

A VARIATIONAL PRINCIPLE FOR DOUBLY NONLINEAR EVOLUTION

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ABSTRACT. The *weighted energy-dissipation* principle stands as a novel variational tool for the study of dissipative evolution and has been already applied to rate-independent systems and gradient flows. We provide here an example of application to a specific yet critical doubly nonlinear equation featuring a super-quadratic dissipation.

1. INTRODUCTION

This note is concerned with the illustration of the so-called *weighted energy-dissipation* (WED) variational principle in the case of a class of doubly nonlinear and degenerate parabolic PDEs. In particular, we are interested in the study of the WED functionals $I_\varepsilon : W^{1,p}(0, T; L^p(\Omega)) \rightarrow (-\infty, \infty]$ given, for $\varepsilon > 0$, by

$$I_\varepsilon(u) \doteq \begin{cases} \int_0^T \int_\Omega e^{-t/\varepsilon} \left(\frac{1}{p} |u_t|^p + \frac{1}{\varepsilon q} |\nabla u|^q + \frac{1}{\varepsilon} F(u) \right) & \text{if } u|_{\partial\Omega} = 0 \text{ a.e.} \\ & \text{and } u(0) = u_0, \\ \infty & \text{else.} \end{cases}$$

Here and in the following $\Omega \subset \mathbb{R}^n$ is a non-empty, bounded, open set with smooth boundary, $2 \leq p < q^* \doteq nq/(n-q)^+$, F is smooth and convex, and $u_0 \in W_0^{1,q}(\Omega)$ is smooth. In particular, we ask for $|\nabla u_0|^q + F(u_0) \in L^1(\Omega)$ and $-\Delta_q u_0 + f(u_0) \in L^{p'}(\Omega)$ where $\Delta_q u_0 \doteq \operatorname{div}(|\nabla u_0|^{q-2} \nabla u_0)$ and $f \doteq F'$ (non-decreasing).

The functional I_ε is convex and lower semicontinuous in $W^{1,p}(0, T; L^p(\Omega)) \cap L^q(0, T; W_0^{1,q}(\Omega))$. Moreover, we immediately check that the sublevels of I_ε are bounded in this space and that I_ε is strictly convex. Hence, I_ε admits a unique global minimizer u_ε .

Our aim is to show the connection of the minimization problem $\min I_\varepsilon$ with the doubly nonlinear degenerate parabolic PDE

$$|u_t|^{p-2} u_t - \Delta_q u + f(u) = 0 \quad \text{in } \Omega \times (0, T). \quad (1)$$

In particular, we show that the minimizers u_ε converge strongly to solutions of the latter. Here is our main result.

Theorem 1.1 (Convergence). *$u_\varepsilon \rightarrow u$ strongly in $W^{1,p}(0, T; L^p(\Omega))$ (up to subsequences) where u is a strong solution of (1) with $u|_{\partial\Omega} = 0$ and $u(0) = u_0$.*

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The interest in this convergence result consists in the possibility of reformulating the differential problem (1) as a (limit of a class of) minimization problem(s). The idea is to possibly reduce the *difficult* PDE problem (1) to the *easier* convex constrained minimization problem for I_ε . In particular, this paves the way to the application of the specific tools of the Calculus of Variations to (1), especially the Direct Method, relaxation, and Γ -convergence.

The WED approach is quite classical in the linear case [6]. Its application to the gradient flow situation ($p = 2$) starts from the paper by Ilmanen [5] on the mean curvature flow. Two examples of relaxation of gradient flows via the WED approach are provided by Conti and Ortiz [3] in the context of microstructure evolution and some general abstract treatment is then given in [9]. The rate-independent case ($p = 1$) has been considered by Mielke and Ortiz [7] and detailed in [8]. Finally, the abstract doubly-nonlinear case ($1 < p < \infty$) is addressed in [2]. The present analysis does not follow from the results of [2] as the nonlinearities here are stronger than the ones which are allowed in [2] and hence require extra care.

The advantage of the WED formalism with respect to former variational approaches [4, 10, 8, 11] is that it relies on a true minimization procedure (plus passage to the limit) and directly applies to doubly nonlinear evolution. In particular, we directly focus on the related Euler system (2) in the following.

We mention that Theorem 1.1 may be much generalized. In particular, smoothness plays indeed little role and the same results hold for a convex, proper, and lower semicontinuous function F and for some more general classes of initial data u_0 . Indeed, even the choice of the q -laplacian as a leading elliptic term in (1) is just illustrative and can be generalized. Of course, a forcing term on the right hand side of (1) can also be included.

2. CAUSAL LIMIT

This section sketches the proof of Theorem 1.1 starting from the following Lemma.

Lemma 2.1. *The only minimizer u_ε of I_ε belongs to the space $W^{1,p}(0, T; L^p(\Omega)) \cap L^q(0, T; W_0^{1,q}(\Omega))$, fulfills $u(0) = u_0$, and is the unique strong solution of the Euler system*

$$-\varepsilon(|u_{\varepsilon,t}|^{p-2}u_{\varepsilon,t})_t + |u_{\varepsilon,t}|^{p-2}u_{\varepsilon,t} - \Delta_q u_\varepsilon + f(u_\varepsilon) = 0 \quad \text{in } \Omega \times (0, T), \quad (2a)$$

$$|u_{\varepsilon,t}(T)|^{p-2}u_{\varepsilon,t}(T) = 0 \quad \text{in } \Omega. \quad (2b)$$

Moreover, there exists a constant $c > 0$ depending on u_0 such that

$$\begin{aligned} & \sup_{t \in [0, T]} \int_\Omega |\nabla u_\varepsilon|^q + \int_0^T \int_\Omega |u_{\varepsilon,t}|^p + \int_0^T \int_\Omega (|-\Delta_q u_\varepsilon + f(u_\varepsilon)|^{p'}) \\ & + \varepsilon^{-p'/2} \int_0^T \int_\Omega |\varepsilon(|u_{\varepsilon,t}|^{p-2}u_{\varepsilon,t})_t|^{p'} + \int_\Omega |u_{\varepsilon,t}(0)|^p \leq c. \end{aligned} \quad (3)$$

The Euler system (2) is *degenerate elliptic in space-time*. In fact, the WED approach consists in the (degenerate) elliptic regularization of (1). Problem (2) is *non causal*, i.e., its solution u_ε at time t depends on the future (see the *final* condition (2b)) as well. As the limiting equation (1) is instead *causal*, we refer to the limit $u_\varepsilon \rightarrow u$ as the *causal limit*. Estimate (3) is the key step for proving Theorem 1.1 and consists in the classical *energy estimate* (first two terms) and a *maximal regularity* estimate for (2) (remainder terms). A proof of Lemma 2.1 is given by time-discretization in Section 3.

By extracting not-labeled subsequences and exploiting $p < q^*$ we have

$$\begin{aligned} u_\varepsilon &\rightharpoonup u && \text{in } W^{1,p}(0, T; L^p(\Omega)), \\ u_\varepsilon &\rightarrow u && \text{in } C([0, T]; L^p(\Omega)), \\ |u_{\varepsilon,t}|^{p-2}u_{\varepsilon,t} &\rightharpoonup \xi && \text{in } L^{p'}(\Omega \times (0, T)), \\ -\Delta_q u_\varepsilon + f(u_\varepsilon) &\rightharpoonup \eta && \text{in } L^{p'}(\Omega \times (0, T)), \\ \varepsilon(|u_{\varepsilon,t}|^{p-2}u_{\varepsilon,t})_t &\rightarrow 0 && \text{in } L^{p'}(\Omega \times (0, T)). \end{aligned}$$

Hence, we can pass to the limit in equation (2a) and get $\xi + \eta = 0$. The identification $\eta = -\Delta_q u + f(u)$ follows immediately from the above convergences and the observation that $u \mapsto -\Delta_q u + f(u)$ is maximal monotone in $L^p(\Omega)$, see [1]. By multiplying (2a) by $u_{\varepsilon,t}$ and integrating in space-time we have

$$\begin{aligned} \int_0^T \int_\Omega |u_{\varepsilon,t}|^p &= \int_0^T \int_\Omega (\varepsilon(|u_{\varepsilon,t}|^{p-2}u_{\varepsilon,t})_t u_{\varepsilon,t} - \int_0^T \int_\Omega (-\Delta_q u_\varepsilon + f(u_\varepsilon))u_{\varepsilon,t} \\ &\stackrel{(2b)}{=} -\frac{\varepsilon}{p'} \int_\Omega |u_{\varepsilon,t}(0)|^p - \int_\Omega \left(\frac{1}{q} |\nabla u_\varepsilon(T)|^q + F(u_\varepsilon(T)) \right) + \int_\Omega \left(\frac{1}{q} |\nabla u_0|^q + F(u_0) \right). \end{aligned}$$

Next, by passing to the lim sup as $\varepsilon \rightarrow 0$ we find that

$$\limsup_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega |u_{\varepsilon,t}|^p \leq - \int_0^T \int_\Omega (-\Delta_q u + f(u))u_t = \int_0^T \int_\Omega \xi u_t.$$

Hence, $\xi = |u_t|^{p-2}u_t$ again by monotonicity and we have also proved that $u_{\varepsilon,t} \rightarrow u_t$ strongly in $L^p(\Omega \times (0, T))$.

3. TIME-DISCRETIZATION

The WED formalism has a discrete counterpart [8, 9], which in turn provides a proof of Lemma 2.1. For a fixed time-step $\tau = T/N$ there exists a unique vector $\{u_\varepsilon^0, u_\varepsilon^1, \dots, u_\varepsilon^N\} \in (W_0^{1,q}(\Omega))^{N+1}$ such that

$$\begin{aligned} -\varepsilon \delta(|\delta u_\varepsilon^{i+1}|^{p-2} \delta u_\varepsilon^{i+1}) + |\delta u_\varepsilon^i|^{p-2} \delta u_\varepsilon^i - \Delta_q u_\varepsilon^i + f(u_\varepsilon^i) &= 0 \quad \text{in } \Omega, \\ \text{for } i &= 1, \dots, N-1, \end{aligned} \tag{4}$$

$$|\delta u_\varepsilon^N|^{p-2} \delta u_\varepsilon^N = 0 \quad \text{in } \Omega \tag{5}$$

where we have used the short-hand notation $\delta v^i \doteq (v^i - v^{i-1})/\tau$. This fact follows from the existence of a unique minimizer of the *discrete* WED functional $I_{\varepsilon\tau} : (W_0^{1,q}(\Omega))^{N+1} \rightarrow (-\infty, \infty]$ given by

$$\begin{aligned} I_{\varepsilon\tau}(u^0, \dots, u^N) &\doteq \sum_{i=1}^N \tau \lambda_{\varepsilon\tau}^i \int_\Omega \frac{1}{p} |\delta u^i|^p + \sum_{i=1}^{N-1} \tau \frac{\lambda_{\varepsilon\tau}^{i+1}}{\varepsilon} \int_\Omega \left(\frac{1}{q} |\nabla u^i|^q + F(u^i) \right) \\ &\text{if } u^0 = u_0 \quad \text{and } I_{\varepsilon\tau}(u^0, \dots, u^N) \doteq \infty \quad \text{else} \end{aligned}$$

where $\lambda_{\varepsilon\tau}^i = (\varepsilon/(\varepsilon + \tau))^i > 0$. The latter functional is strictly convex and coercive and the discrete Euler system (4)-(5) follows from $\partial I_{\varepsilon\tau}(u_\varepsilon^0, \dots, u_\varepsilon^N) \ni 0$.

Henceforth $c > 0$ depends on u_0 but not on ε nor τ . The classical energy estimate *at the final time* reads

$$\sum_{i=1}^{N-1} \tau \int_{\Omega} |\delta u_{\varepsilon}^i|^p + \int_{\Omega} \left(\frac{1}{q} |\nabla u_{\varepsilon}^{N-1}|^q + F(u_{\varepsilon}^{N-1}) \right) \leq c \quad (6)$$

and follows at once by exploiting the final boundary condition (5).

A second estimate exploits the following elementary inequalities, valid indeed for all $x, y \in \mathbb{R}$

$$(|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq c_p |x|^{(p-2)/2}x - |y|^{(p-2)/2}y|^2, \quad (7)$$

$$||x|^{p-2}x - |y|^{p-2}y| \leq C_p (|x|^{(p-2)/2} + |y|^{(p-2)/2}) ||x|^{(p-2)/2}x - |y|^{(p-2)/2}y| \quad (8)$$

where $c_p \doteq 4(p-1)/p^2$ and $C_p \doteq 2(p-1)/p$.

Multiply (4) by $-\tau \delta^2 u_{\varepsilon}^{i+1} \doteq -\tau \delta(\delta u_{\varepsilon}^{i+1})$, integrate on Ω , and sum for $i = 1, \dots, N-1$, getting

$$\begin{aligned} c_p \sum_{i=1}^{N-1} \tau \int_{\Omega} \varepsilon |\delta(|\delta u_{\varepsilon}^{i+1}|^{(p-2)/2} \delta u_{\varepsilon}^{i+1})|^2 &\stackrel{(7)}{\leq} \sum_{i=1}^{N-1} \tau \int_{\Omega} \varepsilon \delta(|\delta u_{\varepsilon}^{i+1}|^{p-2} \delta u_{\varepsilon}^{i+1}) \delta^2 u_{\varepsilon}^{i+1} \\ &\stackrel{(4)}{=} \sum_{i=1}^{N-1} \tau \int_{\Omega} |\delta u_{\varepsilon}^i|^{p-2} \delta u_{\varepsilon}^i \delta^2 u_{\varepsilon}^{i+1} + \sum_{i=1}^{N-1} \tau \int_{\Omega} (-\Delta_q u_{\varepsilon}^i + f(u_{\varepsilon}^i)) \delta^2 u_{\varepsilon}^{i+1} \\ &\leq \frac{1}{p} \int_{\Omega} |\delta u_{\varepsilon}^N|^p - \frac{1}{p} \int_{\Omega} |\delta u_{\varepsilon}^1|^p - \int_{\Omega} (-\Delta_q u_0 + f(u_0)) \delta u_{\varepsilon}^1 \leq -\frac{1}{2p} \int_{\Omega} |\delta u_{\varepsilon}^1|^p + c. \end{aligned} \quad (9)$$

We now use (8) in order to get that

$$\begin{aligned} &|\delta(|\delta u_{\varepsilon}^{i+1}|^{p-2} \delta u_{\varepsilon}^{i+1})|^{p'} \\ &\leq C_p^{p'} \left(|\delta u_{\varepsilon}^{i+1}|^{(p-2)/2} + |\delta u_{\varepsilon}^i|^{(p-2)/2} \right)^{p'} \left| \delta(|\delta u_{\varepsilon}^{i+1}|^{(p-2)/2} \delta u_{\varepsilon}^{i+1}) \right|^{p'}. \end{aligned}$$

Hence, by integrating on Ω and summing on i we deduce that

$$\begin{aligned} &\left(\sum_{i=1}^{N-1} \tau \int_{\Omega} |\varepsilon \delta(|\delta u_{\varepsilon}^{i+1}|^{p-2} \delta u_{\varepsilon}^{i+1})|^{p'} \right)^{1/p'} \\ &\leq C_p \varepsilon^{1/2} \left[\left(\sum_{i=1}^{N-1} \tau \int_{\Omega} |\delta u_{\varepsilon}^{i+1}|^p \right)^{(p-2)/(2p)} + \left(\sum_{i=1}^{N-1} \tau \int_{\Omega} |\delta u_{\varepsilon}^i|^p \right)^{(p-2)/(2p)} \right] \times \\ &\times \left(\sum_{i=1}^{N-1} \tau \int_{\Omega} \varepsilon |\delta(|u_{\varepsilon}^{i+1}|^{(p-2)/2} u_{\varepsilon}^{i+1})|^2 \right)^{1/2} \stackrel{(6)+(9)}{\leq} c \varepsilon^{1/2}. \end{aligned} \quad (10)$$

By comparison in (4) and estimates (6) and (10) we get

$$\sum_{i=1}^{N-1} \tau \int_{\Omega} |-\Delta_q u_{\varepsilon}^i + f(u_{\varepsilon}^i)|^{p'} \leq c. \quad (11)$$

Finally, by performing again the energy estimate (now summing for $1 \leq i \leq m < N-1$), we get

$$\max_{1 \leq i \leq N} \int_{\Omega} \left(\frac{1}{q} |\nabla u_{\varepsilon}^i|^q + F(u_{\varepsilon}^i) \right) \leq c. \quad (12)$$

Estimates (6), (10)-(12) are sufficient in order to pass to the limit into (2) as $\tau \rightarrow 0$ [2]. Hence, we deduce that the (time-interpolant of the) solution of (4)-(5) converges to a solution u of (2) and that estimate (3) holds. In order to conclude our argument, we may check that *all solutions* of (2) (which have been now proved to exist) are critical for I_ε [2]. As the minimizer of I_ε is unique, we deduce that it necessarily solves (2) and fulfills (3). Namely Lemma 2.1 holds true.

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