^2_{L^\infty(\Omega)} + \frac{\tau^i}{4}\,\|\vartheta^i\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.4}$$

Regarding I_9, with the help of (2.2) and (2.29), we deduce

$$\begin{aligned} I_9 &\le \tau^i \int_\Omega |g^i(\cdot, \vartheta^i, \chi^i) - g^i(\cdot, 0, 0) + g^i(\cdot, 0, 0)|\,|\vartheta^i| \\ &\le \tau^i \int_\Omega C_g(|\vartheta^i| + |\chi^i|)|\vartheta^i| + |g^i(\cdot, 0, 0)|\,|\vartheta^i| \\ &\le \tau^i C_g(\|\vartheta^i\|_{L^2(\Omega)} + \|\chi^i\|_{L^2(\Omega)})\|\vartheta^i\|_{L^2(\Omega)} + \tau^i \|g^i(\cdot, 0, 0)\|_{L^2(\Omega)}\|\vartheta^i\|_{L^2(\Omega)} \\ &\le \tau^i C_g\|\vartheta^i\|^2_{L^2(\Omega)} + \tau^i|\Omega|\|\vartheta^i\|_{L^2(\Omega)} + \frac{\tau^i}{2}\|g^i(\cdot, 0, 0)\|^2_{L^2(\Omega)} + \frac{\tau^i}{2}\|\vartheta^i\|^2_{L^2(\Omega)}. \end{aligned}$$

Noticing that

$$\|g^i(\cdot, 0, 0)\|^2_{L^2(\Omega)} \le \frac{1}{(\tau^i)^2}\int_{t^{i-1}}^{t^i} dt \int_{t^{i-1}}^{t^i} \|g_{00}(\cdot, t)\|^2_{L^2(\Omega)} dt,$$

we easily obtain

$$I_9 \le \tau^i C(1 + \|\vartheta^i\|^2_{L^2(\Omega)}) + \|g_{00}\|^2_{L^2(t^{i-1}, t^i; L^2(\Omega))}. \tag{4.5}$$

Then, combining (4.4) and (4.5) with (4.3) we infer

$$\begin{aligned} &\frac{1}{2}\|\vartheta^i\|^2_{L^2(\Omega)} + \frac{1}{2}\|\vartheta^i - \vartheta^{i-1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\vartheta^{i-1}\|^2_{L^2(\Omega)} + \tau^i \int_\Omega |\nabla \vartheta^i|^2 \\ &\qquad \le \tau^i C(1 + \|\vartheta^i\|^2_{L^2(\Omega)}) + \|g_{00}\|^2_{L^2(t^{i-1}, t^i; L^2(\Omega))}, \end{aligned} \tag{4.6}$$

by virtue of (4.2). Setting

$$a_m = \frac{1}{2}\|\vartheta^m\|^2_{L^2(\Omega)} + \sum_{i=1}^m \tau^i \int_\Omega |\nabla \vartheta^i|^2, \tag{4.7}$$

the sum of (4.6) for $i = 1, \ldots, m$ yields

$$\frac{1}{2}\|\vartheta^m\|^2_{L^2(\Omega)} - \frac{1}{2}\|\vartheta_{0\mathcal{P}}\|^2_{L^2(\Omega)} + \sum_{i=1}^{m} \tau^i \int_\Omega |\nabla \vartheta^i|^2 \le CT + C \sum_{i=1}^{m} \tau^i \|\vartheta^i\|^2_{L^2(\Omega)} + \|g_{00}\|^2_{L^2(Q)},$$

and from (2.26) and (4.7) it results that

$$a_m \le C \left(1 + \sum_{i=1}^{m} \tau^i a_i\right) \qquad \text{for } m = 1, \ldots, N.$$

At this point, we can let $\tau < (2C)^{-1}$, where C is just the constant of the previous relation, and clearly have

$$a_m \le 2C \left(1 + \sum_{i=1}^{m-1} \tau^i a_i\right) \qquad \text{for } m = 1, \ldots, N.$$

Hence, applying Gronwall's lemma in a discrete form (see, e.g., [10, Proposition 2.2.1]), we recover the bounds

$$\|\vartheta^i\|^2_{L^2(\Omega)} \le C \qquad \text{for } i = 0, \ldots, N\ ,$$

$$\sum_{i=1}^{N} \tau^i \int_\Omega |\nabla \vartheta^i|^2 \ \le\ C,$$

that is,

$$\|\vartheta_{\mathcal{P}}\|_{L^\infty(0,T;L^2(\Omega))\ \cap\ L^2(0,T;H^1(\Omega))} \ \le\ C. \tag{4.8}$$

4.3 Third a priori estimate

Multiply (2.17) by the function $\vartheta^i - \vartheta^{i-1}$ and integrate over Ω, getting

$$\frac{1}{\tau^i}\|\vartheta^i - \vartheta^{i-1}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla \vartheta^i\|^2_{\mathbf{L^2(\Omega)}} + \frac{1}{2}\|\nabla(\vartheta^i - \vartheta^{i-1})\|^2_{\mathbf{L^2(\Omega)}}$$
$$-\frac{1}{2}\|\nabla \vartheta^{i-1}\|^2_{\mathbf{L^2(\Omega)}} = I_{10} + I_{11}, \tag{4.9}$$

where $\mathbf{L^2(\Omega)}$ stands for $(L^2(\Omega))^d$, and

$$I_{10} \;=\; -\int_\Omega \lambda'(\chi^i) \frac{\chi^i - \chi^{i-1}}{\tau^i} (\vartheta^i - \vartheta^{i-1}),$$
$$I_{11} \;=\; \int_\Omega g^i(\cdot, \vartheta^i, \chi^i)(\vartheta^i - \vartheta^{i-1}).$$

On account of (4.2), we may deal with I_{10} as follows

$$\begin{aligned} I_{10} &\le M_{\lambda'} \left\| \frac{\chi^i - \chi^{i-1}}{\tau^i} \right\|_{L^2(\Omega)} \|\vartheta^i - \vartheta^{i-1}\|_{L^2(\Omega)} \\ &\le \frac{\tau^i}{2}(M_{\lambda'})^2 |\Omega| \left\| \frac{\chi^i - \chi^{i-1}}{\tau^i} \right\|^2_{L^\infty(\Omega)} + \frac{1}{2\tau^i}\|\vartheta^i - \vartheta^{i-1}\|^2_{L^2(\Omega)} \\ &\le C\tau^i + \frac{1}{2\tau^i}\|\vartheta^i - \vartheta^{i-1}\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.10}$$

Performing similar calculations for I_{11} (cf. also (4.5)), we obtain

$$\begin{aligned} I_{11} &\leq \|g^i(\cdot,\vartheta^i,\chi^i)\|_{L^2(\Omega)}\|\vartheta^i-\vartheta^{i-1}\|_{L^2(\Omega)} \\ &= \|g^i(\cdot,\vartheta^i,\chi^i)-g^i(\cdot,0,0)+g^i(\cdot,0,0)\|_{L^2(\Omega)}\|\vartheta^i-\vartheta^{i-1}\|_{L^2(\Omega)} \\ &\leq C_g\left(\|\vartheta^i\|_{L^2(\Omega)}+\|\chi^i\|_{L^2(\Omega)}\right)\|\vartheta^i-\vartheta^{i-1}\|_{L^2(\Omega)} \\ &+ \frac{1}{\sqrt{\tau^i}}\|g_{00}\|_{L^2(t^{i-1},t^i;L^2(\Omega))}\|\vartheta^i-\vartheta^{i-1}\|_{L^2(\Omega)} \\ &\leq C\tau^i+\|g_{00}\|^2_{L^2(t^{i-1},t^i;L^2(\Omega))}+\frac{1}{4\tau^i}\|\vartheta^i-\vartheta^{i-1}\|^2_{L^2(\Omega)} \end{aligned} \tag{4.11}$$

because of (2.2) and (4.8). By virtue of (4.10) and (4.11), from (4.9) we deduce

$$\begin{aligned} \frac{\tau^i}{4}\left\|\frac{\vartheta^i-\vartheta^{i-1}}{\tau^i}\right\|^2_{L^2(\Omega)}+\frac{1}{2}\|\nabla\vartheta^i\|^2_{\mathbf{L^2(\Omega)}}+\frac{1}{2}\|\nabla(\vartheta^i-\vartheta^{i-1})\|^2_{\mathbf{L^2(\Omega)}}-\frac{1}{2}\|\nabla\vartheta^{i-1}\|^2_{\mathbf{L^2(\Omega)}} \\ \leq C\tau^i+\|g_{00}\|^2_{L^2(t^{i-1},t^i;L^2(\Omega))}. \end{aligned} \tag{4.12}$$

Then, adding for $i=1,\ldots,m$, we find

$$\frac{1}{4}\sum_{i=1}^m\tau^i\left\|\frac{\vartheta^i-\vartheta^{i-1}}{\tau^i}\right\|^2_{L^2(\Omega)}+\frac{1}{2}\|\nabla\vartheta^m\|^2_{\mathbf{L^2(\Omega)}}-\frac{1}{2}\|\nabla\vartheta_{0\mathcal{P}}\|^2_{\mathbf{L^2(\Omega)}} \leq CT+\|g_{00}\|^2_{L^2(Q)},$$

whence, owing to (2.26), we conclude that

$$\sum_{i=1}^m\tau^i\left\|\frac{\vartheta^i-\vartheta^{i-1}}{\tau^i}\right\|^2_{L^2(\Omega)}+\|\nabla\vartheta^m\|^2_{\mathbf{L^2(\Omega)}}\leq C \qquad \text{for } m=1,\ldots,N.$$

Taking the maximum with respect to m, we obtain

$$\left\|\frac{\partial\widehat{\vartheta}_{\mathcal{P}}}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}+\|\nabla\vartheta_{\mathcal{P}}\|_{L^\infty(0,T;\mathbf{L^2(\Omega)})}\leq C. \tag{4.13}$$

4.4 Conclusion of the proof of Theorem 2.5

Arguing as in (4.11), we observe that

$$\begin{aligned} \|g_{\mathcal{P}}(\cdot,\cdot,\vartheta_{\mathcal{P}},\chi_{\mathcal{P}})\|^2_{L^2(Q)} &= \sum_{i=1}^N\tau^i\|g^i(\cdot,\vartheta^i,\chi^i)\|^2_{L^2(\Omega)} \\ &\leq C\left(\|\vartheta_{\mathcal{P}}\|^2_{L^2(Q)}+\|\chi_{\mathcal{P}}\|^2_{L^2(Q)}\right)+2\|g_{00}\|^2_{L^2(Q)}. \end{aligned}$$

Then, with the help of (2.1), (2.2), (4.2), (4.8), (4.13) and by comparison in (2.35) we get

$$\|\Delta\vartheta_{\mathcal{P}}\|^2_{L^2(0,T;L^2(\Omega))}\leq C. \tag{4.14}$$

Consequently, (2.38), (4.8), (4.13) and standard elliptic estimates ensure that

$$\|\vartheta_{\mathcal{P}}\|_{L^2(0,T;H^2(\Omega))} \leq \; C. \tag{4.15}$$

On the other hand, by Definition **2.4** and (2.26) it is straightforward to verify that

$$\|\widehat{\vartheta}_{\mathcal{P}}\|_{C^0([0,T];H^1(\Omega))} \leq \; C,$$

while (4.15) directly yields

$$\|\widehat{\vartheta}_{\mathcal{P}}\|_{L^2(0,T;H^2(\Omega))} \leq \; C.$$

Therefore, recalling also (4.1), (4.2), and (4.8), it turns out that estimate (2.40) is completely proved.

5 Passage to the limit

This section is devoted to the proof of Theorem **2.6**. Thanks to well-known compactness results, from (2.40) we obtain existence of $\vartheta, \widehat{\vartheta}, \chi, \widehat{\chi}, \xi$ such that, at least for a sequence of partitions $\mathcal{P}$ of $[0,T]$ with diameter $\tau \to 0$, one has

$$\begin{aligned}
\vartheta_{\mathcal{P}} &\stackrel{*}{\rightharpoonup} \vartheta && \text{in} \quad L^\infty(0,T;H^1(\Omega)) \,\cap\, L^2(0,T;H^2(\Omega)), && (5.1)\\
\widehat{\vartheta}_{\mathcal{P}} &\stackrel{*}{\rightharpoonup} \widehat{\vartheta} && \text{in} \quad H^1(0,T;L^2(\Omega)) \,\cap\, L^\infty(0,T;H^1(\Omega)) && \\
& && \qquad \cap\, L^2(0,T;H^2(\Omega)), && (5.2)\\
\chi_{\mathcal{P}} &\stackrel{*}{\rightharpoonup} \chi && \text{in} \quad L^\infty(Q), && (5.3)\\
\widehat{\chi}_{\mathcal{P}} &\stackrel{*}{\rightharpoonup} \widehat{\chi} && \text{in} \quad W^{1,\infty}(Q), && (5.4)\\
\xi_{\mathcal{P}} &\stackrel{*}{\rightharpoonup} \xi && \text{in} \quad L^\infty(Q). && (5.5)
\end{aligned}$$

We are now interested in controlling the differences $\widehat{\vartheta}_{\mathcal{P}} - \vartheta_{\mathcal{P}}$ and $\widehat{\chi}_{\mathcal{P}} - \chi_{\mathcal{P}}$ with respect to suitable norms. Regarding $\widehat{\vartheta}_{\mathcal{P}} - \vartheta_{\mathcal{P}}$, we infer

$$\begin{aligned}
\|\widehat{\vartheta}_{\mathcal{P}} - \vartheta_{\mathcal{P}}\|^2_{L^2(0,T;L^2(\Omega))} &= \sum_{i=1}^{N} \int_{t^{i-1}}^{t^i} \left(\frac{t-t^i}{\tau^i}\right)^2 \|\vartheta^i - \vartheta^{i-1}\|^2_{L^2(\Omega)} dt \\
&= \sum_{i=1}^{N} \frac{(\tau^i)^3}{3} \left\| \frac{\vartheta^i - \vartheta^{i-1}}{\tau^i} \right\|^2_{L^2(\Omega)} \leq \frac{\tau^2}{3} \left\| \frac{\partial \widehat{\vartheta}_{\mathcal{P}}}{\partial t} \right\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

Then, by (4.13) we deduce that

$$\|\widehat{\vartheta}_{\mathcal{P}} - \vartheta_{\mathcal{P}}\|_{L^2(0,T;L^2(\Omega))} \leq C\tau. \tag{5.6}$$

Since

$$\|\widehat{\vartheta}_{\mathcal{P}} - \vartheta_{\mathcal{P}}\|_{L^\infty(0,T;L^2(\Omega))} \leq \sup_{1\leq i\leq N} \|\vartheta^i - \vartheta^{i-1}\|^2_{L^2(\Omega)} \leq \tau \sum_{i=1}^{N} \tau^i \left\| \frac{\vartheta^i - \vartheta^{i-1}}{\tau^i} \right\|^2_{L^2(\Omega)}$$

and (coming back to (4.12))

$$\|\widehat{\vartheta}_{\mathcal{P}} - \vartheta_{\mathcal{P}}\|^2_{L^2(0,T;H^1(\Omega))} = \sum_{i=1}^{N} \frac{\tau^i}{3} \|\vartheta^i - \vartheta^{i-1}\|^2_{H^1(\Omega)} \leq C\tau,$$

we also point out that

$$\|\widehat{\vartheta}_{\mathcal{P}} - \vartheta_{\mathcal{P}}\|_{L^\infty(0,T;L^2(\Omega))} + \|\widehat{\vartheta}_{\mathcal{P}} - \vartheta_{\mathcal{P}}\|^2_{L^2(0,T;H^1(\Omega))} \leq C\sqrt{\tau}. \tag{5.7}$$

Regarding $\widehat{\chi}_{\mathcal{P}} - \chi_{\mathcal{P}}$, we argue as above and exploit (4.2) to get

$$\|\widehat{\chi}_{\mathcal{P}} - \chi_{\mathcal{P}}\|_{L^\infty(0,T;L^2(\Omega))} \leq \tau \left\| \frac{\partial \widehat{\chi}_{\mathcal{P}}}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C\tau. \tag{5.8}$$

Hence, recalling (5.1)-(5.5), a first consequence of (5.6) and (5.8) is that

$$\vartheta = \widehat{\vartheta}\ ,\ \chi = \widehat{\chi} \quad \text{a.e. in } Q.$$

Now we deduce further convergences from (5.1)-(5.4), which, in particular, entail

$$\frac{\partial}{\partial t}\widehat{\vartheta}_{\mathcal{P}} \rightharpoonup \frac{\partial}{\partial t}\vartheta \quad \text{and} \quad \Delta\vartheta_{\mathcal{P}} \rightharpoonup \Delta\vartheta \quad \text{in} \quad L^2(Q).$$

Moreover, the compactness result in ([17, Corollary 4]) and (5.2) ensure that

$$\widehat{\vartheta}_{\mathcal{P}} \to \vartheta \ \text{ in } \ C^0([0,T];V) \ \cap \ L^2(0,T;H^1(\Omega)), \tag{5.9}$$

for every Banach space V such that $H^1(\Omega) \subset V$ with compact embedding. In particular, using also (5.7), we have

$$\widehat{\vartheta}_{\mathcal{P}} \to \vartheta \ \text{ in } \ C^0([0,T];L^2(\Omega)) \ \cap \ L^2(0,T;H^1(\Omega)), \tag{5.10}$$

$$\vartheta_{\mathcal{P}} \to \vartheta \ \text{ in } \ L^\infty(0,T;L^2(\Omega)) \ \cap \ L^2(0,T;H^1(\Omega)). \tag{5.11}$$

At this point, we only have to show the strong convergences (2.47) and (2.49), which cannot be inferred by compactness from (5.3), (5.4). Then we prove them directly following an argument devised and fully detailed in [14]. We start by observing that (2.36) and (2.37) yield (cf. (2.30))

$$\left(\frac{\partial \widehat{\chi}_{\mathcal{P}}}{\partial t} - \gamma(\vartheta_{\mathcal{P}}, \chi_{\mathcal{P}}), \widehat{\chi}_{\mathcal{P}} - v\right) \ \leq \ \left(\frac{\partial \widehat{\chi}_{\mathcal{P}}}{\partial t} - \gamma(\vartheta_{\mathcal{P}}, \chi_{\mathcal{P}}), \widehat{\chi}_{\mathcal{P}} - \chi_{\mathcal{P}}\right)$$
$$\text{a.e. in } (0,T),\ \forall v \in K, \tag{5.12}$$

where $\gamma(\varphi, r) = -\sigma'(r) - \lambda'(r)/\varphi$ for $\varphi \in \mathbb{R}, r \in [0,1]$ and K is the convex set of functions $v \in L^2(\Omega)$ such that $0 \leq v \leq 1$ a.e. in Ω (as in Subsections **4.1** and **3.2**, respectively). We may now consider two partitions $\mathcal{P}_1, \mathcal{P}_2$, with respective diameters τ_1, τ_2. In the following calculations we semplify notation by dropping the subspript $\mathcal{P}$ and using subscripts $1, 2$ to refer to $\mathcal{P}_1, \mathcal{P}_2$. If we write (5.12) for both

$\mathcal{P}_1, \mathcal{P}_2$ and choose $v = \widehat{\chi}_2, v = \widehat{\chi}_1$, respectively, by adding the two inequalities we infer

$$\frac{1}{2}\frac{d}{dt}\|(\widehat{\chi}_1 - \widehat{\chi}_2)(t)\|^2_{L^2(\Omega)} \le \Big(\gamma(\vartheta_1, \chi_1) - \gamma(\vartheta_2, \chi_2), \widehat{\chi}_1 - \widehat{\chi}_2\Big) + \mathcal{R}_1(t) + \mathcal{R}_2(t), \quad (5.13)$$

where

$$\mathcal{R}_i(t) := \left(\frac{\partial \widehat{\chi}_i}{\partial t} - \gamma(\vartheta_i, \chi_i), \widehat{\chi}_i - \chi_i\right), \quad i = 1, 2.$$

Integrating over $(0,t)$, due to (2.1), (2.34) and (5.8), we obtain

$$\begin{aligned}\|(\widehat{\chi}_1 - \widehat{\chi}_2)(t)\|^2_{L^2(\Omega)} &\le C_3\|\vartheta_1 - \vartheta_2\|^2_{L^2(0,t;L^2(\Omega))} + C\int_0^t \|(\widehat{\chi}_1 - \widehat{\chi}_2)(s)\|^2_{L^2(\Omega)} ds \\ &+ C\tau_1^2 + C\tau_2^2 + \int_0^T \mathcal{R}_1(s)ds + \int_0^T \mathcal{R}_2(s)ds + \|\chi_{0,1} - \chi_{0,2}\|^2_{L^2(\Omega)}, \quad (5.14)\end{aligned}$$

for some constants C_3 and C depending only on data. Our next aim is estimating the quantity $\int_0^T \mathcal{R}(t)dt$ (with obvious notation), for a generic partition $\mathcal{P}$ with diameter τ. Let $t \in]t^{i-1}, t^i]$ and set $\delta^i = (\chi^i - \chi^{i-1})/\tau^i$, $\gamma^i = \gamma(\vartheta^i, \chi^i)$ for $i = 1, \ldots, N$; then $\mathcal{R}(t)$ can be rewritten as (see Definition **2.4**)

$$\begin{aligned}\mathcal{R}(t) &= \Big(\delta^i - \gamma^i, \alpha_i(t)\chi^{i-1} + (1 - \alpha_i(t))\chi^i - \chi^i\Big) \\ &= -\alpha_i(t)\Big(\delta^i - \gamma^i, \chi^i - \chi^{i-1}\Big), \qquad (5.15)\end{aligned}$$

where $\alpha_i(t) := (t^i - t)/\tau^i$. Since $\alpha_i(t) \ge 0$, the inequality (2.30) ensures that $\mathcal{R}(t) \ge 0$. For $i \ge 2$, we consider (2.30) at the previous step, multiply by $-\alpha_i(t)$, and have

$$-\alpha_i(t)\Big(\ \delta^{i-1} - \gamma^{i-1}, \chi^{i-1} - v\ \Big) \ge 0.$$

Choosing $v = \chi^i$ and adding to (5.15) yield

$$\begin{aligned}0 \le \mathcal{R}(t) &\le -\alpha_i(t)\tau^i\Big((\delta^i - \delta^{i-1}, \delta^i) - (\gamma^i - \gamma^{i-1}, \delta^i)\Big) \\ &\le -\frac{\alpha_i(t)}{\tau^i}\tau^2\Big((\delta^i - \delta^{i-1}, \delta^i) - (\gamma^i - \gamma^{i-1}, \delta^i)\Big) \quad \text{whenever } i > 1,\end{aligned}$$

by virtue of $(\tau/\tau^i)^2 \ge 1$. Now, we integrate $\mathcal{R}$ over $[0,T]$. In view of (2.1), (2.34), (4.2), (4.13), by the Lipschitz continuity of the function γ we deduce

$$\begin{aligned}\int_0^T \mathcal{R}(t)dt \le& -\frac{\tau^2}{2}\sum_{i=2}^N \Big((\delta^i - \delta^{i-1}, \delta^i) - (\gamma^i - \gamma^{i-1}, \delta^i)\Big) \\ &- \frac{1}{2}(\tau^1)^2\Big((\delta^1, \delta^1) - (\gamma^1, \delta^1)\Big) \\ \le& -\frac{\tau^2}{2}\sum_{i=2}^N \left(\frac{1}{2}\|\delta^i\|^2_{L^2(\Omega)} + \frac{1}{2}\|\delta^i - \delta^{i-1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\delta^{i-1}\|^2_{L^2(\Omega)}\right) \\ &+ \frac{L_\gamma \tau^2}{2}\sum_{i=2}^N \Big(\|\vartheta^i - \vartheta^{i-1}\|_{L^2(\Omega)} + \tau^i\|\delta^i\|_{L^2(\Omega)}\Big)\|\delta^i\|_{L^2(\Omega)}\end{aligned}$$

$$
\begin{aligned}
& + \frac{1}{2}(\tau^1)^2\|\gamma^1\|_{L^2(\Omega)}\|\delta^1\|_{L^2(\Omega)} \\
\leq \quad & -\frac{\tau^2}{4}\sum_{i=2}^{N}\left(\|\delta^i\|^2_{L^2(\Omega)} - \|\delta^{i-1}\|^2_{L^2(\Omega)}\right) + \frac{L_\gamma\tau^2}{2}\sum_{i=2}^{N}\tau^i\left\|\frac{\vartheta^i-\vartheta^{i-1}}{\tau^i}\right\|^2_{L^2(\Omega)} \\
& + L_\gamma\tau^2\sum_{i=2}^{N}\tau^i\|\delta^i\|^2_{L^2(\Omega)} + \frac{1}{2}(\tau^1)^2\|\gamma^1\|_{L^2(\Omega)}\|\delta^1\|_{L^2(\Omega)}.
\end{aligned}
$$

Noticing that the first sum in last inequality telescopes and recalling (2.1), (4.2) and (4.13), one infers

$$
\int_0^T \mathcal{R}(t)dt \;\leq\; C\tau^2\left(\left\|\frac{\partial\widehat{\chi}_\mathcal{P}}{\partial t}\right\|^2_{L^\infty(Q)} + \left\|\frac{\partial\widehat{\vartheta}_\mathcal{P}}{\partial t}\right\|^2_{L^2(Q)} + \|\gamma\|^2_{L^\infty(Q)}\right) \;\leq\; C\tau^2
$$

This estimate can be estabilished either for $\mathcal{R}_1$ or $\mathcal{R}_2$, so that (5.14) entails

$$
\begin{aligned}
\|(\widehat{\chi}_1-\widehat{\chi}_2)(t)\|^2_{L^2(\Omega)} \;\leq\; & C_3\|\vartheta_1-\vartheta_2\|^2_{L^2(0,t;L^2(\Omega))} \\
& + C\int_0^t \|(\widehat{\chi}_1-\widehat{\chi}_2)(s)\|^2_{L^2(\Omega)}\,ds \\
& + C\tau_1^2 + C\tau_2^2 + \|\chi_{0,1}-\chi_{0,2}\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{5.16}
$$

for all $t \in [0,T]$. Now, (2.28), (5.11), and Gronwall's lemma (see, e.g., the version reported in [1, Thm 2.1]) ensure that $\{\widehat{\chi}_\mathcal{P}\}$ is a Cauchy sequence in $C^0([0,T];L^2(\Omega))$, which is complete. Therefore, the convergence $\widehat{\chi}_\mathcal{P} \stackrel{*}{\rightharpoonup} \chi$ in $W^{1,\infty}(Q)$ allows us to conclude

$$
\widehat{\chi}_\mathcal{P} \to \chi \quad \text{in} \quad C^0([0,T];L^2(\Omega)),
\tag{5.17}
$$

and, thanks to (5.8),

$$
\chi_\mathcal{P} \to \chi \text{ in } \; L^\infty(0,T;L^2(\Omega)).
\tag{5.18}
$$

We now prove that $\xi \in \partial I(\chi)$ a.e. in Q. Relations (5.5) and (5.18) imply

$$
\limsup_{\tau\searrow 0}\iint_Q \xi_\mathcal{P}\chi_\mathcal{P} \;=\; \iint_Q \xi\chi,
$$

whence it suffices to recall, for instance, ([4, Proposition 2.5, p. 27]).

The convergence

$$
g_\mathcal{P}(\cdot,\cdot,\vartheta_\mathcal{P},\chi_\mathcal{P}) \longrightarrow g(\cdot,\cdot,\vartheta,\chi) \;\text{ in }\; L^2(0,T;L^2(\Omega)),
$$

turns out to be a consequence of the technical lemma proved in the Appendix. Regarding the nonlinearities $\lambda'(\chi_\mathcal{P})$, $\sigma'(\chi_\mathcal{P})$, $\lambda'(\chi_\mathcal{P})/\vartheta_\mathcal{P}$ in equations (2.35), (2.36), we observe that they are bounded and Lipschitz continuous with respect to their arguments $\chi_\mathcal{P},\vartheta_\mathcal{P}$ thanks to (2.1) and (2.34). Then, due to the convergences (5.11) and (5.18), we have that

$$
\begin{aligned}
\lambda'(\chi_\mathcal{P}) \to \lambda'(\chi) \quad & \text{in} \quad L^\infty(0,T;L^2(\Omega)), \\
\sigma'(\chi_\mathcal{P}) \to \sigma'(\chi) \quad & \text{in} \quad L^\infty(0,T;L^2(\Omega)), \\
\lambda'(\chi_\mathcal{P})/\vartheta_\mathcal{P} \to \lambda'(\chi)/\vartheta \quad & \text{in} \quad L^\infty(0,T;L^2(\Omega)).
\end{aligned}
\tag{5.19}
$$

Moreover, (5.19) and (5.4) entail

$$\lambda'(\chi_{\mathcal{P}})\frac{\partial \widehat{\chi}_{\mathcal{P}}}{\partial t} \overset{*}{\rightharpoonup} \lambda'(\chi)\chi_t = (\lambda(\chi))_t \quad \text{in} \;\; L^\infty(0,T,L^2(\Omega)).$$

We are now able to pass to the limit in scheme (2.35)-(2.39) and to verify that limit functions ϑ, χ, ξ yield a solution to (1.1), (1.3), (1.4), (2.13), (2.14). In this way, an alternative proof of the existence result in [8] has been obtained. In addition, referring to Proposition **2.1** for uniqueness of such solution, it results that the convergences of this section hold not only for a suitable sequence of partitions, but for the whole family of grids as $\tau \to 0$. This completes the proof of Theorem **2.6**.

6 Error estimates

In this section we derive estimates (2.54) and (2.55), and thus prove Theorem **2.8**.

6.1 First error estimate

Recalling (2.28) and passing to the limit as $\tau_2 \to 0$ in (5.16), we have

$$\begin{aligned} \|(\widehat{\chi}_{\mathcal{P}} - \chi)(t)\|^2_{L^2(\Omega)} \;\; &\leq \;\; C_3\,\|\vartheta_{\mathcal{P}} - \vartheta\|^2_{L^2(0,t;L^2(\Omega))} \\ &+ \;\; C\int_0^t \|(\widehat{\chi}_{\mathcal{P}} - \chi)(s)\|^2_{L^2(\Omega)}\,ds \;+\; C\tau^2 \end{aligned} \tag{6.1}$$

for any partition $\mathcal{P}$ with diameter τ. We may now integrate (1.1) over $(0,t)$ and get

$$\vartheta(\cdot,t) - \vartheta_0 + \lambda(\chi(\cdot,t)) - \lambda(\chi_0) - \int_0^t \Delta\vartheta(\cdot,s)ds = \int_0^t g(\cdot,s,\vartheta(\cdot,s),\chi(\cdot,s))ds,$$

for $t \in (0,T)$. Integrating (2.35) over $(0,t)$, easy calculations provide

$$\begin{aligned} &\widehat{\vartheta}_{\mathcal{P}}(\cdot,t) - \vartheta_{0\mathcal{P}} + \lambda(\widehat{\chi}_{\mathcal{P}}(\cdot,t)) - \lambda(\chi_{0\mathcal{P}}) - \int_0^t \Delta\vartheta_{\mathcal{P}}(\cdot,s)ds \\ &\qquad = \int_0^t \Big(\lambda'(\widehat{\chi}_{\mathcal{P}}(\cdot,s)) - \lambda'(\chi_{\mathcal{P}}(\cdot,s))\Big)\frac{\partial \widehat{\chi}_{\mathcal{P}}}{\partial t}(\cdot,s)ds \\ &\qquad + \int_0^t g_{\mathcal{P}}(\cdot,s,\vartheta_{\mathcal{P}}(\cdot,s),\chi_{\mathcal{P}}(\cdot,s))ds. \end{aligned}$$

By subtracting the last two equations, multiplying by the function $\vartheta - \vartheta_{\mathcal{P}}$, and integrating over $\Omega \times (0,t)$, we infer

$$\|\vartheta - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} + \frac{1}{2}\int_\Omega \left|\nabla\left(\int_0^t (\vartheta - \vartheta_{\mathcal{P}})(\cdot,s)ds\right)\right|^2 \leq \sum_{i=12}^{18} I_i \tag{6.2}$$

for any $t \in [0,T]$, where

$$
\begin{aligned}
I_{12} &= \int_0^t\!\!\int_\Omega |\vartheta_{0\mathcal{P}} - \vartheta_0|\, |\vartheta - \vartheta_{\mathcal{P}}|, \\
I_{13} &= \int_0^t\!\!\int_\Omega |\lambda(\chi_{0\mathcal{P}}) - \lambda(\chi_0)|\, |\vartheta - \vartheta_{\mathcal{P}}|, \\
I_{14} &= \int_0^t\!\!\int_\Omega |\lambda(\widehat{\chi}_{\mathcal{P}}) - \lambda(\chi)|\, |\vartheta - \vartheta_{\mathcal{P}}|, \\
I_{15} &= \int_0^t\!\!\int_\Omega |\widehat{\vartheta}_{\mathcal{P}} - \vartheta_{\mathcal{P}}|\, |\vartheta - \vartheta_{\mathcal{P}}|, \\
I_{16} &= \int_0^t\!\!\int_\Omega \left(\int_0^s |\lambda'(\widehat{\chi}_{\mathcal{P}}) - \lambda'(\chi_{\mathcal{P}})| \left|\frac{\partial \widehat{\chi}_{\mathcal{P}}}{\partial t}\right|\right) |\vartheta - \vartheta_{\mathcal{P}}|, \\
I_{17} &= \int_0^t\!\!\int_\Omega \left(\int_0^s |g(\cdot,\cdot,\vartheta,\chi) - g(\cdot,\cdot,\vartheta_{\mathcal{P}},\chi_{\mathcal{P}})|\right) |\vartheta - \vartheta_{\mathcal{P}}|, \\
I_{18} &= \int_0^t\!\!\int_\Omega \left(\int_0^s |g(\cdot,\cdot,\vartheta_{\mathcal{P}},\chi_{\mathcal{P}}) - g_{\mathcal{P}}(\cdot,\cdot,\vartheta_{\mathcal{P}},\chi_{\mathcal{P}})|\right) |\vartheta - \vartheta_{\mathcal{P}}|.
\end{aligned}
$$

Conditions (2.1), (2.27), (2.28), (5.8), (5.6) allow us to control $I_{12}, I_{13}, I_{14}, I_{15}$ as follows

$$
\begin{aligned}
I_{12} &\le \frac{1}{16}\|\vartheta - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} + 4T\|\vartheta_{0\mathcal{P}} - \vartheta_0\|^2_{L^2(\Omega)} \\
&\le \frac{1}{16}\|\vartheta - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2, && (6.3) \\
I_{13} &\le \frac{1}{16}\|\vartheta - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} + 4TL_\lambda^2\|\chi_{0\mathcal{P}} - \chi_0\|^2_{L^2(\Omega)} \\
&\le \frac{1}{16}\|\vartheta - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2, && (6.4) \\
I_{14} &\le \frac{1}{16}\|\vartheta - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} + 4L_\lambda^2\|\chi - \widehat{\chi}_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} \\
&\le \frac{1}{16}\|\vartheta - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} + 4L_\lambda^2\|\chi - \chi_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2, && (6.5) \\
I_{15} &\le \frac{1}{16}\|\vartheta - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} + 4\|\widehat{\vartheta}_{\mathcal{P}} - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} \\
&\le \frac{1}{16}\|\vartheta - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2. && (6.6)
\end{aligned}
$$

Regarding I_{16}, from (4.2) and (5.8) we have that

$$
\begin{aligned}
I_{16} &\le C\int_0^t\!\!\int_\Omega \left(\int_0^s |(\widehat{\chi}_{\mathcal{P}} - \chi_{\mathcal{P}})(\cdot,r)|\, |(\vartheta - \vartheta_{\mathcal{P}})(\cdot,s)| dr\right) ds \\
&\le C\int_0^t \|\widehat{\chi}_{\mathcal{P}} - \chi_{\mathcal{P}}\|^2_{L^2(0,s;L^2(\Omega))} ds + \frac{1}{16}\|\vartheta - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} \\
&\le \frac{1}{16}\|\vartheta - \vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2. && (6.7)
\end{aligned}
$$

Owing to (2.2), we may control I_{17} as follows

$$
I_{17} \le \int_0^t\!\!\int_\Omega C_g \left(\int_0^s \left(|(\vartheta - \vartheta_{\mathcal{P}})(\cdot,r)| + |(\chi - \chi_{\mathcal{P}})(\cdot,r)|\right) |(\vartheta - \vartheta_{\mathcal{P}})(\cdot,s)| dr\right) ds
$$

$$
\begin{aligned}
&= C_g \int_0^t\!\!\int_\Omega \left(\int_0^s |(\chi-\chi_{\mathcal P})(\cdot,r)|\,|(\vartheta-\vartheta_{\mathcal P})(\cdot,s)|dr\right) ds\\
&+ C_g \int_0^t\!\!\int_\Omega \left(\int_0^s |(\vartheta-\vartheta_{\mathcal P})(\cdot,r)|\,|(\vartheta-\vartheta_{\mathcal P})(\cdot,s)|dr\right) ds.
\end{aligned}
$$

By means of easy calculations we obtain

$$
\begin{aligned}
I_{17} \;&\le\; \frac{1}{8}\|\vartheta-\vartheta_{\mathcal P}\|^2_{L^2(0,t;L^2(\Omega))} \;+\; C\,\|\chi-\chi_{\mathcal P}\|^2_{L^2(0,t;L^2(\Omega))}\\
&+\; C\int_0^t \|\vartheta-\vartheta_{\mathcal P}\|^2_{L^2(0,s;L^2(\Omega))}ds. \qquad (6.8)
\end{aligned}
$$

Regarding I_{18}, we have that

$$
\begin{aligned}
I_{18} \;&\le\; \int_0^t \left(\int_0^s \|g(r,\vartheta_{\mathcal P},\chi_{\mathcal P}) - g_{\mathcal P}(r,\vartheta_{\mathcal P},\chi_{\mathcal P})\|_{L^2(\Omega)}\|(\vartheta-\vartheta_{\mathcal P})(s)\|_{L^2(\Omega)}dr\right) ds\\
&\le\; \int_0^t \|g(\cdot,\vartheta_{\mathcal P},\chi_{\mathcal P}) - g_{\mathcal P}(\cdot,\vartheta_{\mathcal P},\chi_{\mathcal P})\|_{L^1(0,T;L^2(\Omega))}\|(\vartheta-\vartheta_{\mathcal P})(s)\|_{L^2(\Omega)}ds\\
&\le\; \frac{1}{16}\|\vartheta-\vartheta_{\mathcal P}\|^2_{L^2(0,t;L^2(\Omega))} \;+\; C\|g(\cdot,\vartheta_{\mathcal P},\chi_{\mathcal P}) - g_{\mathcal P}(\cdot,\vartheta_{\mathcal P},\chi_{\mathcal P})\|^2_{L^1(0,T;L^2(\Omega))}.
\end{aligned}
$$

From (4.13) we deduce that $\vartheta_{\mathcal P} \in BV(0,T;L^2(\Omega))$. In particular, note

$$
Var_{[0,T],L^2(\Omega)}[\vartheta_{\mathcal P}] = \sup_{\mathcal P}\sum_{i=1}^N \|\vartheta^i - \vartheta^{i-1}\|_{L^2(\Omega)} \le \sup_{\mathcal P}\left\|\frac{\partial\widehat{\vartheta}_{\mathcal P}}{\partial t}\right\|_{L^1(0,T;L^2(\Omega))} \le C.
$$

In the same way, by virtue of (4.2) we also recover that $\chi_{\mathcal P} \in BV(0,T;L^2(\Omega))$. Therefore, owing to (2.52)-(2.53) we infer

$$
\begin{aligned}
&\|g(\cdot,\vartheta_{\mathcal P},\chi_{\mathcal P}) - g_{\mathcal P}(\cdot,\vartheta_{\mathcal P},\chi_{\mathcal P})\|_{L^1(0,T;L^2(\Omega))}\\
&\quad=\; \sum_{i=1}^N \int_{t^{i-1}}^{t^i} \left\|g(s,\vartheta^i,\chi^i) - \frac{1}{\tau^i}\int_{t^{i-1}}^{t^i} g(r,\vartheta^i,\chi^i)dr\right\|_{L^2(\Omega)} ds\\
&\quad\le\; \sum_{i=1}^N \frac{1}{\tau^i}\int_{t^{i-1}}^{t^i}\left(\int_{t^{i-1}}^{t^i}\|g(s,\vartheta^i,\chi^i) - g(r,\vartheta^i,\chi^i)\|_{L^2(\Omega)}dr\right) ds\\
&\quad\le\; \sum_{i=1}^N \tau^i\, Var_{[t^{i-1},t^i];L^2(\Omega)}[g(\cdot,\cdot,\vartheta_{\mathcal P},\chi_{\mathcal P})]\\
&\quad\le\; \tau\, Var_{[0,T];L^2(\Omega)}[g(\cdot,\cdot,\vartheta_{\mathcal P},\chi_{\mathcal P})] \;\le\; C\tau.
\end{aligned}
$$

Thus, we obtain

$$
I_{18} \le \frac{1}{16}\|\vartheta-\vartheta_{\mathcal P}\|^2_{L^2(0,t;L^2(\Omega))} + C\tau^2. \qquad (6.9)
$$

Hence, collecting (6.3)-(6.9) and recalling (2.27)-(2.28), inequality (6.2) finally yields

$$
\begin{aligned}
&\|\vartheta-\vartheta_{\mathcal P}\|^2_{L^2(0,t;L^2(\Omega))} + \int_\Omega \left|\nabla\left(\int_0^t (\vartheta-\vartheta_{\mathcal P})(\cdot,s)ds\right)\right|^2\\
&\quad\le C\left(\int_0^t \|\vartheta-\vartheta_{\mathcal P}\|^2_{L^2(0,s;L^2(\Omega))}ds \;+\; \|\chi-\chi_{\mathcal P}\|^2_{L^2(0,t;L^2(\Omega))} \;+\; \tau^2\right).
\end{aligned}
$$

Now we multiply (6.1) by $1/(2C_3)$, add it to (6.2), and deduce

$$\frac{1}{2}\|\vartheta-\vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))}+\frac{1}{2C_3}\|(\chi-\widehat{\chi}_{\mathcal{P}})(t)\|^2_{L^2(\Omega)}+\int_\Omega\left|\nabla\left(\int_0^t(\vartheta-\vartheta_{\mathcal{P}})(\cdot,s)ds\right)\right|^2$$
$$\leq C\left(\int_0^t\|\vartheta-\vartheta_{\mathcal{P}}\|^2_{L^2(0,s;L^2(\Omega))}ds\ +\ \int_0^t\|(\chi-\widehat{\chi}_{\mathcal{P}})(s)\|^2_{L^2(\Omega)}ds\ +\ \tau^2\right),$$

for all $t\in[0,T]$. Gronwall's lemma now help us to provide the estimate

$$\|\vartheta-\vartheta_{\mathcal{P}}\|_{L^2(0,t;L^2(\Omega))}+\|(\chi-\widehat{\chi}_{\mathcal{P}})(t)\|_{L^2(\Omega)}+\left\|\int_0^t(\vartheta-\vartheta_{\mathcal{P}})(\cdot,s)ds\right\|_{H^1(\Omega)}\leq C\tau.$$

Consequently, thanks to (5.6) we readily derive (2.54). We observe that this estimate is optimal with respect to the order of convergence, with respect to regularity for the phase parameter (refer to [13]), and it involves only computable quantities which depend solely on data (in particular, note that they depend also on the final time T). Moreover, no *a priori* constraints restrict the choice of consecutive time-steps, which could be tailored according to other *a posteriori* error estimates (depending on the discrete solution as well).

6.2 Second error estimate.

We take the difference between (1.1) and (2.35), multiply by the function $\vartheta-\widehat{\vartheta}_{\mathcal{P}}$, and integrate over $\Omega\times(0,t)$.

$$\frac{1}{2}\|(\vartheta-\widehat{\vartheta}_{\mathcal{P}})(t)\|^2_{L^2(\Omega)}+\int_0^t\|\nabla(\vartheta-\widehat{\vartheta}_{\mathcal{P}})(t)\|^2_{\mathbf{L^2(\Omega)}}\leq\frac{1}{2}\|\vartheta_0-\vartheta_{0\mathcal{P}}\|^2_{L^2(\Omega)}+\sum_{i=19}^{21}I_i,\quad(6.10)$$

where

$$\begin{aligned}I_{19}&=\int_0^t\int_\Omega|g_{\mathcal{P}}(\cdot,\cdot,\vartheta_{\mathcal{P}},\chi_{\mathcal{P}})-g(\cdot,\cdot,\vartheta,\chi)|\,|\vartheta-\widehat{\vartheta}_{\mathcal{P}}|,\\ I_{20}&=\int_0^t\int_\Omega\left|\lambda'(\chi)\frac{\partial\chi}{\partial t}-\lambda'(\chi_{\mathcal{P}})\frac{\partial\widehat{\chi}_{\mathcal{P}}}{\partial t}\right|\,|\vartheta-\widehat{\vartheta}_{\mathcal{P}}|,\\ I_{21}&=\left|\int_0^t\int_\Omega(\Delta(\widehat{\vartheta}_{\mathcal{P}}-\vartheta_{\mathcal{P}}))(\vartheta-\widehat{\vartheta}_{\mathcal{P}})\right|.\end{aligned}$$

Arguing for I_{19} as for I_{17},I_{18}, by virtue of conditions (2.1), (2.2), (2.52)-(2.54), (4.2), (4.14), (5.6), and (5.8) it is straightforward to obtain

$$\begin{aligned}I_{19}&\leq C\left(\|\vartheta-\vartheta_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))}+\|\chi-\chi_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))}\right)\\&\qquad+C\|\vartheta-\widehat{\vartheta}_{\mathcal{P}}\|^2_{L^2(0,t;L^2(\Omega))}+C\tau^2\ \leq\ C\tau^2\\ I_{20}&\leq C\|\vartheta-\widehat{\vartheta}_{\mathcal{P}}\|_{L^1(0,t;L^2(\Omega))}\leq\ C\|\vartheta-\widehat{\vartheta}_{\mathcal{P}}\|_{L^2(0,t;L^2(\Omega))}\ \leq\ C\tau,\\ I_{21}&\leq\frac{\tau}{2}\|\Delta(\vartheta_{\mathcal{P}}-\widehat{\vartheta}_{\mathcal{P}})\|^2_{L^2(Q)}+\ \frac{1}{2\tau}\|\vartheta-\widehat{\vartheta}_{\mathcal{P}}\|^2_{L^2(Q)}\ \leq\ C\tau.\end{aligned}$$

Then, recalling (2.27), estimate (6.10) implies

$$\frac{1}{2}\|(\vartheta - \widehat{\vartheta}_{\mathcal{P}})(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla(\vartheta - \widehat{\vartheta}_{\mathcal{P}})(t)\|^2_{\mathbf{L^2(\Omega)}} \le C\tau,$$

and (2.55) immediately follows.

7 Appendix

For the sake of completeness, we include the following technical result, which is used in Section 5 to pass to the limit in the right hand side of (2.35). In fact, we do not know whether its proof is reported in the literature.

Lemma 7.1 *Let Ω be an open set of $\mathbb{R}^N$ $(N \ge 1)$, $T > 0$, and $Q = \Omega \times (0,T)$. Consider a Carathéodory function $g : \Omega \times (0,T) \times \mathbb{R} \longrightarrow \mathbb{R}$ such that there exists a constant $\Lambda > 0$ fulfilling*

$$g(\cdot,\cdot,u) \in L^2(Q) \quad \forall u \in \mathbb{R}, \tag{7.1}$$

$$|g(x,t,u_1) - g(x,t,u_2)| \le \Lambda\, |u_1 - u_2| \quad \text{for a.e. } (x,t) \in Q, \quad \forall u_1, u_2 \in \mathbb{R} \tag{7.2}$$

and some element $v \in L^2(Q)$. Let $\mathcal{P} = \{0 = t^0, t^1, \ldots, t^{N-1}, t^N = T\}$ denote an arbitrary partition of $[0,T]$ with diameter $\tau = \max_{1\le i\le N} \tau^i$, where $\tau^i = t^i - t^{i-1}$. For $i = 1, \ldots, N$ we define

$$g^i(x,u) = \frac{1}{\tau^i}\int_{t^{i-1}}^{t^i} g(x,\xi,u)\, d\xi, \qquad \text{for a.e. } x \in \Omega, \quad \forall u \in \mathbb{R}\ ,$$

and, for every $t \in (t^{i-1}, t^i]$, $x \in \Omega$, $u \in \mathbb{R}$, set

$$g_{\mathcal{P}}(x,t,u) = g^i(x,u).$$

Then, for any family of functions $v_{\mathcal{P}} \in L^2(Q)$ such that

$$v_{\mathcal{P}} \longrightarrow v \quad \text{in } L^2(Q),$$

we have

$$g_{\mathcal{P}}(\cdot,\cdot,v_{\mathcal{P}}(\cdot,\cdot)) \longrightarrow g(\cdot,\cdot,v(\cdot,\cdot)) \quad \text{in } L^2(Q)$$

whatever is the sequence of partitions $\mathcal{P}$ with diameters tending to 0.

Remark 7.2 In particular, this result justifies the passage to the limit within the nonlinearity $g_{\mathcal{P}}(\cdot,\cdot,\vartheta_{\mathcal{P}},\chi_{\mathcal{P}})$ in (2.35).

Proof. Let us consider the application $G : (0,T) \times L^2(\Omega) \longrightarrow L^2(\Omega)$ defined by

$$G(t,u)(x) = g(x,t,u(x)) \quad \text{for a.e. } x \in \Omega.$$

Owing to (7.1) G is well defined for a.e. $t \in (0,T)$ and all $u \in L^2(\Omega)$. Moreover, (7.1) and (7.2) entail

$$G(\cdot,u) \in L^2(0,T;L^2(\Omega)) \quad \forall u \in \mathbb{R}, \tag{7.3}$$

$$\|G(t,u_1) - G(t,u_2)\|_{L^2(\Omega)} \leq \Lambda \, \|u_1 - u_2\|_{L^2(\Omega)}$$
$$\text{for a.e. } t \in (0,T) \,, \quad \forall u_1, u_2 \in L^2(\Omega). \tag{7.4}$$

Next, we construct $G_{\mathcal{P}}$ exactly as G, but starting from $g_{\mathcal{P}}$. Note that the function $G_{\mathcal{P}}$ satisfies (7.3) and (7.4) either, with the same constant Λ. Namely, it is immediately clear that

$$G_{\mathcal{P}}(\cdot,u) \in L^2(0,T;L^2(\Omega)) \quad \forall u \in \mathbb{R}, \tag{7.5}$$

$$\|G_{\mathcal{P}}(t,u_1) - G_{\mathcal{P}}(t,u_2)\|_{L^2(\Omega)} \leq \Lambda \, \|u_1 - u_2\|_{L^2(\Omega)}$$
$$\text{for a.e. } t \in (0,T) \,, \quad \forall u_1, u_2 \in L^2(\Omega). \tag{7.6}$$

Separability of $L^2(\Omega)$ allows us to fix a dense and countable subset D of $L^2(\Omega)$. Referring now to [4, p. 140], for instance, and letting $u \in D$, it is not difficult to realize that

$$G_{\mathcal{P}}(t,u) \longrightarrow G(t,u) \quad \text{in } L^2(\Omega), \quad \text{for a.e. } t \in (0,T),$$

as well as (using, e.g., an extension of Lebesgue's dominated convergence theorem, see below)

$$G_{\mathcal{P}}(\cdot,u) \longrightarrow G(\cdot,u) \quad \text{in } L^2(0,T;L^2(\Omega)).$$

Then, let us term N_u the subset of $(0,T)$ where there is no pointwise convergence, namely

$$N_u = (0,T) \setminus \{t \in (0,T) \text{ such that } G_{\mathcal{P}}(t,u) \longrightarrow G(t,u)\}.$$

Taking the union $N = \bigcup_{v \in D} N_v$, it turns out that $|N| = 0$, because N is a countable union of subsets with zero Lebesgue measure. Moreover

$$\forall t \in (0,T) \setminus N, \quad G_{\mathcal{P}}(t,u(t)) \longrightarrow G(t,u(t)) \ \text{ in } L^2(\Omega),$$

for all $u \in L^2(0,T;L^2(\Omega))$ such that for every $t \in (0,T)$ one has $u(t) \in D$. At this point, let $u, u_\varepsilon \in L^2(0,T;L^2(\Omega))$ such that $u_\varepsilon(t) \in D$ for every $t \in (0,T)$ and besides, for a.e. $t \in (0,T)$, $\|u(t) - u_\varepsilon(t)\|_{L^2(\Omega)} \longrightarrow 0$ as $\varepsilon \to 0$. We call M the set (of zero Lebesgue measure) which contains $t \in (0,T)$ such that $u_\varepsilon(t) \not\longrightarrow u(t)$. The choice of such sequence $\{u_\varepsilon\}$ is possible due to density of D in $L^2(\Omega)$. Thanks to (7.4) and (7.6), we have

$$\begin{aligned}
\|G_{\mathcal{P}}(t,u(t)) - G(t,u(t))\|_{L^2(\Omega)} &\leq \|G_{\mathcal{P}}(t,u(t)) - G_{\mathcal{P}}(t,u_\varepsilon(t))\|_{L^2(\Omega)} \\
&+ \ \|G_{\mathcal{P}}(t,u_\varepsilon(t)) - G(t,u_\varepsilon(t))\|_{L^2(\Omega)} + \|G(t,u_\varepsilon(t)) - G(t,u(t))\|_{L^2(\Omega)} \\
&\leq \ 2\Lambda \|u_\varepsilon(t) - u(t)\|_{L^2(\Omega)} + \|G_{\mathcal{P}}(t,u_\varepsilon(t)) - G(t,u_\varepsilon(t))\|_{L^2(\Omega)}, \\
&\qquad \text{for a.e. } t \in (0,T).
\end{aligned}$$

Our next aim is showing that, for a fixed $t \in (0,T) \setminus (N \cup M)$, the second member of the previous inequality tends to zero. In particular, for any $\delta > 0$, we may take $\hat{\varepsilon}$ such that, for every $\varepsilon \leq \hat{\varepsilon}$ the first term of the previous inequality is controlled by $\delta/2$. Now we select $\hat{\tau}$ such that, for every $\tau < \hat{\tau}$,

$$\|G_{\mathcal{P}}(t, u_{\hat{\varepsilon}}(t)) - G(t, u_{\hat{\varepsilon}}(t))\|_{L^2(\Omega)} \leq \frac{\delta}{2}.$$

We conclude that, for every $u \in L^2(0,T; L^2(\Omega))$, there holds

$$G_{\mathcal{P}}(t, u(t)) \longrightarrow G(t, u(t)) \quad \text{for a.e. } t \in (0,T). \tag{7.7}$$

Conclusion of the proof is now plainly obtained. Indeed, from (7.6) we obviously infer

$$\begin{aligned} &\|G_{\mathcal{P}}(\cdot, v_{\mathcal{P}}(\cdot)) - G(\cdot, v(\cdot))\|_{L^2(Q)} \\ \leq\ &\|G_{\mathcal{P}}(\cdot, v_{\mathcal{P}}(\cdot)) - G_{\mathcal{P}}(\cdot, v(\cdot))\|_{L^2(Q)} + \|G_{\mathcal{P}}(\cdot, v(\cdot)) - G(\cdot, v(\cdot))\|_{L^2(Q)} \\ \leq\ &\Lambda \|v_{\mathcal{P}} - v\|_{L^2(Q)} + \|G_{\mathcal{P}}(\cdot, v(\cdot)) - G(\cdot, v(\cdot))\|_{L^2(Q)} \end{aligned} \tag{7.8}$$

and the last term can be handled as follows,

$$\begin{aligned} \|G_{\mathcal{P}}(t, v(t)) - G(t, v(t))\|_{L^2(\Omega)} &\leq \|G_{\mathcal{P}}(t, v(t)) - G_{\mathcal{P}}(t, 0)\|_{L^2(\Omega)} \\ &+ \|G_{\mathcal{P}}(t,0) - G(t,0)\|_{L^2(\Omega)} + \|G(t,0) - G(t, v(t))\|_{L^2(\Omega)} \\ &\leq 2\Lambda \|v(t)\|_{L^2(\Omega)} + \|G_{\mathcal{P}}(t,0) - G(t,0)\|_{L^2(\Omega)}, \end{aligned}$$

by means of (7.4) and (7.6). Hence, as it is well-known that $G_{\mathcal{P}}(\cdot, 0) \longrightarrow G(\cdot, 0)$ in $L^2(0,T; L^2(\Omega))$ and a.e. in $(0,T)$, property (7.7) and the application of an extended dominated convergence theorem (cf., e.g., [18, p. 1015]) yield

$$G_{\mathcal{P}}(\cdot, v(\cdot)) \longrightarrow G(\cdot, v(\cdot)) \quad \text{in} \quad L^2(0,T; L^2(\Omega)).$$

Thus, owing to (7.8), the lemma is completely proved.

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