Structure result for steady-state solutions of a one-dimensional Frémond model of SMA

Ken SHIRAKAWA

Department of Information Environment Design and Integration,
School of Information Environment, Tokyo Denki University,
2-1200 Muzai-Gakuendai, Inzai, Chiba, 270-1382, Japan,

E-mail: sirakawa@sie.dendai.ac.jp

Ulisse STEFANELLI

Istituto di Matematica Applicata e Tecnologie Informatiche - CNR,
Via Ferrata 1, I-27100 Pavia, Italy,

E-mail: ulisse@ian.pv.cnr.it

Abstract

This note deals with the complete characterization of solutions to an elliptic system of variational inequalities. The latter model arises in the study of the long-time behavior of shape memory materials and is suitable of describing a variety of experimentally observed phenomena.

Key words: shape memory alloys, steady-state solutions, structure of the solution set.

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1. Introduction

The interest in so-called *smart materials* has incredibly grown in the last decades. In particular, a wide range of physical phenomena seems to lead to crucial engineering applications and has motivated a number of contributions concerning of the modeling, the analysis and the approximation.

A great effort in this direction has been devoted to the study of the properties of *shape memory materials*. The latter are metallic alloys that exhibit a surprising thermo-mechanical behavior. At a fixed temperature and in a small deformation realm, the material shows a highly nonlinear stress-strain diagram, possibly including elastic regions as well as one or more disjoint hysteretic loops (*pseudo-plastic* behavior). This is the so-called *super-elastic* effect. On the other hand, whenever the temperature is allowed to change, one may force a mechanically deformed material to recover its original shape just by thermal means, i.e. heating or cooling. This is the so-called *shape memory* effect. Let us just stress that this kind of behavior is common to a large number of alloys and even quite ordinary steels present it, although to a smaller extent [26].

Both the above mentioned phenomena turn out to be extremely interesting from the engineering point of view. Indeed, they are actually exploited in order to realize a variety of actuators (also of microscopic size) or build special structures. The field of application of shape memory technologies ranges nowadays from bio-engineering to structures-engineering and aerospace sciences.

At the microscopic level, the thermo-mechanical behavior for a shape memory alloy is interpreted as the effect of a structural *solid-solid phase transition* [1, 14, 17, 18]. Indeed, the material is locally regarded as a mixture of variants of the metallic crystal with different symmetries. In particular, we distinguish between an *austenite*, a highly symmetric metallic configuration, and several variants of *martensites*. The latter are configurations which show less symmetry and are internally twinned, i.e. they consist of several variants related by symmetry. As a matter of fact it is a common observation that these martensitic variants develop highly organized structures arranging often in a regular quasi-periodic manner. Namely, quite usual experimental observations show strip-like and spike-like martensitic structures growing inside an austenitic region due to the cooling of the sample. This is a very important point since our analysis will show that the model captures part of this organized behavior.

We now focus on the so-called Frémond model for shape memory alloys [15, 16]. The latter was originally introduced in the three dimensional setting and takes as state variable for the system the absolute temperature ϑ, the displacement of the body $\boldsymbol{u}$, and the volumetric phase proportions β_i of the crystallographic variants. Indeed, one assumes from the very beginning that only two martensitic variants are present beside one austenite and indicate the respective proportions as β_1, β_2, and β_A, respectively. Of course nowadays we know that this assumption is extremely reductive since, in some particular alloy, up to 24 martensitic variants have been observed. Nevertheless our somehow crude simplification is still suitable of describing the basic features of the physical phenomenon [9]. We moreover assume that the phases possibly coexist at each point of the body, that no overlapping between different phases can occur, and that no void appears in the mixture. Indeed, we might regard the phase proportions β_1, β_2, and β_A as some kind of local average on the quantity of material in each phase, respectively. In the Frémond model we focus on the macroscopic behavior of the alloy by introducing a *free energy* functional. Hence, the phase proportions β_1, β_2, and β_A are actually regarded as thermodynamic quantities and are constrained to fulfill the obvious relation

$$0 \leq \ \beta_1, \beta_2, \beta_A \ \leq 1, \quad \beta_1 + \beta_2 + \beta_A = 1. \tag{1.1}$$

We prefer not to go into a deeper detail with respect to the modelization and refer the reader at once to the paper [9] where the full discussion on the modeling of the case $\lambda = 0$ was originally devised. As for the diffusion case $\lambda > 0$ the reader might be referred to the paper [2]. Indeed, we limit ourselves to introduce the partial differential equations that turn out to govern the evolution in time $t > 0$ of a one-dimensional sample of shape memory material (wire) that occupies the region $\Omega := (0, 1)$. Namely, we will exploit (1.1) in order to eliminate β_A by introducing the new variables

$$\chi_1 := \beta_1 + \beta_2, \quad \chi_2 := \beta_1 - \beta_2.$$

Of course the set $\{\chi_1 = 1\}$ corresponds to the situation where no austenite is present, the set $\{\chi_1 = \chi_2\}$ corresponds to the set where just the first variant of martensite is present etc. Taking into account the latter position, we aim now to consider the coupling of the *conservation equation for internal energy*, the *conservation of momentum* (taken as usual in its quasi-stationary form), and the *phase-flow* relation as follows

$$\partial_t\big(c_0\vartheta - L\chi_1\big) + \partial_t\big((\alpha(\vartheta) - \vartheta\alpha'(\vartheta))\chi_2 u_x\big) - h\vartheta_{xx} = \alpha(\vartheta)\chi_2 u_{xt}, \tag{1.2}$$

$$\partial_x\big(u_x + \beta\alpha(\vartheta)\chi_2\big) = 0, \tag{1.3}$$

$$\mu\partial_t\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} - \lambda\partial_{xx}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \begin{pmatrix}l(\vartheta-\vartheta^*)\\ \alpha(\vartheta)u_x\end{pmatrix} + \partial I_K(\chi_1,\chi_2) \ni \begin{pmatrix}0\\0\end{pmatrix}. \tag{1.4}$$

Of course in the latter system u stands for the one-dimensional longitudinal displacement of the shape memory wire and we used standard notations for partial derivatives. Moreover, the quantities c_0, L, h, β, μ, λ, l, and ϑ^* are positive parameters whose physical meaning may be found in [9]. The function α represents the *thermal expansion* of the system and is known to be regular, non-negative, non-increasing, constant outside the interval $[0,\vartheta_c]$, and vanishing in ϑ_c (the latter is known as *Curie temperature*).

Let K be the closed triangle in $\mathbb{R}^2$ given by

$$K := \left\{\ (\chi_1,\chi_2) \in \mathbb{R}^2 \ \big|\ 0 \le \chi_1 \le 1,\ |\chi_2| \le \chi_1\ \right\}, \tag{1.5}$$

and I_K be the indicator function on K, namely

$$I_K(\chi_1,\chi_2) := \begin{cases} 0 & \text{if } (\chi_1,\chi_2)\in K,\\ +\infty & \text{otherwise.}\end{cases}$$

Moreover, let the symbol $\partial I_K : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ denote the subgradient (subdifferential) of I_K, namely

$$(\xi_1,\xi_2) \in \partial I_K(\chi_1,\chi_2) \iff (\chi_1,\chi_2)\in K$$
$$\text{and}\quad \xi_1(\chi_1-\nu_1) + \xi_2(\chi_2-\nu_2) \ge 0 \quad \forall(\nu_1,\nu_2)\in K.$$

In particular, we stress that, owing to the definition of χ_1 and χ_2, the first condition in the above right hand side is indeed equivalent to (1.1).

We shall now give a brief review of the present analytical results on the Frémond model for shape memory alloys. First of all we quote again the paper [9], where the well-posedness of a suitably variational formulation of the problem (1.2)-(1.4) in three-dimensions with $\lambda = 0$ and with the energy equation (1.2) completely linearized ($\alpha \equiv 0$ just in (1.2)) is deduced. The full problem for $\lambda = 0$ has been proved to be well-posed both in the one-dimensional setting [12] and in the three-dimensional setting [4, 6]. A variety of existence results for somehow related models are also present in the literature [8, 13, 19]. Among these we wish to quote models complying with the full momentum equation [7, 24] and models addressing the case of generalized heat fluxes with memory [2, 3]. Moreover some qualitative property of the solutions are deduced in [11, 21] and details in the direction of a possible numerical approximation are also discussed in [22, 23].

Of course, in view of real applications, the issue of the asymptotic behavior of the solutions to (1.2)-(1.4) for large times is crucial. A first result in this direction has been achieved in [10] where the authors study the system (1.2)-(1.4) coupled with the boundary conditions

$$
\begin{aligned}
h\vartheta_x(0,t) - k\big(\vartheta(0,t) - \bar{\vartheta}\big) &= 0 \quad \text{for a.e. } t > 0, && (1.6)\\
h\vartheta_x(1,t) + k\big(\vartheta(1,t) - \bar{\vartheta}\big) &= 0 \quad \text{for a.e. } t > 0, && (1.7)\\
u(0,t) &= 0 \quad \text{for any } t > 0, && (1.8)\\
\big(u_x + \beta\alpha(\vartheta)\chi_2\big)(1,t) - \beta g(t) &= 0 \quad \text{for a.e. } t > 0, && (1.9)\\
\chi_{i,x}(0,t) = \chi_{i,x}(1,t) &= 0 \quad \text{for a.e. } t > 0, \quad i = 1,2, && (1.10)\\
\vartheta(x,0) &= \vartheta_0 \quad \text{for any } x \in \Omega, && (1.11)\\
\chi_i(x,0) &= \chi_{i,0} \quad \text{for any } x \in \Omega, \quad i = 1,2, && (1.12)
\end{aligned}
$$

where k is a positive heat exchange coefficient, while the constant $\bar{\vartheta} > 0$ and $g : (0, +\infty) \to \mathbb{R}$ account for the interactions with the medium surrounding the domain. In particular, $\bar{\vartheta}$ stands for an external temperature and g is a traction. Denoting by $H := L^2(\Omega)$ and $V := \{u \in H^1(\Omega) \ : \ u(0) = 0\}$ endowed with the standard scalar products, and assuming suitable regularity on data, the result [10, Thm. 2.3] entails that the $\omega-$limit set $(\vartheta, u, \chi_1, \chi_2)$ in $H \times V \times H \times H$ defined by

$$
\begin{aligned}
\omega\big(\vartheta, u, \chi_1, \chi_2\big) := \Big\{ &\big(\vartheta_\infty, u_\infty, \chi_{1,\infty}, \chi_{2,\infty}\big) \in H \times V \times H \times H \quad \text{such that there exists}\\
&\text{a sequence of positive real numbers } \{t_n\} \text{ with } t_n \longrightarrow +\infty \text{ and}\\
&\big(\vartheta(t_n), u(t_n), \chi_1(t_n), \chi_2(t_n)\big) \longrightarrow \big(\vartheta_\infty, u_\infty, \chi_{1,\infty}, \chi_{2,\infty}\big) \text{ in } H \times V \times H \times H \Big\}, \qquad (1.13)
\end{aligned}
$$

is non-empty, compact, and connected. Moreover, any point $(\vartheta_\infty, u_\infty, \chi_{1,\infty}, \chi_{2,\infty})$ of $\omega(\vartheta, u, \chi_1, \chi_2)$ is a solution of the stationary problem

$$\vartheta_\infty = \bar{\vartheta} \quad \text{a.e. in } \Omega, \qquad (1.14)$$

$$u_\infty(x) = -\beta\alpha(\bar{\vartheta}) \int_0^x \chi_{2,\infty}(\xi)\, d\xi \quad \forall x \in (0,1), \qquad (1.15)$$

$$-\lambda\partial_{xx}\begin{pmatrix}\chi_{1,\infty}\\ \chi_{2,\infty}\end{pmatrix} + \begin{pmatrix} l(\bar{\vartheta} - \vartheta^*)\\ -\beta(\alpha(\bar{\vartheta}))^2\chi_{2,\infty}\end{pmatrix} + \partial I_K(\chi_{1,\infty}, \chi_{2,\infty}) \ni \begin{pmatrix}0\\0\end{pmatrix} \quad \text{a.e. in } \Omega, \qquad (1.16)$$

$$\chi_{i,\infty,x}(0) = \chi_{i,\infty,x}(1) = 0, \quad i = 1,2. \qquad (1.17)$$

The main novelty of this paper is that of focusing on the structure of the solution set of the latter nonlinear elliptic system. Of course, since ϑ_∞ is known and (1.15) holds, it suffices to study the behavior of the unknowns $(\chi_{1,\infty}, \chi_{2,\infty})$.

We wish to quote at this point the paper [20] where the first analysis of the structure of the solution set to steady-state Frémond model is carried out. In particular, in the above cited paper the author focuses in the characterization of one-dimensional steady-state solutions but in a somehow different framework. Indeed, a fourth order momentum balance equation is coupled to a non-diffusive ($\lambda = 0$) phase transition dynamics. Then, the resulting nonlinear equation for the scalar strain is studied by means of ODE techniques. Here we analyze instead the case of diffusion for the phases together with a second order momentum balance (see (1.3)) which seems to be suitable of describing the behavior of one-dimensional wires and for which we already have an asymptotic result. As an effect of this different perspective, we have to deal with the full phase system instead of the single momentum equation.

Our main result is the complete characterization of all the possible solutions to the elliptic stationary system (1.14)-(1.17) with respect to different external temperatures $\bar{\vartheta}$. In particular, we will point out how, in a prescribed temperature range, the solution set contains elements that show a deeply structured alternance of martensitic variants, in complete agreement with the experimental evidence. Moreover, we pose the ground base for the development of a stability theory for the phase configuration with respect to thermal perturbation. The latter problem is of course very important from the applicative point of view and has been already addressed by the first author [25]. In particular, a suitable concept of *set stability* is introduced and the complete behavior of the *solution flow* with respect to the external temperature $\bar{\vartheta}$ is devised.

In the forthcoming sections we will also present some graphs for the possible explicit solutions. These graphs are just intended to suggest the behavior of the latter solutions since the values of the parameters appearing in the Frémond model have not been completely determined so far.

This is the plan of the paper. In the forthcoming Section 2 we introduce the statement of our main structure result whose proof will be detailed in Sections 3 and 4. Eventually, we gather some technical remarks in the Appendix.

2. Statement of the main result

Let $0 < \vartheta^* < \vartheta_c$, λ and l be given positive constants, and $J := (a, b)$ be any bounded open interval in $\mathbb{R}$ with $-\infty < a < b < +\infty$.

In this paper, for any (fixed) positive constant $\bar{\vartheta}$, we consider the following boundary value problem of a vector-valued differential equation, of the form:

$$(P)_J^{\bar{\vartheta}} \quad \begin{cases} -\lambda \dfrac{d^2}{dx^2}\begin{pmatrix} \chi_1^{\bar{\vartheta}} \\ \chi_2^{\bar{\vartheta}} \end{pmatrix} + \partial I_K(\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}) \ni \begin{pmatrix} l(\vartheta^* - \bar{\vartheta}) \\ \gamma(\bar{\vartheta})\chi_2^{\bar{\vartheta}} \end{pmatrix} \quad \text{a.e. in } J, \\ \\ (\chi_i^{\bar{\vartheta}})_x(a) = (\chi_i^{\bar{\vartheta}})_x(b) = 0, \ i = 1, 2, \end{cases}$$

where ∂I_K is the subdifferential of I_K in $\mathbb{R}^2$.

$(P)_J^{\bar{\vartheta}}$ is motivated by the steady-state problem (1.14)-(1.17). The function γ corresponds to the term $\beta\alpha(\cdot)^2$ as in (1.16). More precisely, $\gamma : [0, +\infty) \longrightarrow \mathbb{R}$ is a continuous and strictly decreasing function such that $\operatorname{supp} \gamma \subset [0, \vartheta_c]$.

Definition 2.1 A pair $\{\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}\}$ of functions $\chi_i^{\bar{\vartheta}}$, $i = 1, 2$, on $\overline{J} = [a, b]$ is called a solution of $(P)_J^{\bar{\vartheta}}$, if $\chi_i^{\bar{\vartheta}} \in H^2(J)$, $(\chi_i^{\bar{\vartheta}})_x(a) = (\chi_i^{\bar{\vartheta}})_x(b) = 0$, $i = 1, 2$, and there exist two functions $\xi_i^{\bar{\vartheta}} \in L^2(J)$, $i = 1, 2$, such that

$$\begin{pmatrix} \xi_1^{\bar{\vartheta}} \\ \xi_2^{\bar{\vartheta}} \end{pmatrix} \in \partial I_K(\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}) \text{ and } -\lambda \frac{d^2}{dx^2}\begin{pmatrix} \chi_1^{\bar{\vartheta}} \\ \chi_2^{\bar{\vartheta}} \end{pmatrix} + \begin{pmatrix} \xi_1^{\bar{\vartheta}} \\ \xi_2^{\bar{\vartheta}} \end{pmatrix} = \begin{pmatrix} l(\vartheta^* - \bar{\vartheta}) \\ \gamma(\bar{\vartheta})\chi_2^{\bar{\vartheta}} \end{pmatrix} \quad \text{a.e. in } J,$$

For the description of the main result, it is convenient to introduce the following notation.

Definition 2.2 Let $T := [\tau_*, \tau^*]$ be any compact interval in $\mathbb{R}$ with $\tau_* < \tau^*$, $\kappa > 0$, and $\tau_* < \tau < \tau^*$. For any nonnegative integer n, we define a subclass $S_{J,T}(\kappa, \tau; n)$ in $H^2(J)$ as follows.

(I) $S_{J,T}(\kappa,\tau;0) := \{\tau_*, \tau, \tau^*\}$.

(II) For $n \in \mathbb{N}$, $\chi \in S_{J,T}(\kappa,\tau;n)$ if and only if there exist a constant $c \in \mathbb{R}$, a partition

$$a - \sqrt{\kappa}\pi =: x_0 < x_1 < \cdots < x_k < \cdots < x_n < x_{n+1} := b$$

and n-th open intervals $J_k := (x_k, x_k + \sqrt{\kappa}\pi)$, $k = 1, \cdots, n$, such that:

(i) $\chi(x) = (-1)^{k-1} c \cos\left(\dfrac{x - x_k}{\sqrt{\kappa}}\right) + \tau$ for $x \in J_k$, $k = 1, \cdots, n$;

(ii) $|c| \le \min\{\tau^* - \tau, \tau - \tau_*\}$;

(iii) $x_k + \sqrt{\kappa}\pi \le x_{k+1}$ for $k = 0, 1, \cdots, n$;

(iv) if $x_k + \sqrt{\kappa}\pi < x_{k+1}$ for some $0 \le k \le n$, then $c = \min\{\tau^* - \tau, \tau - \tau_*\}$, and $\chi \equiv \tau^*$ or $\chi \equiv \tau_*$ on the corresponding interval $[x_k + \sqrt{\kappa}\pi, x_{k+1}]$.

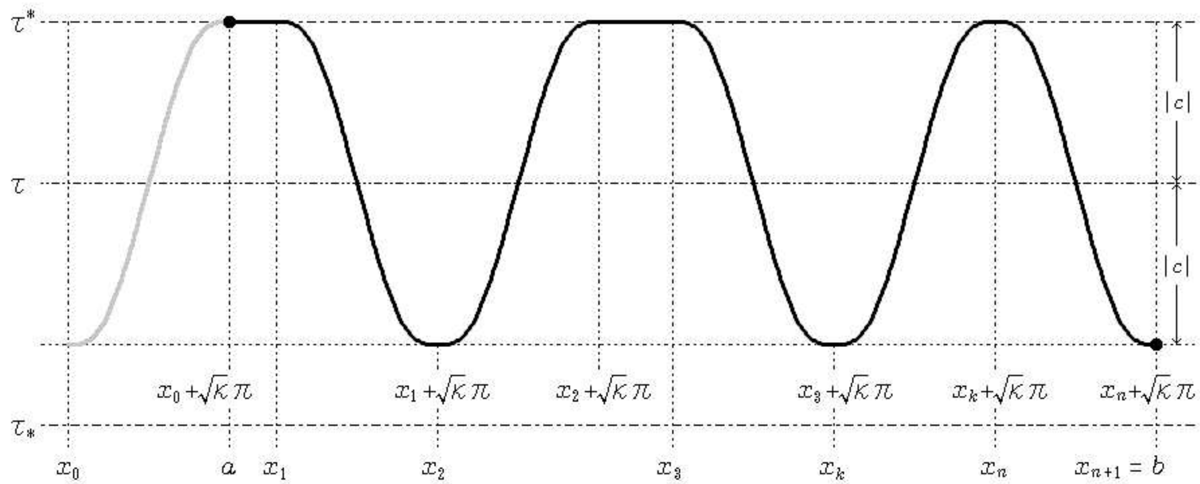

Figure 1: graph of an element of $S_{J,T}(\kappa,\tau;n)$ for $n \ne 0$

Remark 2.1 We notice that

$$n^*(\kappa) := \sup\left\{ \, n \,\middle|\, S_{J,T}(\kappa,\tau;n) \ne \emptyset \, \right\} < +\infty. \tag{2.1}$$

In fact, since

$$n^*(\kappa)\sqrt{\kappa}\pi = \sum_{k=1}^{n^*(\kappa)} |J_k| \le |J|,$$

we have

$$0 \le n^*(\kappa) \le \frac{|J|}{\sqrt{\kappa}\pi} < +\infty.$$

Here, let us define

$$S_{J,T}(\kappa,\tau) := \begin{cases} \{\tau^*\}, \ \text{if } \tau < \tau_*, \\ \bigcup_{n=0}^{n^*(\kappa)} S_{J,T}(\kappa,\tau;n), \ \text{if } \tau_* \le \tau \le \tau^*, \\ \{\tau_*\}, \ \text{if } \tau > \tau^*. \end{cases} \tag{2.2}$$

Remark 2.2 In the investigation of the asymptotic behavior for Allen-Cahn equations, in [5] the authors studied the structure of solutions of the following boundary value problem of a single equation:

$$\begin{cases} -\kappa\chi_{xx} + \partial I_T(\chi) \ni \chi - \tau \text{ a.e. in } J, \\ \chi_x(a) = \chi_x(b) = 0, \end{cases} \tag{2.3}$$

where ∂I_T is the subdifferential of the indicator function I_T on the closed interval T. According to their result, the subclass $S_{J,T}(\kappa,\tau)$ in $H^2(J)$ is nothing but the class of solutions of (2.3). The structure result obtained in [5] also gives many useful informations to our vector valued case.

Now our main result is stated as follows.

Theorem 2.1 *(Structure of solutions of $(P)^{\bar\vartheta}_J$) Let $J := (a,b)$ be any bounded and open interval in $\mathbb{R}$. Let K_0, K_1, K_+ and K_- be subsets of the boundary ∂K of the triangle K as in (1.5), defined by:*

$$\begin{cases} K_0 := \{(0,0),(1,1),(1,-1)\}, \ K_1 := \{ \ (\eta,\zeta) \in \partial K \ \big| \ \eta = 1 \ \}, \\ K_+ := \{ \ (\eta,\zeta) \in \partial K \ \big| \ \eta = \zeta \ \} \ \textit{and} \ \ K_- := \{ \ (\eta,\zeta) \in \partial K \ \big| \ \eta = -\zeta \ \}, \end{cases} \tag{2.4}$$

respectively. Then, a pair $\{\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta}\}$ is a solution of $(P)^{\bar\vartheta}_J$, if and only if:

(s1) (the case of $0 < \bar\vartheta < \vartheta^*$) $(\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta}) \in K_1$ on $\overline{J}$, namely

$$\chi_1^{\bar\vartheta} \equiv 1 \ \textit{and} \ |\chi_2^{\bar\vartheta}| \le 1 \ \textit{on} \ \overline{J},$$

and $\chi_2^{\bar\vartheta}$ is in the class $S_{J,[-1,1]}\left(\frac{\lambda}{\gamma(\bar\vartheta)},0\right)$ as in (2.2) with the open interval J, $T = [-1,1]$, $\kappa = \frac{\lambda}{\gamma(\bar\vartheta)}$ and $\tau = 0$ (see also Fig. 2);

(s2) (the case of $\bar\vartheta = \vartheta^*$) $\chi_1^{\bar\vartheta} \equiv$ const. on $\overline{J}$, and $\chi_2^{\bar\vartheta}$ is in the class $S_{J,[-1,1]}\left(\frac{\lambda}{\gamma(\bar\vartheta)},0\right)$;

(s3) (the case of $\vartheta^* < \bar\vartheta < \vartheta_c$) $(\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta}) \in K_+ \cup K_-$ on $\overline{J}$, namely

$$0 \le \chi_1^{\bar\vartheta} \le 1 \ \textit{and} \ |\chi_2^{\bar\vartheta}| = \chi_1^{\bar\vartheta} \ \textit{on} \ \overline{J},$$

and $\chi_1^{\bar\vartheta}$ is in the class $S_{J,[0,1]}\left(\frac{2\lambda}{\gamma(\bar\vartheta)},\frac{l(\bar\vartheta-\vartheta^*)}{\gamma(\bar\vartheta)}\right)$ as in (2.2) with the open interval J, $T = [0,1]$, $\kappa = \frac{2\lambda}{\gamma(\bar\vartheta)}$ and $\tau = \frac{l(\bar\vartheta-\vartheta^*)}{\gamma(\bar\vartheta)}$, in particular, if $\frac{l(\bar\vartheta-\vartheta^*)}{\gamma(\bar\vartheta)} \ge 1$, then

$$\chi_1^{\bar\vartheta} = \chi_2^{\bar\vartheta} \equiv 0 \ \textit{on} \ \overline{J};$$

(s4) (the case of $\bar{\vartheta} \geq \vartheta_c$) $\chi_1^{\bar{\vartheta}} = \chi_2^{\bar{\vartheta}} \equiv 0$ on $\overline{J}$.

Of course, we remark that our result agrees completely with the expected physical behavior. Indeed, for high external temperatures, only austenite is present ($\chi_1 = 0$) while, at subcritical temperatures $0 < \bar{\vartheta} < \vartheta^*$ the material is completed into the martersite phases.

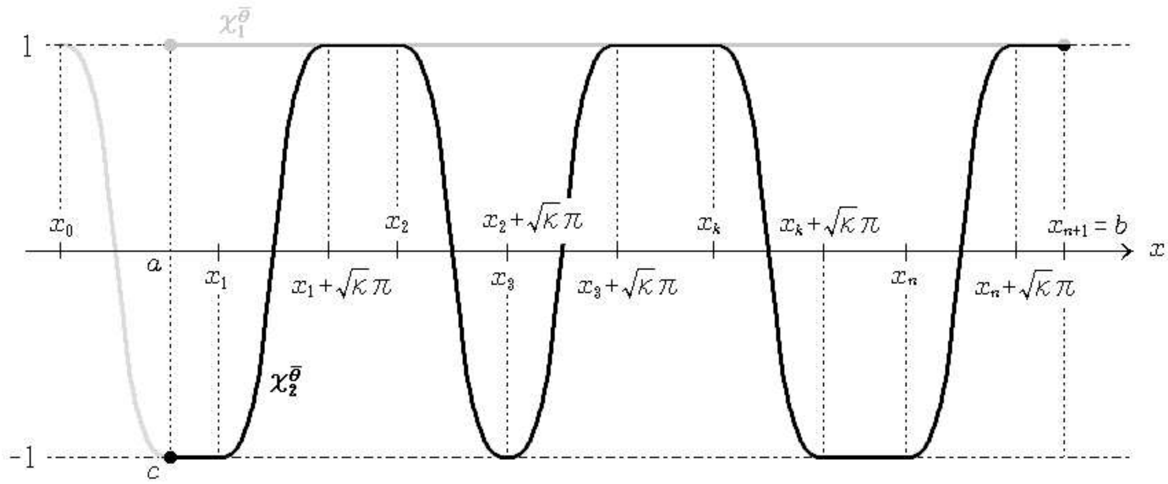

Figure 2: Theorem 2.1, case (s1)

3. Proof of (s1), (s2) and (s4) of Theorem 2.1

In this section, we give the proof of Theorem 2.1 except for the assertion (s3). For the proof, we prepare some lemmas.

Lemma 3.1 *Assume that $\bar{\vartheta} \leq \vartheta^*$ and $\{\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}\}$ is the solution of $(P)_J^{\bar{\vartheta}}$. Then,*

$$\chi_1^{\bar{\vartheta}}(x) = \chi_1^{\bar{\vartheta}}(a) \text{ for any } x \in \overline{J}.$$

Proof. First, let us assume that

$$\chi_1^{\bar{\vartheta}}(a) < \chi_1^{\bar{\vartheta}}(x_0) \leq 1 \text{ for some } x_0 \in (a, b], \tag{3.1}$$

and denote by (a, x_*) with $x_* > a$ the maximal open interval such that

$$\chi_1^{\bar{\vartheta}}(x) < \chi_1^{\bar{\vartheta}}(x_*) = \chi_1^{\bar{\vartheta}}(x_0) \leq 1 \text{ for any } x \in [a, x_*).$$

Then, since $(\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}) \in K \setminus K_1$ for any $x \in [a, x_*)$, we see from Lemma 5.1 that

$$\xi_1^{\bar{\vartheta}} := \lambda(\chi_1^{\bar{\vartheta}})_{xx} + l(\vartheta^* - \bar{\vartheta}) \leq 0 \text{ on } [a, x_*).$$

Furthermore, by the assumption and the boundary condition,

$$(\chi_1^{\bar\vartheta})_x(x) = (\chi_1^{\bar\vartheta})_x(a) + \int_a^x (\chi_1^{\bar\vartheta})_{xx}(y)\ dy = \frac{1}{\lambda}\int_a^x \{\xi_1^{\bar\vartheta}(y) + l(\bar\vartheta - \vartheta^*)\}\ dy \le 0$$
$$\text{for any } x \in [a, x_*).$$

Therefore, $\chi_1^{\bar\vartheta}(x_0) = \chi_1^{\bar\vartheta}(x_*) \le \chi_1^{\bar\vartheta}(a)$, which contradicts (3.1). Thus

$$\chi_1^{\bar\vartheta}(x) \le \chi_1^{\bar\vartheta}(a) \text{ for any } x \in \overline{J}. \tag{3.2}$$

Secondly, let us assume that

$$\chi_1^{\bar\vartheta}(x_1) < \chi_1^{\bar\vartheta}(a) \quad \text{for some } x_1 \in (a, b],$$

and denote by $(\underline{x}_1, \overline{x}_1)$ with $\underline{x}_1 < \overline{x}_1$ the maximal open interval such that

$$x_1 \in (\underline{x}_1, \overline{x}_1) \text{ and } \chi_1^{\bar\vartheta}(y) < \chi_1^{\bar\vartheta}(a) \quad \text{for any } y \in (\underline{x}_1, \overline{x}_1).$$

Then

$$(\chi_1^{\bar\vartheta})_x(\overline{x}_1) = (\chi_1^{\bar\vartheta})_x(\underline{x}_1) = 0. \tag{3.3}$$

In fact, if $\underline{x}_1 = a$ (resp. $\overline{x}_1 = b$), then it immediately follows from the boundary condition that

$$(\chi_1^{\bar\vartheta})_x(\underline{x}_1) = (\chi_1^{\bar\vartheta})_x(a) = 0 \text{ (resp. } (\chi_1^{\bar\vartheta})_x(\overline{x}_1) = (\chi_1^{\bar\vartheta})_x(b) = 0).$$

If $\underline{x}_1 > a$ (resp. $\overline{x}_1 < b$), then by the definition of the interval $(\underline{x}_1, \overline{x}_1)$,

$$\chi_1^{\bar\vartheta}(\underline{x}_1) = \chi_1^{\bar\vartheta}(a) \ \left(\text{resp. } \chi_1^{\bar\vartheta}(\overline{x}_1) = \chi_1^{\bar\vartheta}(a)\right).$$

Since $\chi_1^{\bar\vartheta} \in H^2(J)$, we see from (3.2) that

$$(\chi_1^{\bar\vartheta})_x(\underline{x}_1) = \lim_{x \nearrow \underline{x}_1} \frac{\chi_1^{\bar\vartheta}(x) - \chi_1^{\bar\vartheta}(\underline{x}_1)}{x - \underline{x}_1} \ge 0 \left(\text{resp. } (\chi_1^{\bar\vartheta})_x(\overline{x}_1) = \lim_{x \nearrow \overline{x}_1} \frac{\chi_1^{\bar\vartheta}(x) - \chi_1^{\bar\vartheta}(\overline{x}_1)}{x - \overline{x}_1} \ge 0\right)$$

and

$$(\chi_1^{\bar\vartheta})_x(\underline{x}_1) = \lim_{x \searrow \underline{x}_1} \frac{\chi_1^{\bar\vartheta}(x) - \chi_1^{\bar\vartheta}(\underline{x}_1)}{x - \underline{x}_1} \le 0 \left(\text{resp. } (\chi_1^{\bar\vartheta})_x(\overline{x}_1) = \lim_{x \searrow \overline{x}_1} \frac{\chi_1^{\bar\vartheta}(x) - \chi_1^{\bar\vartheta}(\overline{x}_1)}{x - \overline{x}_1} \le 0\right).$$

Thus, we have (3.3).

Also, since $(\chi_1^{\bar\vartheta}(x), \chi_2^{\bar\vartheta}(x)) \in K \setminus K_1$ for any $x \in (\underline{x}_1, \overline{x}_1)$, we see from Lemma 5.1 that

$$\lambda(\chi_1^{\bar\vartheta})_{xx} = \xi_1^{\bar\vartheta} + l(\bar\vartheta - \vartheta^*) \le 0 \text{ in } (\underline{x}_1, \overline{x}_1).$$

Now, by (3.3),

$$\int_{\underline{x}_1}^{\overline{x}_1} |(\chi_1^{\bar\vartheta})_{xx}(y)|\ dy = -\int_{\underline{x}_1}^{\overline{x}_1} (\chi_1^{\bar\vartheta})_{xx}(y)\ dy = -\lambda(\chi_1^{\bar\vartheta})_x(\overline{x}_1) + \lambda(\chi_1^{\bar\vartheta})_x(\underline{x}_1) = 0,$$

which implies

$$\chi_1^{\bar\vartheta} \equiv \chi_1^{\bar\vartheta}(a) \text{ in } (\underline{x}_1, \overline{x}_1), \text{ in particular } \chi_1^{\bar\vartheta}(x_1) = \chi_1^{\bar\vartheta}(a).$$

It contradicts the assumption. ∎

Proof of (s1) and (s2) of Theorem 2.1: Let $\bar\vartheta \le \vartheta^*$, and $X_{\bar\vartheta}$ be a $\bar\vartheta$-dependent set given by

$$X_{\bar\vartheta} := \begin{cases} \{1\}, \ \text{if } \bar\vartheta < \vartheta^*, \\ [0,1], \ \text{if } \bar\vartheta = \vartheta^*. \end{cases}$$

Then, by Remark 2.2, it is enough to show that $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ is a solution of $(P)_J^{\bar\vartheta}$ if and only if $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ is a solution of the following system

$$\begin{cases} \chi_1^{\bar\vartheta} \equiv c_{\bar\vartheta} \text{ on } \overline{J} \text{ for a constant } c_{\bar\vartheta} \in X_{\bar\vartheta}, \\ -c_{\bar\vartheta} \le \chi_2^{\bar\vartheta} \le c_{\bar\vartheta} \text{ on } \overline{J}, \\ -\lambda(\chi_2^{\bar\vartheta})_{xx} + \xi_2^{\bar\vartheta} = \gamma(\bar\vartheta)\chi_2^{\bar\vartheta}, \ \text{a.e. in } J, \\ \xi_2^{\bar\vartheta} \in \partial I_{[-1,1]}(\chi_2^{\bar\vartheta}), \ \text{a.e. in } J, \\ \chi_i^{\bar\vartheta}(a) = \chi_i^{\bar\vartheta}(b) = 0, \ i = 1,2. \end{cases} \tag{3.4}$$

First, we assume $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ is a solution of $(P)_J^{\bar\vartheta}$. Then, we find two functions $\xi_i^{\bar\vartheta} \in L^2(J)$, $i = 1, 2$, such that

$$\begin{pmatrix} \xi_1^{\bar\vartheta} \\ \xi_2^{\bar\vartheta} \end{pmatrix} \in \partial I_K(\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}), \ \text{a.e. in } J, \tag{3.5}$$

$$\xi_1^{\bar\vartheta} = \lambda(\chi_1^{\bar\vartheta})_{xx} + l(\vartheta^* - \bar\vartheta), \ \text{a.e. in } J \tag{3.6}$$

and

$$\xi_2^{\bar\vartheta} = \lambda(\chi_2^{\bar\vartheta})_{xx} + \gamma(\bar\vartheta)\chi_2^{\bar\vartheta}, \ \text{a.e. in } J. \tag{3.7}$$

Here, by Lemma 3.1, $\chi_1^{\bar\vartheta} \equiv c_{\bar\vartheta}$ for some constant $0 \le c_{\bar\vartheta} \le 1$, so that

$$\xi_1^{\bar\vartheta} = \lambda(\chi_1^{\bar\vartheta})_{xx} + l(\vartheta^* - \bar\vartheta) = l(\vartheta^* - \bar\vartheta) \begin{cases} > 0, \ \text{if } \bar\vartheta < \vartheta^*, \\ = 0, \ \text{if } \bar\vartheta = \vartheta^*. \end{cases} \tag{3.8}$$

Thus, it follows from (3.5), (3.8) and Lemma 5.1 that $c_{\bar\vartheta} \in X_{\bar\vartheta}$.

Next, since $(\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}) \in K$ on $\overline{J}$,

$$-c_{\bar\vartheta} \equiv -\chi_1^{\bar\vartheta} \le \chi_2^{\bar\vartheta} \le \chi_1^{\bar\vartheta} \equiv c_{\bar\vartheta} \text{ on } \overline{J}. \tag{3.9}$$

On account of (3.5), (3.8)-(3.9) and Lemma 5.1, we obtain that

$$\xi_2^{\bar\vartheta}(x) \begin{cases} = 0, \ \text{if } \bar\vartheta \le \vartheta^* \text{ and } -1 \le -c_{\bar\vartheta} < \chi_2^{\bar\vartheta}(x) < c_{\bar\vartheta} \le 1, \\ = (\xi_1^{\bar\vartheta}(x) + \xi_2^{\bar\vartheta}(x)) - \xi_1^{\bar\vartheta}(x) = 0, \ \text{if } \bar\vartheta = \vartheta^* \text{ and } \chi_2^{\bar\vartheta}(x) = c_{\bar\vartheta} < 1, \\ = \xi_1^{\bar\vartheta}(x) - (\xi_1^{\bar\vartheta}(x) - \xi_2^{\bar\vartheta}(x)) = 0, \ \text{if } \bar\vartheta = \vartheta^* \text{ and } \chi_2^{\bar\vartheta}(x) = -c_{\bar\vartheta} > -1, \\ \ge 0, \ \text{if } \bar\vartheta \le \vartheta^* \text{ and } \chi_2^{\bar\vartheta}(x) = c_{\bar\vartheta} = 1, \\ \le 0, \ \text{if } \bar\vartheta \le \vartheta^* \text{ and } \chi_2^{\bar\vartheta}(x) = -c_{\bar\vartheta} = -1, \end{cases}$$

so that

$$\xi_2^{\bar\vartheta} \in \partial I_{[-1,1]}(\chi_2^{\bar\vartheta}) \text{ in } J. \tag{3.10}$$

Now, by (3.6)-(3.10), Lemma 5.1 and the boundary condition, we conclude that $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ is a solution of (3.4).

Conversely, we assume that $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ is a solution of (3.4) and give a function $\xi_1^{\bar\vartheta} \in L^2(J)$ by (3.8). Then, we see from (3.8), (3.10) and Lemma 5.1 that (3.5) holds. Now, noting the boundary condition, we conclude that $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ is a solution of $(P)_J^{\bar\vartheta}$. ■

Next, we show (s4) of Theorem 2.1. Let us define $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a set-valued function, by putting

$$\Phi(x) := \begin{cases} (-\infty, +\infty), \text{ if } x = 0, \\ [0, +\infty), \text{ if } x > 0, \\ \emptyset, \text{ otherwise.} \end{cases}$$

and consider the following boundary value problem

$$\begin{cases} -\lambda \hat\chi_{xx} + \hat\xi = l(\vartheta^* - \bar\vartheta), \text{ a.e. in } J, \\ \hat\xi \in \Phi(\hat\chi) \text{ in } J, \\ \hat\chi_x(a) = \hat\chi_x(b) = 0. \end{cases} \tag{3.11}$$

Definition 3.1 A function $\hat\chi : \hat J \longrightarrow \mathbb{R}$ is called a solution of (3.11), if $\hat\chi \in H^2(J)$ and there is a function $\hat\xi \in L^2(J)$ such that

$$\hat\xi \in \Phi(\hat\chi) \text{ and } \hat\xi = \lambda\hat\chi_{xx} + l(\vartheta^* - \bar\vartheta), \text{ a.e. in } J.$$

Remark 3.1 By the definition, any solution $\hat\chi$ of (3.11) satisfies $\hat\chi \geq 0$ on $\overline{J}$.

Lemma 3.2 *Assume that $\bar\vartheta > \vartheta^*$. Then, a function $\hat\chi \in H^2(J)$ is a solution of (3.11) if and only if $\hat\chi \equiv 0$ on $\overline{J}$.*

Proof. It is easy to check that the constant 0 is a solution of (3.11) with a function $\hat\xi \equiv l(\vartheta^* - \bar\vartheta) < 0$ in J.

Now, let us assume that there is a nonzero solution $\hat\chi$ of (3.11). Then, since $\hat\chi \in H^2(J)$, the set $Y_* := \{ x \in J \mid \hat\chi(x) > 0 \}$ is a disjoint union of connected and open subintervals of J. Here let $I_* := (\underline{y}_*, \overline{y}_*)$ be any connected component of Y_*. Then,

$$\hat\chi_x(\underline{y}_*) = \hat\chi_x(\overline{y}_*) = 0.$$

In fact, if $\underline{y}_* = a$ (resp. $\overline{y}_* = b$), then the boundary condition, we immediately have

$$\hat\chi_x(\underline{y}_*) = 0 \text{ (resp. } \hat\chi_x(\overline{y}_*) = 0). \tag{3.12}$$

If $\underline{y}_* > a$ (resp. $\overline{y}_* < b$), then by Remark 3.1,

$$\frac{\hat\chi(x) - \hat\chi(\underline{y}_*)}{x - \underline{y}_*} \begin{cases} \geq 0, \text{ if } x > \underline{y}_*, \\ \leq 0, \text{ if } x < \underline{y}_*, \end{cases} \left(\text{resp. } \frac{\hat\chi(x) - \hat\chi(\overline{y}_*)}{x - \overline{y}_*} \begin{cases} \geq 0, \text{ if } x > \overline{y}_*, \\ \leq 0, \text{ if } x < \overline{y}_*, \end{cases} \right).$$

So, letting $x \to \underline{y}_*$ (resp. $x \to \overline{y}_*$), we obtain (3.12).

On the other hand, by (3.11),

$$\lambda\hat{\chi}_{xx} = \hat{\xi} + l(\bar{\vartheta} - \vartheta^*) > 0 \text{ on } I_*,$$

so that

$$\hat{\chi}_x(\overline{y}_*) - \hat{\chi}_x(\underline{y}_*) = \frac{1}{\lambda}\int_{\underline{y}_*}^{\overline{y}_*} (\hat{\xi}(t) + l(\bar{\vartheta} - \vartheta^*))\, dt > 0.$$

It contradicts (3.12). ■

Proof of (s4) of Theorem 2.1: Let us assume $\bar{\vartheta} \geq \vartheta_c$. Then, since $\gamma(\bar{\vartheta}) = 0$, the problem $(P)_J^{\bar{\vartheta}}$ is reformulated to the following system:

$$-\lambda(\chi_1^{\bar{\vartheta}})_{xx} + \xi_1^{\bar{\vartheta}} = l(\vartheta^* - \overline{\vartheta}), \text{ a.e. in } J, \tag{3.13}$$

$$-\lambda(\chi_2^{\bar{\vartheta}})_{xx} + \xi_2^{\bar{\vartheta}} = 0, \text{ a.e. in } J, \tag{3.14}$$

$$\begin{pmatrix} \xi_1^{\bar{\vartheta}} \\ \xi_2^{\bar{\vartheta}} \end{pmatrix} \in \partial I_K(\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}), \text{ a.e. in } J, \tag{3.15}$$

$$(\chi_i^{\bar{\vartheta}})_x(a) = (\chi_i^{\bar{\vartheta}})_x(b) = 0, \; i = 1, 2. \tag{3.16}$$

Cleary, $\chi_1^{\bar{\vartheta}} = \chi_2^{\bar{\vartheta}} \equiv 0$ is a solution of the system (3.13)-(3.16).

Conversely, let $\{\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}\}$ be any solution of (3.13)-(3.16). Then, taking the sum of (resp. the difference between) both sides of (3.13) and (3.14), we have

$$\begin{cases} -\lambda(\chi_1^{\bar{\vartheta}} + \chi_2^{\bar{\vartheta}})_{xx} + (\xi_1^{\bar{\vartheta}} + \xi_2^{\bar{\vartheta}}) = l(\vartheta^* - \bar{\vartheta}), \text{ a.e. in } J \\ \left(\text{resp. } -\lambda(\chi_1^{\bar{\vartheta}} - \chi_2^{\bar{\vartheta}})_{xx} + (\xi_1^{\bar{\vartheta}} - \xi_2^{\bar{\vartheta}}) = l(\vartheta^* - \bar{\vartheta}), \text{ a.e. in } J\right). \end{cases} \tag{3.17}$$

Here, by (3.15) and Lemma 5.1,

$$\xi_1^{\bar{\vartheta}}(x) + \xi_2^{\bar{\vartheta}}(x) \geq 0, \text{ if } \chi_1^{\bar{\vartheta}}(x) + \chi_2^{\bar{\vartheta}}(x) > 0$$
$$\left(\text{resp. } \xi_1^{\bar{\vartheta}}(x) - \xi_2^{\bar{\vartheta}}(x) \geq 0, \text{ if } \chi_1^{\bar{\vartheta}}(x) - \chi_2^{\bar{\vartheta}}(x) > 0 \right),$$

so that

$$\xi_1^{\bar{\vartheta}} + \xi_2^{\bar{\vartheta}} \in \Phi(\chi_1^{\bar{\vartheta}} + \chi_2^{\bar{\vartheta}}) \; \left(\text{resp. } \xi_1^{\bar{\vartheta}} - \xi_2^{\bar{\vartheta}} \in \Phi(\chi_1^{\bar{\vartheta}} - \chi_2^{\bar{\vartheta}})\right), \text{ a.e. in } J. \tag{3.18}$$

On account of (3.16)~(3.18), $\chi_1^{\bar{\vartheta}} + \chi_2^{\bar{\vartheta}}$ (resp. $\chi_1^{\bar{\vartheta}} - \chi_2^{\bar{\vartheta}}$) is a solution of (3.11). Therefore, it follows from Lemma 3.2 that

$$\chi_1^{\bar{\vartheta}} + \chi_2^{\bar{\vartheta}} = \chi_1^{\bar{\vartheta}} - \chi_2^{\bar{\vartheta}} \equiv 0 \text{ on } \overline{J}.$$

It implies that $\chi_1^{\bar{\vartheta}} = \chi_2^{\bar{\vartheta}} \equiv 0$ on $\overline{J}$. ■

4. Proof of (s3) of Theorem 2.1

In this section, we prove the remaining assertion. Since the proof is a very long story, we divide it by several lemmas.

Lemma 4.1 *Let $\vartheta^* \le \bar\vartheta < \vartheta_c$, and $\chi_i^{\bar\vartheta} \in H^2(J)$, $i = 1, 2$, be two functions such that*

$$(\chi_1^{\bar\vartheta}(x), \chi_2^{\bar\vartheta}(x)) \in K_+ \ \textit{(resp. } (\chi_1^{\bar\vartheta}(x), \chi_2^{\bar\vartheta}(x)) \in K_-\textit{) for any } x \in \overline{J}.$$

Then, $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ is a solution of $(P)_J^{\bar\vartheta}$ if and only if

$$\begin{cases} \chi_1^{\bar\vartheta} = \chi_2^{\bar\vartheta} \ \textit{(resp. } \chi_1^{\bar\vartheta} = -\chi_2^{\bar\vartheta}\textit{) on } \overline{J}, \\ -2\lambda(\chi_1^{\bar\vartheta})_{xx} + \bar\xi_1^{\bar\vartheta} = \gamma(\bar\vartheta)\chi_1^{\bar\vartheta} + l(\vartheta^* - \bar\vartheta), \ \textit{a.e. in } J, \\ \bar\xi_1^{\bar\vartheta} \in \partial I_{[0,1]}(\chi_1^{\bar\vartheta}), \ \textit{a.e. in } J, \\ (\chi_1^{\bar\vartheta})_x(a) = (\chi_2^{\bar\vartheta})_x(b) = 0. \end{cases} \tag{4.1}$$

Proof. We show only the case of $(\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}) \in K_+$ on $\overline{J}$, since the another case is similarly obtained.

First, let us assume that $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ is a solution of $(P)_J^{\bar\vartheta}$. Then, we find two functions $\xi_i^{\bar\vartheta} \in L^2(J)$, $i = 1, 2$, which satisfy (3.5)-(3.7). Here, by the assumption and Lemma 5.1,

$$\chi_1^{\bar\vartheta} = \chi_2^{\bar\vartheta} \text{ and } \bar\xi_1^{\bar\vartheta} := \xi_1^{\bar\vartheta} + \xi_2^{\bar\vartheta} \in \partial I_{[0,1]}(\chi_1^{\bar\vartheta}), \text{ a.e. in } J.$$

Therefore, adding the both sides of (3.6) and (3.7),

$$-2\lambda(\chi_1^{\bar\vartheta})_{xx} + \bar\xi_1^{\bar\vartheta} = \gamma(\bar\vartheta)\chi_1^{\bar\vartheta} + l(\vartheta^* - \bar\vartheta) \text{ and } \bar\xi_1^{\bar\vartheta} \in \partial I_{[0,1]}(\chi_1^{\bar\vartheta}) \text{ in } J.$$

Thus, $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ satisfies (4.1).

Conversely, let us assume that $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ satisfies (4.1), and put two functions $\xi_i^{\bar\vartheta}$, $i = 1, 2$, by (3.6) and (3.7). Then, since $\chi_1^{\bar\vartheta} = \chi_2^{\bar\vartheta}$ a.e. in J, it is easily checked that

$$\xi_1^{\bar\vartheta} + \xi_2^{\bar\vartheta} = \bar\xi_1^{\bar\vartheta} \in \partial I_{[0,1]}(\chi_1^{\bar\vartheta}), \text{ a.e. in } J, \tag{4.2}$$

and

$$\xi_1^{\bar\vartheta} - \xi_2^{\bar\vartheta} = l(\vartheta^* - \bar\vartheta) - \gamma(\bar\vartheta)\chi_1^{\bar\vartheta}, \text{ a.e. in } J.$$

By (4.1),

$$\lambda(\chi_1^{\bar\vartheta})_{xx} = \frac{(\xi_1^{\bar\vartheta} + \xi_2^{\bar\vartheta}) - \gamma(\bar\vartheta)\chi_1^{\bar\vartheta} - l(\vartheta^* - \bar\vartheta)}{2} \begin{cases} \le -\dfrac{l(\vartheta^* - \bar\vartheta)}{2}, \text{ if } 0 \le \chi_1^{\bar\vartheta} < 1, \\ \ge -\dfrac{\gamma(\bar\vartheta)}{2}, \text{ if } \chi_1^{\bar\vartheta} = 1, \end{cases}$$

so that

$$\begin{cases} \xi_1^{\bar\vartheta} &= \lambda(\chi_1^{\bar\vartheta})_{xx} + l(\vartheta^* - \bar\vartheta) \le -\dfrac{l(\vartheta^* - \bar\vartheta)}{2} + l(\vartheta^* - \bar\vartheta) \\ &= \dfrac{l(\vartheta^* - \bar\vartheta)}{2} \le 0, \text{ if } 0 \le \chi_1^{\bar\vartheta} < 1, \\ \xi_2^{\bar\vartheta} &= \lambda(\chi_2^{\bar\vartheta})_{xx} + \gamma(\bar\vartheta)\chi_2^{\bar\vartheta} = \lambda(\chi_1^{\bar\vartheta})_{xx} + \gamma(\bar\vartheta)\chi_1^{\bar\vartheta} \\ &\ge -\dfrac{\gamma(\bar\vartheta)}{2} + \gamma(\bar\vartheta) = \dfrac{\gamma(\bar\vartheta)}{2} \ge 0, \text{ if } \chi_1^{\bar\vartheta} = 1. \end{cases} \tag{4.3}$$

Now, on account of (4.2)-(4.3) and Lemma 5.1, we obtain (3.5). Thus, $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ is a solution of $(P)_J^{\bar\vartheta}$. ■

On the base of the above lemma, we also have the following corollary.

Corollary 4.1 *Let $\vartheta^* \le \bar\vartheta < \vartheta_c$, $\tilde J$ be any open subinterval in J, and $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ be a solution of $(P)_J^{\bar\vartheta}$. If $(\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}) \in K_+$ (resp. K_-) on $\overline{\tilde J}$, then $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ satisfies:*

$$\begin{cases} \chi_1^{\bar\vartheta} = \chi_2^{\bar\vartheta} \ (\text{resp. } \chi_1^{\bar\vartheta} = -\chi_2^{\bar\vartheta}) \text{ on } \overline{\tilde J}, \\ -2\lambda(\chi_1^{\bar\vartheta})_{xx} + \tilde\xi_1^{\bar\vartheta} = \gamma(\bar\vartheta)\chi_1^{\bar\vartheta} + l(\vartheta^* - \bar\vartheta), \ \text{a.e. in } \tilde J, \\ \tilde\xi_1^{\bar\vartheta} \in \partial I_{[0,1]}(\chi_1^{\bar\vartheta}), \ \text{a.e. in } \tilde J. \end{cases}$$

Proposition 4.1 *If $\vartheta^* < \bar\vartheta < \vartheta_c$ and $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ is a solution of $(P)_J^{\bar\vartheta}$, then*

$$(\chi_1^{\bar\vartheta}(x), \chi_2^{\bar\vartheta}(x)) \in K_+ \cup K_- \text{ for any } x \in \overline{J}.$$

Since the proof of the above proposition is quite extended, we divide the proof in some lemmas. Let us assume $\vartheta^* < \bar\vartheta < \vartheta_c$, and take any solution $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ of $(P)_J^{\bar\vartheta}$. Then, we first have the following lemma.

Lemma 4.2 *$(\chi_1^{\bar\vartheta}(x), \chi_2^{\bar\vartheta}(x)) \in \overset{\circ}{K} \cup K_+ \cup K_-$ for any $x \in \overline{J}$, where $\overset{\circ}{K}$ is the interior of the triangle K.*

Proof. Assume that $(\chi_1^{\bar\vartheta}(y_0), \chi_2^{\bar\vartheta}(y_0)) \notin \overset{\circ}{K} \cup K_+ \cup K_-$ for some $y_0 \in \overline{J}$, namely

$$\chi_1^{\bar\vartheta}(y_0) = 1 \text{ and } -1 < \chi_2^{\bar\vartheta}(y_0) < 1. \tag{4.4}$$

Then, $(\chi_1^{\bar\vartheta})_x(y_0) = 0$. In fact, by the assumption,

$$\frac{\chi_1^{\bar\vartheta}(x) - \chi_1^{\bar\vartheta}(y_0)}{x - y_0} = \frac{\chi_1^{\bar\vartheta}(x) - 1}{x - y_0} \begin{cases} \le 0, & \text{if } x > y_0, \\ \ge 0, & \text{if } x < y_0. \end{cases}$$

So, letting $x \to y_0$ yields that $(\chi_1^{\bar\vartheta})_x(y_0) = 0$.

By (4.4), we find a (small) positive number δ_0 such that

$$(\chi_1^{\bar\vartheta}(x), \chi_2^{\bar\vartheta}(x)) \notin K_+ \cup K_- \text{ for any } x \in (y_0 - \delta_0, y_0 + \delta_0) \cap J.$$

Here,

$$\lambda(\chi_1^{\bar\vartheta})_{xx} = \xi_1^{\bar\vartheta} + l(\bar\vartheta - \vartheta^*) > 0 \text{ in } (y_0 - \delta_0, y_0 + \delta_0) \cap J,$$

so that $\chi_1^{\bar\vartheta}$ is strictly convex in $(y_0 - \delta_0, y_0 + \delta_0) \cap J$. It implies that

$$\chi_1^{\bar\vartheta}(x) > \chi_1^{\bar\vartheta}(y_0) = 1 \text{ for any } x \in (y_0 - \delta_0, y_0 + \delta_0) \cap J.$$

It is a contradiction, since $(\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}) \in K$ a.e. in J. ■

Now, for the proof of Proposition 4.1, we would like to show that

$$(\chi_1^{\bar\vartheta}(x), \chi_2^{\bar\vartheta}(x)) \notin \overset{\circ}{K} \ \text{ for any } x \in \overline{J}. \tag{4.5}$$

So let us assume that

$$(\chi_1^{\bar\vartheta}(y_1), \chi_2^{\bar\vartheta}(y_1)) \in \overset{\circ}{K} \ \text{ for some } y_1 \in \overline{J}.$$

Then we find a connected open interval $I := (\tilde{x}_0, \tilde{x}_1)$ such that

$$(\chi_1^{\bar\vartheta}(x), \chi_2^{\bar\vartheta}(x)) \in \overset{\circ}{K} \ \text{ for any } x \in I, \tag{4.6}$$

by putting

$$\tilde{x}_0 := \inf \left\{ \ x \le y_1 \ \middle| \ (\chi_1^{\bar\vartheta}(y), \chi_2^{\bar\vartheta}(y)) \in \overset{\circ}{K} \ \text{ for any } y \in [x, y_1] \ \right\}$$

and

$$\tilde{x}_1 := \sup \left\{ \ x \ge y_1 \ \middle| \ (\chi_1^{\bar\vartheta}(y), \chi_2^{\bar\vartheta}(y)) \in \overset{\circ}{K} \ \text{ for any } y \in [y_1, x] \ \right\}.$$

Then, since $\{\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}\}$ is a solution of $(P)_J^{\bar\vartheta}$, we have

$$\begin{cases} \lambda(\chi_1^{\bar\vartheta})_{xx} = \xi_1^{\bar\vartheta} - l(\vartheta^* - \bar\vartheta) = l(\bar\vartheta - \vartheta^*) > 0, \ \text{a.e. in } I, \\ -\lambda(\chi_2^{\bar\vartheta})_{xx} = -\xi_2^{\bar\vartheta} + \gamma(\bar\vartheta)\chi_2^{\bar\vartheta} = \gamma(\bar\vartheta)\chi_2^{\bar\vartheta}, \ \text{a.e. in } I. \end{cases} \tag{4.7}$$

It implies that

$$\chi_1^{\bar\vartheta} \text{ is a strictly convex parabola in } I, \tag{4.8}$$

and

$$\chi_2^{\bar\vartheta} \text{ is a sin-curve, which is symmetric with respect to } x\text{-axis, in } I. \tag{4.9}$$

Lemma 4.3 *Let $I = (\tilde{x}_0, \tilde{x}_1)$ be the interval as in (4.6). Then, $a < \tilde{x}_0 < \tilde{x}_1 < b$.*

Proof. Let us assume that $\tilde{x}_0 = a$ (resp. $\tilde{x}_1 = b$). Then, we see from the boundary condition, (4.8) and (4.9) that

$$\chi_1^{\bar\vartheta} \text{ is strictly increasing (resp. decreasing) in } I,$$

and

$$|\chi_2^{\bar\vartheta}(x)| \le |\chi_2^{\bar\vartheta}(\tilde{x}_0)| = |\chi_2^{\bar\vartheta}(a)| \ (\text{resp. } |\chi_2^{\bar\vartheta}(x)| \le |\chi_2^{\bar\vartheta}(\tilde{x}_1)| = |\chi_2^{\bar\vartheta}(b)|) \text{ for any } x \in I.$$

Since $(\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}) \in K$ on $\overline{J}$,

$$|\chi_2^{\bar\vartheta}(\tilde{x}_1)| = \lim_{x \nearrow \tilde{x}_1} |\chi_2^{\bar\vartheta}(x)| \le |\chi_2^{\bar\vartheta}(\tilde{x}_0)| \le \chi_1^{\bar\vartheta}(\tilde{x}_0) < \lim_{x \nearrow \tilde{x}_1} \chi_1^{\bar\vartheta}(x) = \chi_1^{\bar\vartheta}(\tilde{x}_1)$$

$$\left(\text{resp. } |\chi_2^{\bar\vartheta}(\tilde{x}_0)| = \lim_{x \searrow \tilde{x}_0} |\chi_2^{\bar\vartheta}(x)| \le |\chi_2^{\bar\vartheta}(\tilde{x}_1)| \le \chi_1^{\bar\vartheta}(\tilde{x}_1) < \lim_{x \searrow \tilde{x}_0} \chi_1^{\bar\vartheta}(x) = \chi_1^{\bar\vartheta}(\tilde{x}_0) \right)$$

Therefore,

$$(\chi_1^{\bar\vartheta}(\tilde{x}_1), \chi_2^{\bar\vartheta}(\tilde{x}_1)) \in \overset{\circ}{K} \ (\text{resp. } (\chi_1^{\bar\vartheta}(\tilde{x}_1), \chi_2^{\bar\vartheta}(\tilde{x}_1)) \in \overset{\circ}{K}\ ).$$

By the definition of $\tilde{x}_1$ (resp. $\tilde{x}_0$), $\tilde{x}_1 = b$ (resp. $\tilde{x}_0 = a$), so that

$$(\chi_1^{\bar\vartheta})_x(\tilde{x}_0) = (\chi_1^{\bar\vartheta})_x(\tilde{x}_1) = 0.$$

It contradicts (4.8). ■

By Lemma 4.3, we also conclude that

$$(\chi_1^{\bar\vartheta}(\tilde x_i), \chi_2^{\bar\vartheta}(\tilde x_i)) \in K_+ \cup K_-, \ \ i = 0, 1. \tag{4.10}$$

Lemma 4.4 *Let $\tilde x \in J$. If $(\chi_1^{\bar\vartheta}(\tilde x), \chi_2^{\bar\vartheta}(\tilde x)) \in K_+$ (resp. K_-), then*

$$(\chi_1^{\bar\vartheta})_x(\tilde x) = (\chi_2^{\bar\vartheta})_x(\tilde x) \ \ \left(\textit{resp. } (\chi_1^{\bar\vartheta})_x(\tilde x) = -(\chi_2^{\bar\vartheta})_x(\tilde x)\right),$$

in particular,

$$(\chi_1^{\bar\vartheta})_x(\tilde x) = (\chi_2^{\bar\vartheta})_x(\tilde x) = 0, \ \ \textit{if } \chi_1^{\bar\vartheta}(\tilde x) = \chi_2^{\bar\vartheta}(\tilde x) = 0.$$

Proof. If $(\chi_1^{\bar\vartheta}(\tilde x), \chi_2^{\bar\vartheta}(\tilde x)) \in K_+$ (resp. K_-), then

$$-\chi_1^{\bar\vartheta}(\tilde x) = -\chi_2^{\bar\vartheta}(\tilde x) \ \text{(resp. } -\chi_1^{\bar\vartheta}(\tilde x) = \chi_2^{\bar\vartheta}(\tilde x)\text{)}. \tag{4.11}$$

Since $(\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}) \in K$, a.e. in $\overline{J}$,

$$\chi_1^{\bar\vartheta}(x) \geq \chi_2^{\bar\vartheta}(x) \ \text{(resp. } \chi_1^{\bar\vartheta}(x) \geq -\chi_2^{\bar\vartheta}(x)\text{) for any } x \in \overline{J}. \tag{4.12}$$

Adding the both sides of (4.11) and (4.12),

$$\chi_1^{\bar\vartheta}(x) - \chi_1^{\bar\vartheta}(\tilde x) \geq \chi_2^{\bar\vartheta}(x) - \chi_2^{\bar\vartheta}(\tilde x) \ \ \left(\text{resp. } \chi_1^{\bar\vartheta}(x) - \chi_1^{\bar\vartheta}(\tilde x) \geq -(\chi_2^{\bar\vartheta}(x) - \chi_2^{\bar\vartheta}(\tilde x))\right),$$

so that

$$\frac{\chi_1^{\bar\vartheta}(x) - \chi_1^{\bar\vartheta}(\tilde x)}{x - \tilde x} \begin{cases} \geq \dfrac{\chi_2^{\bar\vartheta}(x) - \chi_2^{\bar\vartheta}(\tilde x)}{x - \tilde x}, & \text{if } x > \tilde x, \\[2ex] \leq \dfrac{\chi_2^{\bar\vartheta}(x) - \chi_2^{\bar\vartheta}(\tilde x)}{x - \tilde x}, & \text{if } x < \tilde x \end{cases}$$

$$\left(\text{resp. } \frac{\chi_1^{\bar\vartheta}(x) - \chi_1^{\bar\vartheta}(\tilde x)}{x - \tilde x} \begin{cases} \geq -\dfrac{\chi_2^{\bar\vartheta}(x) - \chi_2^{\bar\vartheta}(\tilde x)}{x - \tilde x}, & \text{if } x > \tilde x, \\[2ex] \leq -\dfrac{\chi_2^{\bar\vartheta}(x) - \chi_2^{\bar\vartheta}(\tilde x)}{x - \tilde x}, & \text{if } x < \tilde x \end{cases}\right).$$

Now, letting $x \to \tilde x$, we have the lemma. ■

Lemma 4.5 *Let $I = (\tilde x_0, \tilde x_1)$ be the interval given in (4.6). Then,*

$$(\chi_2^{\bar\vartheta})_x > 0 \ \textit{on } \overline{I} \ \textit{or } (\chi_2^{\bar\vartheta})_x < 0 \ \textit{on } \overline{I},$$

namely $\chi_2^{\bar\vartheta}$ is strictly increasing or strictly decreasing on $\overline{I}$.

Proof. Assume that $(\chi_2^{\bar\vartheta})_x(\tilde y_0) = 0$ for some $\tilde y_0 \in \overline{I}$. If $\tilde y_0 = \tilde x_0$ (resp. $\tilde y_0 = \tilde x_1$), then by (4.10) and Lemma 4.4,

$$(\chi_1^{\bar\vartheta})_x(\tilde x_0) = (\chi_2^{\bar\vartheta})_x(\tilde x_0) = 0 \ \ \left(\text{resp. } (\chi_1^{\bar\vartheta})_x(\tilde x_1) = (\chi_2^{\bar\vartheta})_x(\tilde x_1) = 0\right).$$

Therefore, we can show a contradiction just as in Lemma 4.3.

Now, let us assume that $\tilde{y}_0 \in I$. If $(\chi_1^{\bar{\vartheta}})_x(\tilde{y}_0) \geq 0$ (resp. $(\chi_1^{\bar{\vartheta}})_x(\tilde{y}_0) < 0$), then we see from (4.8)-(4.9) that

$$\begin{cases} \chi_1^{\bar{\vartheta}} \text{ is strictly increasing in } (\tilde{y}_0, \tilde{x}_1) \text{ (resp. strictly decreasing in } (\tilde{x}_0, \tilde{y}_0)), \\ |\chi_2^{\bar{\vartheta}}(x)| \leq |\chi_2^{\bar{\vartheta}}(\tilde{y}_0)| \text{ for any } x \in I. \end{cases}$$

Since $(\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}) \in \overset{\circ}{K}$ in I,

$$|\chi_2^{\bar{\vartheta}}(\tilde{x}_1)| = \lim_{x \nearrow \tilde{x}_1} |\chi_2^{\bar{\vartheta}}(x)| \leq |\chi_2^{\bar{\vartheta}}(\tilde{y}_0)| < \chi_1^{\bar{\vartheta}}(\tilde{y}_0) < \lim_{x \nearrow \tilde{x}_1} \chi_1^{\bar{\vartheta}}(x) = \chi_1^{\bar{\vartheta}}(\tilde{x}_1).$$

$$\left(\text{resp. } |\chi_2^{\bar{\vartheta}}(\tilde{x}_0)| = \lim_{x \searrow \tilde{x}_0} |\chi_2^{\bar{\vartheta}}(x)| \leq |\chi_2^{\bar{\vartheta}}(\tilde{y}_0)| < \chi_1^{\bar{\vartheta}}(\tilde{y}_0) < \lim_{x \searrow \tilde{x}_0} \chi_1^{\bar{\vartheta}}(x) = \chi_1^{\bar{\vartheta}}(\tilde{x}_0).\right)$$

Therefore,

$$(\chi_1^{\bar{\vartheta}}(\tilde{x}_1), \chi_2^{\bar{\vartheta}}(\tilde{x}_1)) \in \overset{\circ}{K} \left(\text{resp. } (\chi_1^{\bar{\vartheta}}(\tilde{x}_0), \chi_2^{\bar{\vartheta}}(\tilde{x}_0)) \in \overset{\circ}{K}\right).$$

It contradicts (4.10). ■

Lemma 4.6 *Let $I = (\tilde{x}_0, \tilde{x}_1)$ be the interval as in (4.6). Then, there is a point $z_0 \in I$ such that $(\chi_1^{\bar{\vartheta}})_x(z_0) = 0$.*

Proof. Let us assume that

$$(\chi_1^{\bar{\vartheta}})_x > 0 \text{ (resp. } (\chi_1^{\bar{\vartheta}})_x < 0) \text{ in } I.$$

Then, by (4.10)

$$|\chi_2^{\bar{\vartheta}}(\tilde{x}_0)| = \chi_1^{\bar{\vartheta}}(\tilde{x}_0) < \chi_1^{\bar{\vartheta}}(\tilde{x}_1) = |\chi_2^{\bar{\vartheta}}(\tilde{x}_1)| \ \left(\text{resp. } |\chi_2^{\bar{\vartheta}}(\tilde{x}_0)| = \chi_1^{\bar{\vartheta}}(\tilde{x}_0) > \chi_1^{\bar{\vartheta}}(\tilde{x}_1) = |\chi_2^{\bar{\vartheta}}(\tilde{x}_1)|\right).$$

So, we see from (4.9) that

$$|(\chi_2^{\bar{\vartheta}})_x(\tilde{x}_0)| > |(\chi_2^{\bar{\vartheta}})_x(\tilde{x}_1)| \ \left(\text{resp. } |(\chi_2^{\bar{\vartheta}})_x(\tilde{x}_0)| < |(\chi_2^{\bar{\vartheta}})_x(\tilde{x}_1)|\right) \tag{4.13}$$

On the other hand, by (4.8), (4.10) and Lemma 4.4,

$$|(\chi_2^{\bar{\vartheta}})_x(\tilde{x}_0)| = (\chi_1^{\bar{\vartheta}})_x(\tilde{x}_0) < (\chi_1^{\bar{\vartheta}})_x(\tilde{x}_1) = |(\chi_2^{\bar{\vartheta}})_x(\tilde{x}_1)|$$

$$\left(\text{resp. } |\chi_2^{\bar{\vartheta}}(\tilde{x}_0)| = -(\chi_1^{\bar{\vartheta}})_x(\tilde{x}_0) > -(\chi_1^{\bar{\vartheta}})_x(\tilde{x}_1) = |(\chi_2^{\bar{\vartheta}})_x(\tilde{x}_1)|\right).$$

It contradicts (4.13). ■

Lemma 4.7 *Let $I = (\tilde{x}_0, \tilde{x}_1)$ be the interval as in (4.6). Then, $\chi_1^{\bar{\vartheta}}(\tilde{x}_0) = \chi_1^{\bar{\vartheta}}(\tilde{x}_1)$.*

Proof. If $\chi_1^{\bar{\vartheta}}(\tilde{x}_0) < \chi_1^{\bar{\vartheta}}(\tilde{x}_1)$ (resp. $\chi_1^{\bar{\vartheta}}(\tilde{x}_0) > \chi_1^{\bar{\vartheta}}(\tilde{x}_1)$), then

$$\chi_1^{\bar{\vartheta}}(\tilde{x}_0) - \chi_1^{\bar{\vartheta}}(z_0) < \chi_1^{\bar{\vartheta}}(\tilde{x}_1) - \chi_1^{\bar{\vartheta}}(z_0) \ \left(\text{resp. } \chi_1^{\bar{\vartheta}}(\tilde{x}_0) - \chi_1^{\bar{\vartheta}}(z_0) > \chi_1^{\bar{\vartheta}}(\tilde{x}_1) - \chi_1^{\bar{\vartheta}}(z_0)\right),$$

where z_0 is the same as in Lemma 4.6. By (4.8),

$$|(\chi_1^{\bar{\vartheta}})_x(\tilde{x}_0)| < |(\chi_1^{\bar{\vartheta}})_x(\tilde{x}_1)| \ \left(\text{resp. } |(\chi_1^{\bar{\vartheta}})_x(\tilde{x}_0)| > |(\chi_1^{\bar{\vartheta}})_x(\tilde{x}_1)|\right) \tag{4.14}$$

On the other hand, by (4.10),

$$|\chi_2^{\bar\vartheta}(\tilde x_0)| = \chi_1^{\bar\vartheta}(\tilde x_0) < \chi_1^{\bar\vartheta}(\tilde x_1) = |\chi_2^{\bar\vartheta}(\tilde x_1)| \ \left(\text{resp. } |\chi_2^{\bar\vartheta}(\tilde x_0)| = \chi_1^{\bar\vartheta}(\tilde x_0) > \chi_1^{\bar\vartheta}(\tilde x_1) = |\chi_2^{\bar\vartheta}(\tilde x_1)|\right).$$

So, it follows from Lemma 4.4 and (4.9) that

$$|(\chi_1^{\bar\vartheta})_x(\tilde x_0)| = |(\chi_2^{\bar\vartheta})_x(\tilde x_0)| > |(\chi_2^{\bar\vartheta})_x(\tilde x_1)| = |(\chi_1^{\bar\vartheta})_x(\tilde x_1)|$$
$$\left(\text{resp. } |(\chi_1^{\bar\vartheta})_x(\tilde x_0)| = |(\chi_2^{\bar\vartheta})_x(\tilde x_0)| < |(\chi_2^{\bar\vartheta})_x(\tilde x_1)| = |(\chi_1^{\bar\vartheta})_x(\tilde x_1)|\right)$$

It contradicts (4.14). ■

On basis of the above lemmas, we have the following corollary.

Corollary 4.2 *Let $I = (\tilde x_0, \tilde x_1)$ be the interval as in (4.6). Then, the following three statements hold.*

(a) If $(\chi_1^{\bar\vartheta}(\tilde x_0), \chi_2^{\bar\vartheta}(\tilde x_0)) \in K_+$ (resp. K_-), then $(\chi_1^{\bar\vartheta}(\tilde x_1), \chi_2^{\bar\vartheta}(\tilde x_1)) \in K_-$ (resp. K_+).

(b) $$(\chi_1^{\bar\vartheta})_x(x) = \begin{cases} < 0, & \text{if } \tilde x_0 \le x < \dfrac{\tilde x_1 + \tilde x_0}{2}, \\ = 0, & \text{if } x = \dfrac{\tilde x_1 + \tilde x_0}{2}, \\ > 0, & \text{if } \dfrac{\tilde x_1 + \tilde x_0}{2} < x \le \tilde x_1. \end{cases}$$

(c) $\chi_2^{\bar\vartheta}(\tilde x_1) = -\chi_2^{\bar\vartheta}(\tilde x_0)$, $|\chi_2^{\bar\vartheta}(x)| \le |\chi_2^{\bar\vartheta}(\tilde x_0)| = |\chi_2^{\bar\vartheta}(\tilde x_1)|$ and

$$|(\chi_2^{\bar\vartheta})_x(x)| \le \left|(\chi_2^{\bar\vartheta})_x\left(\frac{\tilde x_1 + \tilde x_0}{2}\right)\right| \ \text{for any } x \in \overline{I}, \text{ and } \chi_2^{\bar\vartheta}\left(\frac{\tilde x_1 + \tilde x_0}{2}\right) = 0.$$

Proof. By (4.10), Lemmas 4.5 and 4.7,

$$|\chi_2^{\bar\vartheta}(\tilde x_0)| = \chi_1^{\bar\vartheta}(\tilde x_0) = \chi_1^{\bar\vartheta}(\tilde x_1) = |\chi_2^{\bar\vartheta}(\tilde x_1)| \text{ and } \chi_2^{\bar\vartheta}(\tilde x_1) = -\chi_2^{\bar\vartheta}(\tilde x_0) \neq 0, \tag{4.15}$$

which implies the assertion (a) holds.

Also, by (4.15), we find a point $\tilde z_0 \in I$ such that $\chi_2^{\bar\vartheta}(\tilde z_0) = 0$. Hence, we see from (4.8), (4.9) and symmetricities of parabola and sin-curves that

$$z_0 = \tilde z_0 = \frac{\tilde x_0 + \tilde x_1}{2}.$$

Now, it is not so difficult to check assertions (b) and (c). ■

Lemma 4.8 *Let $I = (\tilde x_0, \tilde x_1)$ be the interval as in (4.6). Then, There exist two points $\tilde x^*$ and $\tilde x_*$ such that:*

(i) $a \le \tilde x_* < \tilde x_0 < \tilde x_1 < \tilde x^* \le b$;

(ii) $(\chi_1^{\bar\vartheta})_x < 0$ on $(\tilde x_*, \tilde x_0]$ and $(\chi_1^{\bar\vartheta})_x > 0$ on $[\tilde x_1, \tilde x^*)$;

(iii) if $(\chi_1^{\bar\vartheta}(\tilde x_0), \chi_2^{\bar\vartheta}(\tilde x_0)) \in K_+$ *(resp.* K_-*), then*

$$(\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}) \in K_+ \text{ (resp. } K_-\text{) on } [\tilde x_*, \tilde x_0] \text{ and } (\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}) \in K_- \text{ (resp. } K_+\text{) on } [\tilde x_1, \tilde x^*];$$

(iv) $(\chi_i^{\bar\vartheta})_x(\tilde x_*) = (\chi_i^{\bar\vartheta})_x(\tilde x^*) = 0$, $i = 1, 2$.

Proof. We show only the case of $(\chi_1^{\bar\vartheta}(\tilde x_0), \chi_2^{\bar\vartheta}(\tilde x_0)) \in K_+$, since the another case is similarly obtained.

Let us define two sets $\tilde X_+$ and $\tilde X_-$ by putting

$$\tilde X_+ := \left\{ \ \tilde x \ge \tilde x_1 \ \middle| \ (\chi_1^{\bar\vartheta})_x(\tilde x) = 0 \ \right\} \text{ and } \tilde X_- := \left\{ \ \tilde x \le \tilde x_0 \ \middle| \ (\chi_1^{\bar\vartheta})_x(\tilde x) = 0 \ \right\}.$$

Then, $\tilde X_+$ and $\tilde X_-$ are nonempty and closed subset in J, since $\chi_1^{\bar\vartheta} \in H^2(J)$, $b \in \tilde X_+$ and $a \in \tilde X_-$.

Here let us put

$$\tilde x^* := \inf \tilde X_+ \text{ and } \tilde x_* := \sup \tilde X_-.$$

Then by (b) of Corollary 4.2, we have

$$a \le \tilde x_* < \tilde x_0 < \tilde x_1 < \tilde x^* \le b,$$

$$(\chi_1^{\bar\vartheta})_x < 0 \text{ on } (\tilde x_*, \tilde x_0] \text{ and } (\chi_1^{\bar\vartheta})_x > 0 \text{ on } [\tilde x_1, \tilde x^*), \tag{4.16}$$

and

$$(\chi_1^{\bar\vartheta})_x(\tilde x_*) = (\chi_1^{\bar\vartheta})_x(\tilde x^*) = 0 \tag{4.17}$$

Thus assertions (i) and (ii) hold.

Next, by (a) and (b) of Corollary 4.2,

$$(\chi_1^{\bar\vartheta}(\tilde x_0), \chi_2^{\bar\vartheta}(\tilde x_0)) \in K_+ \setminus (0,0) \text{ and } (\chi_1^{\bar\vartheta}(\tilde x_1), \chi_2^{\bar\vartheta}(\tilde x_1)) \in K_- \setminus (0,0). \tag{4.18}$$

Now, it follows from (4.16) that the assertion (iii) holds, namely in this case,

$$(\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}) \in K_+ \text{ on } [\tilde x_*, \tilde x_0] \text{ and } (\chi_1^{\bar\vartheta}, \chi_2^{\bar\vartheta}) \in K_- \text{ on } [\tilde x_1, \tilde x^*]. \tag{4.19}$$

In fact, if there are two points $\tilde y^*$ and $\tilde y_*$ such that

$$\tilde x_* \le \tilde y_* < \tilde x_0 < \tilde x_1 < \tilde y^* \le \tilde x^*, \ (\chi_1^{\bar\vartheta}(\tilde y^*), \chi_2^{\bar\vartheta}(\tilde y^*)) \notin K_- \text{ and } (\chi_1^{\bar\vartheta}(\tilde y_*), \chi_2^{\bar\vartheta}(\tilde y_*)) \notin K_+,$$

then by (4.16) and (4.18), we find two points $\hat x^*$ and $\hat x_*$ such that

$$\tilde x_* \le \tilde y_* \le \hat x_* < \tilde x_0 < \tilde x_1 < \hat x^* \le \tilde y^* \le \tilde x^*,$$

$$(\chi_1^{\bar\vartheta}(\hat x^*), \chi_2^{\bar\vartheta}(\hat x^*)) \in \overset{\circ}{K} \ \text{ and } (\chi_1^{\bar\vartheta}(\hat x_*), \chi_2^{\bar\vartheta}(\hat x_*)) \in \overset{\circ}{K}.$$

So, we also find two connected intervals

$$\hat I^* := (\hat x_0^*, \hat x_1^*) \text{ and } \hat I_* := (\hat x_{*,0}, \hat x_{*,1}),$$

such that

$$\tilde x_* \le \hat x_{*,0} < \hat x_* < \hat x_{*,1} \le \tilde x_0 < \tilde x_1 \le \hat x_0^* < \hat x^* < \hat x_1^* \le \tilde x^*, \tag{4.20}$$

$$(\chi_1^{\bar\vartheta}(x),\chi_2^{\bar\vartheta}(x))\in\overset{\circ}{K}\ \text{ for any } x\in\hat I^*\cup\hat I_*,$$
$$(\chi_1^{\bar\vartheta}(\hat x_{*,1}),\chi_2^{\bar\vartheta}(\hat x_{*,1}))\in K_+\cup K_- \text{ and } (\chi_1^{\bar\vartheta}(\hat x_0^*),\chi_2^{\bar\vartheta}(x_0^*))\in K_+\cup K_-,$$

by putting

$$\hat x_{*,0}:=\inf\left\{\ x\le\hat x_*\ \middle|\ (\chi_1^{\bar\vartheta}(y),\chi_2^{\bar\vartheta}(y))\in\overset{\circ}{K}\ \text{ for any } y\in[x,\hat x_*]\ \right\},$$
$$\hat x_{*,1}:=\sup\left\{\ x\ge\hat x_*\ \middle|\ (\chi_1^{\bar\vartheta}(y),\chi_2^{\bar\vartheta}(y))\in\overset{\circ}{K}\ \text{ for any } y\in[\hat x_*,x]\ \right\},$$
$$\hat x_0^*:=\inf\left\{\ x\le\hat x^*\ \middle|\ (\chi_1^{\bar\vartheta}(y),\chi_2^{\bar\vartheta}(y))\in\overset{\circ}{K}\ \text{ for any } y\in[x,\hat x^*]\ \right\},$$
$$\hat x_1^*:=\sup\left\{\ x\ge\hat x^*\ \middle|\ (\chi_1^{\bar\vartheta}(y),\chi_2^{\bar\vartheta}(y))\in\overset{\circ}{K}\ \text{ for any } y\in[\hat x^*,x]\ \right\}.$$

Here by a similar arguments to obtain (a)-(c) in Corollary 4.2, we obtain that

$$(\chi_1^{\bar\vartheta})_x(\hat x_{*,1})>0 \text{ and } (\chi_1^{\bar\vartheta})_x(\hat x_0^*)<0.$$

It contradicts (4.16).

Finally, on account of Lemma 4.4, (4.17) and (4.19), the assertion (iv) is immediately obtained. ■

Proof of Proposition 4.1: Now, we shall show Proposition 4.1 by a contradiction. Let $\vartheta^*<\bar\vartheta<\vartheta_c$ and $\{\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta}\}$ be a solution satisfying (4.6).

First, we note the detailed profile of the solution $\{\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta}\}$ on the closed interval $[\tilde x_*,\tilde x_0]$. For a simplicity, we put

$$\left\{\begin{array}{l}\chi_{\bar\vartheta}:=\chi_1^{\bar\vartheta}(\tilde x_0),\ \ \chi'_{\bar\vartheta}:=(\chi_1^{\bar\vartheta})_x(\tilde x_0),\\[2ex] p_{\bar\vartheta}:=\sqrt{\dfrac{\gamma(\bar\vartheta)}{2\lambda}},\ \ q_{\bar\vartheta}:=\dfrac{l(\bar\vartheta-\vartheta^*)}{\gamma(\bar\vartheta)}\ \text{ and } r_{\bar\vartheta}:=\chi_1^{\bar\vartheta}(\tilde x_*)-q_{\bar\vartheta}.\end{array}\right.\tag{4.21}$$

Then, on account of Remark 2.2, Lemma 4.8 and Corollaries 4.1 and 4.2, we have

$$p_{\bar\vartheta}>0,\ \ 0<q_{\bar\vartheta}<1 \text{ and } 0<r_{\bar\vartheta}\le\min\{q_{\bar\vartheta},1-q_{\bar\vartheta}\},\tag{4.22}$$
$$\chi_{\bar\vartheta}=r_{\bar\vartheta}\ \cos(p_{\bar\vartheta}(\tilde x_0-\tilde x_*))+q_{\bar\vartheta}>q_{\bar\vartheta},$$
$$\chi'_{\bar\vartheta}=-r_{\bar\vartheta}p_{\bar\vartheta}\ \sin(p_{\bar\vartheta}(\tilde x_0-\tilde x_*)).$$

Therefore, the pair $(\chi_{\bar\vartheta},\chi'_{\bar\vartheta})\in\mathbb{R}^2$ is always on the following ellipse:

$$\left(\chi_{\bar\vartheta}-q_{\bar\vartheta}\right)^2=r_{\bar\vartheta}^2-\left(\frac{\chi'_{\bar\vartheta}}{p_{\bar\vartheta}}\right)^2.\tag{4.23}$$

Secondly, we note the detailed profile of the solution $\{\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta}\}$ on the closed interval $\left[\tilde x_0,\frac{\tilde x_0+\tilde x_1}{2}\right]$. On account of the definition of $\tilde x_0$, (4.7), (4.21) and Lemma 4.4, $\chi_1^{\bar\vartheta}$ and $\chi_2^{\bar\vartheta}$ are solutions of the following ordinary differential equations:

$$\left\{\begin{array}{l}\lambda(\chi_1^{\bar\vartheta})_{xx}=l(\bar\vartheta-\vartheta^*)\ \text{ in } \left(\tilde x_0,\dfrac{\tilde x_0+\tilde x_1}{2}\right),\\ \chi_1^{\bar\vartheta}(\tilde x_0)=\chi_{\bar\vartheta},\ \ (\chi_1^{\bar\vartheta})_x(\tilde x_0)=\chi'_{\bar\vartheta}\end{array}\right.$$

and

$$\begin{cases} -\lambda(\chi_2^{\bar\vartheta})_{xx} = \gamma(\bar\vartheta)\chi_2^{\bar\vartheta} \ \text{ in } \left(\tilde x_0, \dfrac{\tilde x_0+\tilde x_1}{2}\right), \\ \chi_2^{\bar\vartheta}(\tilde x_0) = \pm\chi_{\bar\vartheta},\ (\chi_2^{\bar\vartheta})_x(\tilde x_0) = \pm\chi'_{\bar\vartheta}, \end{cases}$$

respectively. Therefore, by basic calculations,

$$\chi_1^{\bar\vartheta}(x) = \frac{l(\bar\vartheta-\vartheta^*)}{2\lambda}\left((x-\tilde x_0) + \frac{\lambda\chi'_{\bar\vartheta}}{l(\bar\vartheta-\vartheta^*)}\right)^2 + \left(\chi_{\bar\vartheta} - \frac{\lambda{\chi'_{\bar\vartheta}}^2}{2l(\bar\vartheta-\vartheta^*)}\right)$$

and

$$\chi_2^{\bar\vartheta}(x) = \mp\sqrt{\chi_{\bar\vartheta}^2 + \frac{\lambda{\chi'_{\bar\vartheta}}^2}{\gamma(\bar\vartheta)}}\sin\left(\sqrt{\frac{\gamma(\bar\vartheta)}{\lambda}}(x-\tilde x_0) + \mathrm{Tan}^{-1}\left(\frac{\chi_{\bar\vartheta}}{\chi'_{\bar\vartheta}}\sqrt{\frac{\gamma(\bar\vartheta)}{\lambda}}\right)\right).$$

By (b) and (c) of Corollary 4.2,

$$(\chi_1^{\bar\vartheta})_x\left(\frac{\tilde x_1+\tilde x_0}{2}\right) = 0 \text{ and } \chi_2^{\bar\vartheta}\left(\frac{\tilde x_1+\tilde x_0}{2}\right) = 0,$$

which implies

$$-\frac{\lambda\chi'_{\bar\vartheta}}{l(\bar\vartheta-\vartheta^*)} = \frac{\tilde x_1-\tilde x_0}{2} = -\sqrt{\frac{\lambda}{\gamma(\bar\vartheta)}}\mathrm{Tan}^{-1}\left(\frac{\chi_{\bar\vartheta}}{\chi'_{\bar\vartheta}}\sqrt{\frac{\gamma(\bar\vartheta)}{\lambda}}\right).$$

Using notations as in (4.21), the above equality can be reduced by

$$\chi_{\bar\vartheta} = \frac{\chi'_{\bar\vartheta}}{\sqrt2 p_{\bar\vartheta}}\tan\left(\frac{\chi'_{\bar\vartheta}}{\sqrt2 p_{\bar\vartheta}q_{\bar\vartheta}}\right), \quad -\frac{p_{\bar\vartheta}q_{\bar\vartheta}}{\sqrt2}\pi < \chi'_{\bar\vartheta} < 0. \tag{4.24}$$

Combining (4.23) and (4.24), we obtain that

$$\frac12\left(\frac{\chi'_{\bar\vartheta}}{\sqrt2 p_{\bar\vartheta}q_{\bar\vartheta}}\tan\left(\frac{\chi'_{\bar\vartheta}}{\sqrt2 p_{\bar\vartheta}q_{\bar\vartheta}}\right)-1\right)^2 = \frac12\left(\frac{r_{\bar\vartheta}}{q_{\bar\vartheta}}\right)^2 - \left(\frac{\chi'_{\bar\vartheta}}{\sqrt2 p_{\bar\vartheta}q_{\bar\vartheta}}\right)^2 \quad \text{and } -p_{\bar\vartheta}r_{\bar\vartheta} < \chi'_{\bar\vartheta} < 0. \tag{4.25}$$

On the other hand, by Lemma 5.2,

$$\frac12(u\tan u - 1)^2 - \left(\frac12 - u^2\right) = u\left(u + \frac12 u\tan^2 u - \tan u\right) > 0 \quad \text{for any } -\frac{1}{\sqrt2} < u < 0.$$

So, by the following change of variable:

$$v = \sqrt2 p_{\bar\vartheta}q_{\bar\vartheta}\,u \ \text{ for } -\frac{1}{\sqrt2} < u < 0 \text{ and } -p_{\bar\vartheta}q_{\bar\vartheta} < v < 0,$$

it follows from (4.22) that

$$\frac12\left(\frac{v}{\sqrt2 p_{\bar\vartheta}q_{\bar\vartheta}}\tan\left(\frac{v}{\sqrt2 p_{\bar\vartheta}q_{\bar\vartheta}}\right)-1\right)^2 > \frac12 - \left(\frac{v}{\sqrt2 p_{\bar\vartheta}q_{\bar\vartheta}}\right)^2 \geq \frac12\left(\frac{r_{\bar\vartheta}}{q_{\bar\vartheta}}\right)^2 - \left(\frac{v}{\sqrt2 p_{\bar\vartheta}q_{\bar\vartheta}}\right)^2 \quad \text{for any } -p_{\bar\vartheta}r_{\bar\vartheta} < v < 0.$$

It contradicts (4.25). ■

Proof of (s3) of Theorem 2.1: First, let us assume that a pair $\{\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta}\}$ of H^2-functions is a solution of $(P)_J^{\bar\vartheta}$. Then, by Proposition 4.1,

$$\chi_1^{\bar\vartheta}(x)=\chi_2^{\bar\vartheta}(x) \text{ or } \chi_1^{\bar\vartheta}(x)=-\chi_2^{\bar\vartheta}(x) \text{ for any } x\in\overline{J}. \tag{4.26}$$

If $(\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta})\in K_+$ on $\overline{J}$ or $(\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta})\in K_-$ on $\overline{J}$, then it immediately follows from Lemma 4.1 that

$$\chi_1^{\bar\vartheta}\in S_{J,[0,1]}\left(\frac{2\lambda}{\gamma(\bar\vartheta)},\frac{l(\bar\vartheta-\vartheta^*)}{\gamma(\bar\vartheta)}\right). \tag{4.27}$$

Now, we consider other cases. Let us assume that

$$\begin{cases} J_+ := \{\ x\in J \mid (\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta})\in K_+\setminus\{(0,0)\}\ \}\neq\emptyset, \\ J_- := \{\ x\in J \mid (\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta})\in K_-\setminus\{(0,0)\}\ \}\neq\emptyset. \end{cases}$$

By the smoothness of $\chi_1^{\bar\vartheta}$ and $\chi_2^{\bar\vartheta}$, J_+ and J_- are disjoint unions of connected and open subintervals of J. Let $I_+ := (\underline{x}_+,\overline{x}_+)$ be any connected component of J_+. Then, we immediately see from Lemma 4.4 that

$$(\chi_i^{\bar\vartheta})_x(\underline{x}_+)=(\chi_i^{\bar\vartheta})_x(\overline{x}_+)=0 \text{ for } i=1,2. \tag{4.28}$$

So, by Lemma 4.1 and Remark 2.2 with $J=I_+$,

$$\chi_1^{\bar\vartheta}\in S_{I_+,[0,1]}\left(\frac{2\lambda}{\gamma(\bar\vartheta)},\frac{l(\bar\vartheta-\vartheta^*)}{\gamma(\bar\vartheta)}\right), \tag{4.29}$$

and

$$|I_+| := |\overline{x}_+-\underline{x}_+|\geq\sqrt{\frac{2\lambda}{\gamma(\bar\vartheta)}}\pi>0.$$

It implies that the number of connected components of J_+ is finite. Therefore, $J\setminus\overline{J_+}$ ($\supset J_-$) is also a disjoint union of a finite number of connected and open intervals of J. Let $I_- := (\underline{x}_-,\overline{x}_-)$ be any connected component of $J\setminus\overline{J_+}$. Then, we see from the definition of J_+ and (4.28) that

$$(\chi_1^{\bar\vartheta},\chi_2^{\bar\vartheta})\in K_- \text{ in } I_- \text{ and } (\chi_i^{\bar\vartheta})_x(\underline{x}_-)=(\chi_i^{\bar\vartheta})_x(\overline{x}_-)=0,\ i=1,2.$$

So, applying Lemma 4.1 with $J=I_-$,

$$\chi_1^{\bar\vartheta}\in S_{I_-,[0,1]}\left(\frac{2\lambda}{\gamma(\bar\vartheta)},\frac{l(\bar\vartheta-\vartheta^*)}{\gamma(\bar\vartheta)}\right). \tag{4.30}$$

On account of (4.29) and (4.30), we obtain (4.27).

Conversely, let us assume (4.26) and (4.27). Then, by Remark 2.2, the set:

$$\hat{Y} := \{\ x\in J \mid \chi_1^{\bar\vartheta}>0\ \}$$

is a disjoint union of a finite number of connected and open subintervals of J. Let $\hat{I} := (\hat{y}_0, \hat{y}_1)$ be any connected component of $\hat{Y}$. Then, we see from (4.26), (4.27) and Remark 2.2 that

$$(\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}) \in K_+ \text{ on } \overline{\hat{I}} \text{ or } (\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}) \in K_- \text{ on } \overline{\hat{I}}.$$

and

$$(\chi_i^{\bar{\vartheta}})_x(\hat{y}_0) = (\chi_i^{\bar{\vartheta}})_x(\hat{y}_1) = 0, \ i = 1, 2.$$

Applying Lemma 4.1 with $J = \hat{I}$, $\{\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}\}$ is a solution of $(P)_{\hat{I}}^{\bar{\vartheta}}$. Now, we conclude that $\{\chi_1^{\bar{\vartheta}}, \chi_2^{\bar{\vartheta}}\}$ is a solution of $(P)_J^{\bar{\vartheta}}$, since

$$\chi_1^{\bar{\vartheta}} = \chi_2^{\bar{\vartheta}} \equiv 0 \text{ on } J \setminus \hat{Y}. \quad \blacksquare$$

5. Appendix

In general, let C be any closed and convex set in $\mathbb{R}^N$ with $1 \le N < +\infty$, and I_C be the indicator function on C. Then it is easy to see from the definition of the subdifferential that

$$\partial I_C(x) = \left\{ \ x^* \in \mathbb{R}^N \ \middle| \ x^* \cdot (y - x) \le 0 \ \text{ for any } y \in C \ \right\}.$$

Roughly speaking, the subdifferential ∂I_C is a set of all vectors which are normal to C.

On basis of the above fact, we immediately have the following lemma.

Lemma 5.1 *Let K be the triangle in $\mathbb{R}^2$ as in (1.5), and I_K be the indicator function on K. Then*

$$\partial I_K(\eta, \zeta) = \begin{cases} 0, \ \textit{if } (\eta, \zeta) \in \overset{\circ}{K}, \\ \left\{ \begin{pmatrix} \eta^* \\ \zeta^* \end{pmatrix} \in \mathbb{R}^2 \ \middle| \ \eta^* \le 0, \ \eta^* + \zeta^* \le 0, \ \eta^* - \zeta^* \le 0 \ \right\}, \ \textit{if } (\eta, \zeta) = (0, 0), \\ \left\{ \begin{pmatrix} \eta^* \\ \zeta^* \end{pmatrix} \in \mathbb{R}^2 \ \middle| \ \eta^* \le 0, \ \zeta^* \ge 0, \ \eta^* + \zeta^* = 0 \ \right\}, \\ \qquad\qquad\qquad \textit{if } (\eta, \zeta) \in K_+ \setminus \{(0, 0), (1, 1)\}, \\ \left\{ \begin{pmatrix} \eta^* \\ \zeta^* \end{pmatrix} \in \mathbb{R}^2 \ \middle| \ \zeta^* \ge 0, \ \eta^* + \zeta^* \ge 0 \ \right\}, \ \textit{if } (\eta, \zeta) = (1, 1), \\ \left\{ \begin{pmatrix} \eta^* \\ \zeta^* \end{pmatrix} \in \mathbb{R}^2 \ \middle| \ \eta^* \ge 0, \ \zeta^* = 0 \ \right\}, \ \textit{if } (\eta, \zeta) \in K_1 \setminus \{(1, 1), (1, -1)\}, \\ \left\{ \begin{pmatrix} \eta^* \\ \zeta^* \end{pmatrix} \in \mathbb{R}^2 \ \middle| \ \zeta^* \le 0, \ \eta^* - \zeta^* \ge 0 \ \right\}, \ \textit{if } (\eta, \zeta) = (1, -1), \\ \left\{ \begin{pmatrix} \eta^* \\ \zeta^* \end{pmatrix} \in \mathbb{R}^2 \ \middle| \ \eta^* \le 0, \ \zeta \le 0, \ \eta^* - \zeta^* = 0 \ \right\}, \\ \qquad\qquad\qquad \textit{if } (\eta, \zeta) \in K_- \setminus \{(1, -1), (0, 0)\}, \\ \emptyset, \ \textit{otherwise.} \end{cases}$$

Next, we consider the following function $f \in C^\infty(-\frac{\pi}{2}, \frac{\pi}{2})$, defined by

$$f(x) := x + \frac{1}{2} x \tan^2 x - \tan x \text{ for any } -\frac{\pi}{2} < x < \frac{\pi}{2}. \tag{5.1}$$

Then, the following property gives an useful information for in the proof of our main result.

Lemma 5.2 *Let f be the function defined by (5.1). Then, f is strictly increasing in $(-\frac{\pi}{2}, \frac{\pi}{2})$, and $f(x) = 0$ if and only if $x = 0$.*

Proof. Taking the derivative of f, we have

$$f'(x) = \frac{2x \tan x - \sin^2 x}{2\cos^2 x} = \frac{\tan x}{\cos^2 x}\left(x - \frac{1}{4}\sin(2x)\right) = \frac{x \tan x}{\cos^2 x}\left(1 - \frac{1}{2}\frac{\sin(2x)}{2x}\right) > 0$$
$$\text{for any } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}.$$

Therefore, f is strictly increasing in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover, since $f(0) = 0$, we conclude that $x = 0$ is a unique solution of the equation $f(x) = 0$. ■

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